

## System Dynamics (22.554 & 24.509)

### IV. Transform and Frequency Domain Methods

#### Introduction

The previous section has detailed the *time domain solution of dynamic systems*. We saw that the state space description of a linear dynamic system focuses on the behavior of the entire system. It addresses the evolution of all state variables as a function of the prescribed initial conditions and input variables. Analysis of the transient response of systems enhances our overall understanding of the system performance, but it does not tell us directly how to influence the system behavior. An important question is "How does one control or alter the response of a particular system?" An alternate approach to viewing the system may give additional insight into the design of systems (or the control of systems). The goal of this section will be to introduce the *frequency domain description of dynamic systems*. Here we will see how to go from the time domain to the frequency domain and vice versa. In the frequency domain description, we will concentrate on the transformation of system inputs to system outputs.

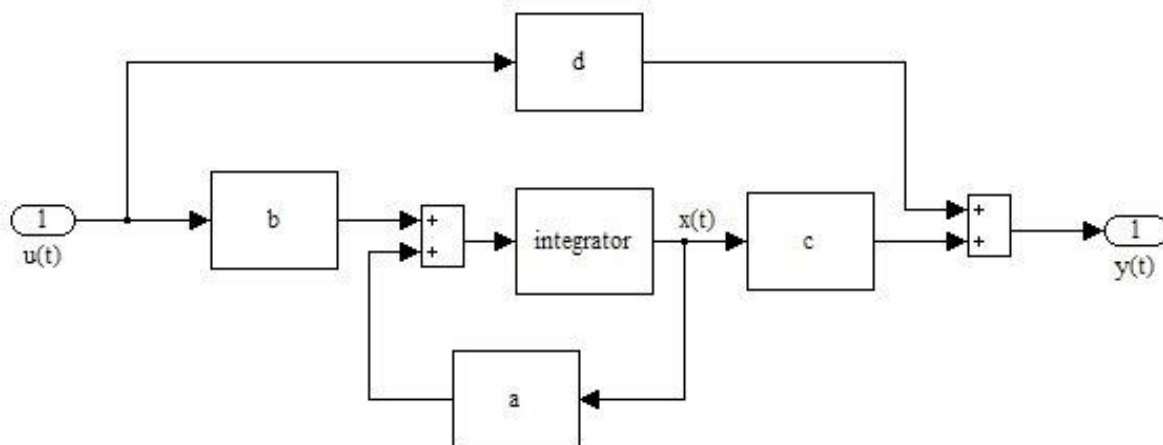
The state variables that completely describe a given system provide considerable information, but normally only a few measurable or special outputs are of interest to the design or control engineer. In this context, we have shown how the system outputs can be written in terms of the state variables and input variables. For linear stationary systems, one has

$$\frac{d}{dt} \underline{x} = \underline{A} \underline{x} + \underline{B} \underline{u} \quad (4.1)$$

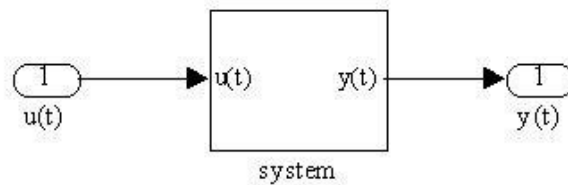
$$\underline{y} = \underline{C} \underline{x} + \underline{D} \underline{u} \quad (4.2)$$

where  $\underline{y} = \underline{y}(t)$  is the vector of system outputs and  $\underline{u} = \underline{u}(t)$  is the vector of system inputs.

If a system has only one input and one output (SISO), it is easy to think of the system as a black box that somehow transforms the input function into an output response. For example, the following system



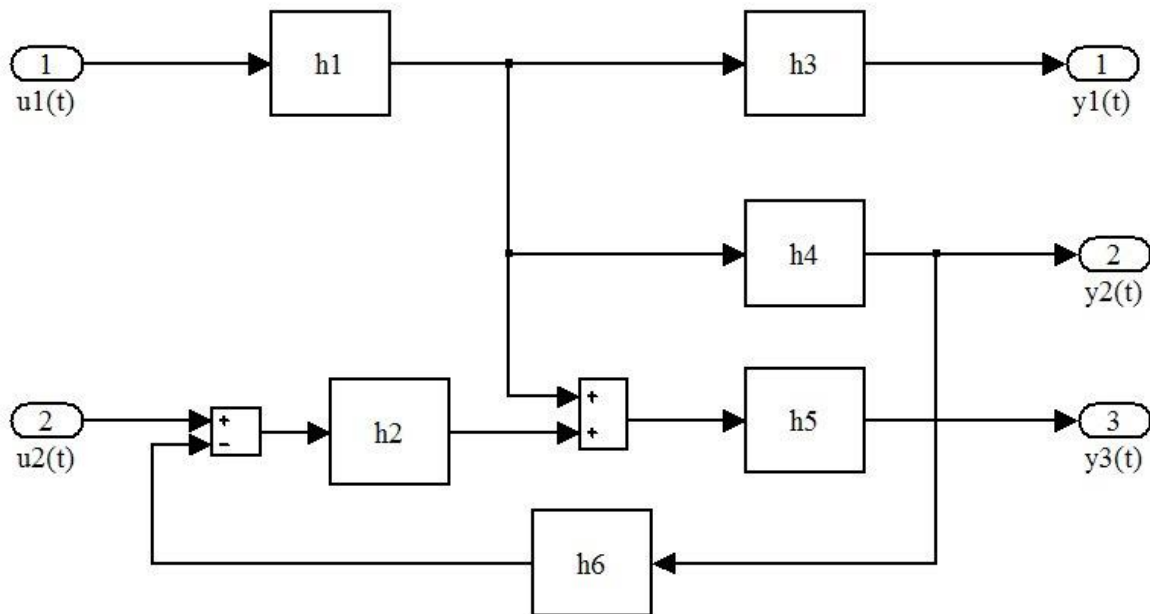
can be represented simply as



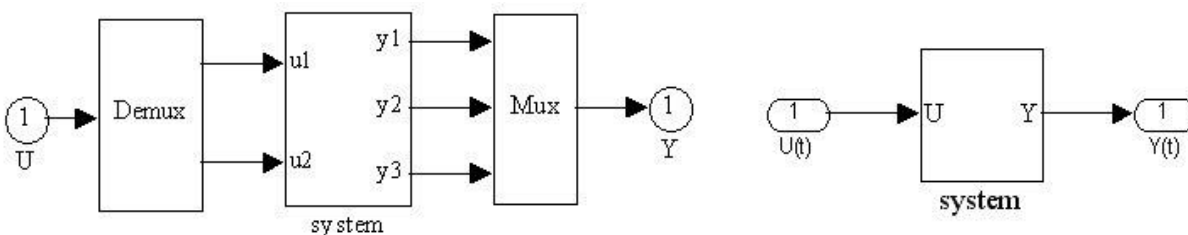
where the input-output relationship within the “system” block is given by

$$x(t) = \int_0^t (ax(\tau) + bu(\tau))d\tau \quad \text{and} \quad y(t) = cx(t) + du(t)$$

If there are multiple inputs and outputs (MIMO), then it is often useful to partition the original system into a number of interconnected subsystems each having a single input and output. Each subsystem can be analyzed as a transformation of input to output and the results can be combined in an appropriate manner (to be discussed shortly). For example, the following system has two inputs, three outputs, and six subsystems with single input-output relationships.



This relatively complex system can be represented as a single block with multiple inputs and outputs or as a block with a single input vector  $U(t) = \underline{u}(t)$  and single output vector  $Y(t) = \underline{y}(t)$ , as shown below.



### Laplace Transforms

For the moment we will shift our study of dynamic systems from a simulation viewpoint to the control or design viewpoint. Our goal will be to introduce methods for the determination of input-output relationships. With this capability we should be able to design a system to have certain desired output characteristics. To do this, we need to be familiar with **Laplace Transforms**.

We will see shortly that the Laplace transform is a useful device for describing input-output relationships. Before showing this fact however, let's become familiar with the definition and manipulation of Laplace transform pairs.

*Definition:* The Laplace transform is defined by the following linear transformation

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st} dt \quad (4.3)$$

where  $s = \sigma + j\omega$  is an arbitrary complex number and  $f(t)$  is of exponential order (i.e.  $f(t)e^{-st}$  is bounded for large  $t$ ).

Note that the above condition on  $f(t)$  is not very restrictive, since  $\sigma$  can be chosen to allow  $f(t)e^{-(\sigma+j\omega)t}$  to be bounded. When  $f(t)$  is increasing faster than exponentially, then an appropriate choice of  $\sigma$  can not be made (i.e.  $\exp(at^2)$  does not have a Laplace transform). However, most systems have transient solutions that are of exponential order or less.

*Definition:* The inverse Laplace transform is given by

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds \quad (4.4)$$

This is a formal definition. In practice, one usually does not need to perform a contour integration in the complex plane. Instead, a "dictionary" of Laplace transform pairs  $[f(t) \Leftrightarrow F(s)]$  is generated and some simple rules allow one to easily convert between the time domain solution,  $f(t)$ , and the frequency or  $s$ -plane solution,  $F(s)$ .

#### Some Laplace Transform Pairs $[f(t) \Leftrightarrow F(s)]$

##### Unit Impulse

$$f(t) = \delta(t) \quad \text{and} \quad F(s) = 1 \quad (4.5)$$

$$\text{where } F(s) = \int_0^{\infty} \delta(t)e^{-st} dt = 1$$

##### Unit Step

$$f(t) = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \quad \text{and} \quad F(s) = \frac{1}{s} \quad (4.6)$$

$$\text{where } F(s) = \int_0^{\infty} (1)e^{-st} dt = \left. \frac{-1}{s} e^{-st} \right|_0^{\infty} = \frac{-1}{s} (0 - 1) = \frac{1}{s}$$

Exponential

$$f(t) = e^{-at} \quad \text{and} \quad F(s) = \frac{1}{s+a} \quad (4.7)$$

$$\text{where } F(s) = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt = \frac{-1}{s+a} e^{-(s+a)t} \Big|_0^{\infty} = \frac{1}{s+a}$$

Ramp

$$f(t) = at \quad \text{and} \quad F(s) = \frac{a}{s^2} \quad (4.8)$$

$$\text{where } F(s) = \int_0^{\infty} at e^{-st} dt = a \left[ \frac{e^{-st}}{s^2} (-st - 1) \right] \Big|_0^{\infty} = \frac{a}{s^2}$$

Superposition

$$f(t) = af_1(t) + bf_2(t) \quad \text{and} \quad F(s) = aF_1(s) + bF_2(s) \quad (4.9)$$

Let's use this relationship to find  $L\{\sin \omega_o t\}$ , where

$$f(t) = \sin \omega_o t = \frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j} \quad \text{and} \quad F(s) = \frac{\omega_o}{s^2 + \omega_o^2} \quad (4.10)$$

$$\text{where } f(t) = f_1(t) + f_2(t) = \frac{e^{j\omega_o t}}{2j} - \frac{e^{-j\omega_o t}}{2j}$$

From eqns. (4.7) and (4.9) we have

$$F_1(s) = \frac{1}{2j} \left( \frac{1}{s - j\omega_o} \right) \quad \text{and} \quad F_2(s) = \frac{-1}{2j} \left( \frac{1}{s + j\omega_o} \right)$$

$$\text{or } F(s) = \frac{1}{2j} \left[ \frac{1}{s - j\omega_o} - \frac{1}{s + j\omega_o} \right] = \frac{1}{2j} \left[ \frac{s + j\omega_o - s + j\omega_o}{s^2 + \omega_o^2} \right] = \frac{\omega_o}{s^2 + \omega_o^2}$$

Note that this same relationship can also be derived from the basic definition in eqn. (4.3).

Time Delay

$$f(t) = f_1(t - t_0)u(t - t_0) \quad \text{and} \quad F(s) = F_1(s)e^{-st_0} \quad (4.11)$$

$$\text{where } F(s) = \int_0^{\infty} f_1(t - t_0)u(t - t_0)e^{-st} dt = \int_{t_0}^{\infty} f_1(t - t_0)e^{-st} dt$$

Letting  $\tau = t - t_0$  and  $d\tau = dt$ , gives

$$F(s) = \int_0^{\infty} f_1(\tau)e^{-s(\tau+t_0)} d\tau = e^{-st_0} \int_0^{\infty} f_1(\tau)e^{-s\tau} d\tau = F_1(s)e^{-st_0}$$

Note that the time delay is given formally by the use of the delayed unit step function,  $u(t - t_0)$  [which is sometimes referred to as the Heaviside function]. If we define a **causal time function** as one that is identically zero for a negative argument [that is,  $f(t - t_0) = 0$  for  $t < t_0$ ], then the use of the delayed unit step notation is really unnecessary. Therefore, we simply denote such a causal time function as  $f(t - t_0)$  and its Laplace transform is  $F(s)e^{-st_0}$  where  $F(s) = \mathcal{L}\{f(t)\}$ .

### Differentiation

$$f(t) = \frac{d}{dt} f_1(t) \quad \text{and} \quad F(s) = sF_1(s) - f_1(0) \quad (4.12)$$

$$\text{where } F(s) = \int_0^\infty \frac{df_1(t)}{dt} e^{-st} dt$$

Integrating this expression by parts gives

$$u = e^{-st} \quad dv = \frac{df_1(t)}{dt} dt \quad \text{and} \quad du = -se^{-st} dt \quad v = f_1(t)$$

$$\text{and } F(s) = f_1(t)e^{-st} \Big|_0^\infty - \int_0^\infty (-se^{-st}) f_1(t) dt = -f_1(0) + sF_1(s)$$

For the 2<sup>nd</sup> derivative, we have

$$f(t) = \frac{d^2 f_1}{dt^2} \quad \text{and} \quad F(s) = s^2 F_1(s) - sf_1(t) \Big|_0 - \frac{df_1(t)}{dt} \Big|_0 \quad (4.13)$$

In general, the  $n^{\text{th}}$  derivative is

$$f(t) = \frac{d^n f_1}{dt^n} \quad \text{and} \quad F(s) = s^n F_1(s) - s^{n-1} f_1(t) \Big|_0 - s^{n-2} \frac{df_1(t)}{dt} \Big|_0 - \dots - \frac{d^{n-1} f_1(t)}{dt^{n-1}} \Big|_0 \quad (4.14)$$

### Integration

$$f(t) = \int_0^t f_1(\tau) d\tau \quad \text{and} \quad F(s) = \frac{1}{s} F_1(s) \quad (4.15)$$

$$\text{where } F(s) = \int_0^\infty \left\{ \int_0^t f_1(\tau) d\tau \right\} e^{-st} dt$$

Integrating by parts gives

$$u = \int_0^t f_1(\tau) d\tau \quad dv = e^{-st} dt \quad \text{and} \quad du = f_1(t) dt \quad v = \frac{-1}{s} e^{-st}$$

$$\text{and } F(s) = \frac{-1}{s} e^{-st} \int_0^t f_1(\tau) d\tau \Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st} f_1(t) dt = \frac{1}{s} F_1(s)$$

The reader should see Table 4.1 and/or consult an available advanced math or dynamic systems text for a more complete list of several important Laplace transform pairs. This particular table, which was taken from **Control and Dynamic Systems** by Takahashi et. al., contains both Laplace

transforms for continuous functions and a set of Z transforms for the corresponding discrete representation (where T is the sampling period). The Z-transform for dealing with discrete systems will be defined shortly.

**Table 4.1 Some Laplace Transform and Z Transform Pairs  
(from Takahashi et. al.)**

|    | $F(s)$                   | $f(t) = L^{-1}[F(s)]$             | $f_{(kT)} = f_k$                          | $Z[f_k] = F(z)$  |
|----|--------------------------|-----------------------------------|---|--|
| 1  | $\frac{1}{s}$            | 1                                 | 1   | $\frac{z}{z-1}$  |
| 2  | $\frac{1}{s^2}$          | t                                 | kT  | $\frac{Tz}{(z-1)^2}$                                   |
| 3  | $\frac{1}{s^3}$          | $\frac{1}{2}t^2$                  | $\frac{1}{2}(kT)^2$                       | $\frac{T^2}{2} \frac{z(z+1)}{(z-1)^3}$                 |
| 4  | $\frac{1}{s+a}$          | $e^{-at}$                         | $c^k, c = e^{-aT}$                        | $\frac{z}{z-c}$  |
| 5  | $\frac{1}{(s+a)^2}$      | $te^{-at}$                        | $(kT)c^k, c = e^{-aT}$                    | $\frac{cTz}{(z-c)^2}$                                  |
| 6  | $\frac{1}{(s+a)^3}$      | $\frac{1}{2}t^2e^{-at}$           | $\frac{1}{2}(kT)^2c^k, c = e^{-aT}$       | $\frac{T^2}{2} \frac{cz(z+c)}{(z-c)^3}$                |
| 7  | $\frac{a}{s(s+a)}$       | $1 - e^{-at}$                     | $1 - c^k, c = e^{-aT}$                    | $\frac{(1-c)z}{(z-1)(z-c)}$                            |
| 8  | $\frac{a}{s^2(s+a)}$     | $\frac{1}{a}[at - (1 - e^{-at})]$ | $\frac{1}{a}[kaT - 1 + c^k], c = e^{-aT}$ | $\frac{Tz}{(z-1)^2} - \frac{(1-c)z}{a(z-1)(z-c)}$      |
| 9  | $\frac{s}{(s+a)^2}$      | $(1-at)e^{-at}$                   | $(1-kaT)c^k, c = e^{-aT}$                 | $\frac{z^2 - c(1+aT)z}{(z-c)^2}$                       |
| 10 | $\frac{a^2}{s(s+a)^2}$   | $1 - (1+at)e^{-at}$               | $1 - (1+kaT)c^k, c = e^{-aT}$             | $\frac{z}{z-1} - \frac{z}{z-c} - \frac{caTz}{(z-c)^2}$ |
| 11 | $\frac{b-a}{(s+a)(s+b)}$ | $e^{-at} - e^{-bt}$               | $c^k - d^k, c = e^{-aT}, d = e^{-bT}$     | $\frac{(c-d)z}{(z-c)(z-d)}$                            |

|    |  |                                       |  |   |
|----|--|---------------------------------------|--|---|
| 12 | $\frac{(b-a)s}{(s+a)(s+b)}$  | $-ae^{-at} + be^{-bt}$                | $-ac^k + bd^k,$<br>$c = e^{-aT}, d = e^{-bT}$                | $\frac{(b-a)z^2 - (bc-ad)z}{(z-c)(z-d)}$  |
| 13 | $\frac{ab}{s(s+a)(s+b)}$   | $1 + \frac{be^{-at} - ae^{-bt}}{a-b}$ | $1 + \frac{bc^k - ad^k}{a-b},$<br>$c = e^{-aT}, d = e^{-bT}$ | $\frac{z}{z-1} + \frac{b}{a-b} \frac{z}{z-c}$<br>$-\frac{a}{a-b} \frac{z}{z-d}$ |
| 14 | $\frac{\beta}{s^2 + \beta^2}$                                      | $\sin \beta t$                        | $\sin k\beta T$  | $\frac{z \sin \beta T}{z^2 - 2z \cos \beta T + 1}$                              |
| 15 | $\frac{s}{s^2 + \beta^2}$  | $\cos \beta t$                        | $\cos k\beta T$  | $\frac{z^2 - z \cos \beta T}{z^2 - 2z \cos \beta T + 1}$                        |
| 16 | $\frac{\beta}{s^2 - \beta^2}$                                      | $\sinh \beta t$                       | $\sinh k\beta T$   | $\frac{z \sinh \beta T}{z^2 - 2z \cosh \beta T + 1}$                            |
| 17 | $\frac{s}{s^2 - \beta^2}$  | $\cosh \beta t$                       | $\cosh k\beta T$   | $\frac{z^2 - z \cosh \beta T}{z^2 - 2z \cosh \beta T + 1}$                      |
| 18 | $\frac{s + \alpha}{(s + \alpha)^2 + \beta^2}$                      | $e^{-\alpha t} \cos \beta t$          | $c^k \cos k\beta T,$<br>$c = e^{-\alpha T}$                  | $\frac{z^2 - cz \cos \beta T}{z^2 - 2cz \cos \beta T + c^2}$                    |
| 19 | $\frac{s + \alpha}{(s + \alpha)^2 + \left(\frac{\pi}{T}\right)^2}$ | $e^{-\alpha t} \cos \frac{\pi t}{T}$  | $(-c^k), \quad c = e^{-\alpha T}$                            | $\frac{z}{z - c}$   |
| 20 | $\frac{\beta}{(s + \alpha)^2 + \beta^2}$                           | $e^{-\alpha t} \sin \beta t$          | $c^k \sin k\beta T,$<br>$c = e^{-\alpha T}$                  | $\frac{cz \sin \beta T}{z^2 - 2cz \cos \beta T + c^2}$                          |

### Inverse Transforms

We have seen that the formal definition of the inverse transform is

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds$$

However, this formal approach is rarely used. Instead, use is made of the simple transform pairs,  $f(t) \Leftrightarrow F(s)$ , many of which were derived in the previous few pages and listed in Table 4.1.

The transformation of a single input to some output function is typically described by an  $n^{\text{th}}$ -order differential equation. Laplace transform solutions to  $n^{\text{th}}$ -order lumped parameter linear time-invariant systems are typically of the form

$$F(s) = \frac{(s - a_1)(s - a_2) \cdots (s - a_m)}{(s - b_1)(s - b_2) \cdots (s - b_n)} = \frac{Z(s)}{P(s)} \quad (4.16)$$

where the  $a_1, a_2, a_3, \dots, a_m$  are referred to as the zeros of  $F(s)$  and  $b_1, b_2, b_3, \dots, b_n$  are the poles of  $F(s)$ . The zeros are the roots of  $Z(s) = 0$  and the poles are the roots of  $P(s) = 0$ .

There are two principal methods for finding  $L^{-1}\{F(s)\}$  when  $F(s)$  is a ratio of polynomials;

**Partial Fraction Expansion** and the **Method of Residues**. These will be illustrated with a few simple examples [again, the student should consult additional reference material, as needed, to get a more complete set of illustrative examples...].

### ***Partial Fraction Expansion***

#### Non-Repeated Linear Factors

$$F(s) = \frac{s^2 + 1}{(s+1)(s+3)(s+4)} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$A = \left. \frac{s^2 + 1}{(s+3)(s+4)} \right|_{s=-1} = \frac{2}{(2)(3)} = \frac{1}{3}$$

$$B = \left. \frac{s^2 + 1}{(s+1)(s+4)} \right|_{s=-3} = \frac{10}{(-2)(1)} = -5$$

$$C = \left. \frac{s^2 + 1}{(s+1)(s+3)} \right|_{s=-4} = \frac{17}{(-3)(-1)} = \frac{17}{3}$$

$$F(s) = \frac{1}{3} \left( \frac{1}{s+1} \right) - 5 \left( \frac{1}{s+3} \right) + \frac{17}{3} \left( \frac{1}{s+4} \right)$$

and

$$f(t) = \frac{1}{3} e^{-t} - 5e^{-3t} + \frac{17}{3} e^{-4t}$$

#### Repeated Linear Factors

$$F(s) = \frac{s^2}{(s+1)^2(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+2)}$$

$$B = \left. \frac{s^2}{s+2} \right|_{s=-1} = \frac{1}{1} = 1$$

$$C = \left. \frac{s^2}{(s+1)^2} \right|_{s=-2} = \frac{4}{1} = 4$$

$$A = \left. \frac{s^2}{(s+1)(s+2)} \right|_{s=-1} = \frac{1}{(0)(1)} - \frac{B}{0} \quad (\text{can't do this})$$



Instead, to compute A, let's evaluate  $F(s)$  at some convenient value of  $s$  -- say  $s = 0$  for this case. Doing this gives

$$0 = A + B + \frac{C}{2}$$

and, upon substitution for the known values of B and C, we have

$$A = -B - \frac{C}{2} = -1 - 2 = -3$$

Now, with knowledge of the coefficients, the inverse Laplace transform of  $F(s)$  is obtained with a simple table lookup. This process gives

$$f(t) = -3e^{-t} + te^{-t} + 4e^{-2t}$$

### Complex Roots and Quadratic Factors

$$F(s) = \frac{2s^2 + 6s + 6}{(s+2)(s^2 + 2s + 2)} = \frac{A}{s+2} + \frac{Bs+C}{s^2 + 2s + 2}$$

Solving for A gives

$$A = \left. \frac{2s^2 + 6s + 6}{s^2 + 2s + 2} \right|_{s=-2} = \frac{8-12+6}{4-4+2} = \frac{2}{2} = 1$$

Now clearing fractions gives

$$2s^2 + 6s + 6 = (1)(s^2 + 2s + 2) + (Bs + C)(s + 2)$$

$$s^2 + 4s + 4 = Bs^2 + 2Bs + Cs + 2C$$

Equating like terms

$$Bs^2 = s^2 \quad 2C = 4 \quad (2B+C)s = 4s$$

$$\text{or} \quad B = 1 \quad C = 2 \quad 2+2 = 4$$

Finally, rewriting the quadratic term as the sum of two squares, gives

$$s^2 + 2\alpha s + (\beta^2 + \alpha^2) = (s + \alpha)^2 + \beta^2 = s^2 + 2s + 2$$

where, for this case,  $\alpha = 1$  and  $\beta = 1$ , gives

$$F(s) = \frac{1}{s+2} + \frac{s+2}{(s+1)^2 + 1^2} = \frac{1}{s+2} + \frac{s+1}{(s+1)^2 + 1^2} + \frac{1}{(s+1)^2 + 1^2}$$

Now, with the creative algebra in the last step to separate the decaying sine and cosine terms as individual components, we have

$$f(t) = e^{-2t} + e^{-t} \cos t + e^{-t} \sin t$$

### *Method of Residues*

The method of partial fractions is conceptually simple, but it becomes rather tedious (from an algebraic viewpoint) when the number of terms becomes large. An alternate method, which is sometimes useful (especially for formal proofs) is the method of residues.

**Statement of Method:** If  $F(s)$  is a ratio of polynomials in  $s$ , then

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{\text{all poles}} \left( \text{residues of } F(s)e^{st} \right) \quad (4.17)$$

where the residue of an  $n^{\text{th}}$ -order pole at  $s = s_1$  is given by

$$R_{s_1} = \frac{1}{(n-1)!} \left[ \frac{d^{n-1}}{ds^{n-1}} \left( (s-s_1)^n F(s)e^{st} \right) \right]_{s=s_1} \quad (4.18)$$

$$\textbf{First Order Pole:} \quad R_{s_1} = \left[ (s-s_1)F(s)e^{st} \right]_{s=s_1} \quad (4.19)$$

$$\textbf{Second Order Pole:} \quad R_{s_1} = \left[ \frac{d}{ds} \left( (s-s_1)^2 F(s)e^{st} \right) \right]_{s=s_1} \quad (4.20)$$

To illustrate the use of the method of residues, let's use the same examples as before.

#### Non-Repeated Linear Factors

$$F(s) = \frac{s^2 + 1}{(s+1)(s+3)(s+4)}$$

$$f(t) = \left. \frac{s^2 + 1}{(s+3)(s+4)} e^{st} \right|_{s=-1} + \left. \frac{s^2 + 1}{(s+1)(s+4)} e^{st} \right|_{s=-3} + \left. \frac{s^2 + 1}{(s+1)(s+3)} e^{st} \right|_{s=-4}$$

$$f(t) = \frac{1}{3}e^{-t} - 5e^{-3t} + \frac{17}{3}e^{-4t}$$

#### Repeated Linear Factors

$$F(s) = \frac{s^2}{(s+1)^2(s+2)}$$

$$\begin{aligned} f(t) &= \left. \frac{s^2}{(s+1)^2} e^{st} \right|_{s=-2} + \left. \frac{d}{ds} \left\{ \frac{s^2}{s+2} e^{st} \right\} \right|_{s=-1} \\ &= 4e^{-2t} + \left\{ \left[ \frac{2s}{s+2} - \frac{s^2}{(s+2)^2} \right] e^{st} + \frac{s^2}{s+2} te^{st} \right\} \bigg|_{s=-1} \end{aligned}$$

$$f(t) = 4e^{-2t} - 3e^{-t} + te^{-t}$$

Complex Roots and Quadratic Factors

$$F(s) = \frac{2s^2 + 6s + 6}{(s+2)(s^2 + 2s + 2)}$$

Note that, given the quadratic polynomial  $as^2 + bs + c = 0$ , the roots are given by

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and for this case the roots of the quadratic term are

$$s = \frac{-2}{2} \pm \frac{\sqrt{4-8}}{2} = -1 \pm j$$

Therefore,  $F(s)$  can be written in terms of linear factors, or

$$F(s) = \frac{2s^2 + 6s + 6}{(s+2)[s - (-1+j)][s - (-1-j)]}$$

$$f(t) = \left( \frac{2s^2 + 6s + 6}{s^2 + 2s + 2} \right) e^{st} \Big|_{s=-2} + \left[ \frac{2s^2 + 6s + 6}{(s+2)[s - (-1+j)]} \right] e^{st} \Big|_{s=-1+j} + \left[ \frac{2s^2 + 6s + 6}{(s+2)[s - (-1-j)]} \right] e^{st} \Big|_{s=-1-j}$$

Expanding the coefficient of the middle term, for example, gives

$$\frac{2(-1+j)^2 + 6(-1+j) + 6}{(1+j)[-1+j+1+j]} = \frac{2(1-2j-1) - 6 + 6j + 6}{2j-2} = \frac{2j}{2j-2} = \frac{j}{-1+j} \frac{(-1-j)}{(-1-j)} = \frac{1-j}{1+1} = \frac{1-j}{2}$$

Evaluating the first and last terms in a similar manner gives

$$f(t) = e^{-2t} + \frac{1}{2}(1-j)e^{(-1+j)t} + \frac{1}{2}(1+j)e^{(-1-j)t}$$

$$= e^{-2t} + \frac{1}{2}e^{-t}(e^{jt} + e^{-jt}) + \frac{j}{2}e^{-t}(e^{-jt} - e^{jt}) = e^{-2t} + \frac{1}{2}e^{-t}(e^{jt} + e^{-jt}) + \frac{1}{2j}e^{-t}(e^{jt} - e^{-jt})$$

$$f(t) = e^{-2t} + e^{-t} \cos t + e^{-t} \sin t$$

Both methods can be complicated algebraically (especially when working with complex arithmetic), but conceptually it is straightforward to find  $f(t)$  given  $F(s)$  and vice versa. The **Partial Fraction Expansion** technique and **Method of Residues** are very powerful, and these techniques can be utilized in most cases of interest when  $F(s)$  is written as a ratio of polynomials.

**Important Note:** My recommendation for dealing with quadratic factors is to definitely avoid the complex arithmetic needed for the method of residues approach (as just illustrated). Instead, the partial fraction expansion method, with the use of the “sum of squares” technique for writing the quadratic factors, is almost always the better way to proceed (easier and faster). Obviously, this is a matter of personal preference since both methods lead to the same result -- however, my experience is that the partial fraction expansion technique is less algebraically tedious, and more often leads to the correct result with less frustration...

### Solution of Differential Equations

One particular application of Laplace transforms that is of primary interest in this course concerns their use in the solution of linear constant coefficient differential equations. The procedure is relatively straightforward, and it involves three simple steps:

1. Take the Laplace transform of each term in the original equation. This simply converts the differential equation into an algebraic equation.
2. Solve the resultant algebraic equation for the unknown dependent variable.
3. Find the time domain solution by taking the inverse Laplace transform.

This procedure can be illustrated nicely with the following simple example.

#### Example 4.1 Laplace Transform Solution to 2<sup>nd</sup> Order System

##### Problem Statement:

Given  $m \frac{d^2x}{dt^2} + kx = f(t)$  with  $x(t)|_0 = \frac{dx(t)}{dt}|_0 = 0$ , solve for  $x(t)$  given a specific  $f(t)$ .

##### Problem Solution:

Taking the Laplace transform of each term gives

$$m \left[ s^2 X(s) - sx(0) - \frac{dx}{dt} \Big|_0 \right] + kX(s) = F(s)$$

$$\text{or} \quad X(s) = \frac{F(s)}{ms^2 + k} + \frac{msx(0) - m \frac{dx}{dt} \Big|_0}{ms^2 + k}$$

and with zero initial conditions, we have

$$X(s) = G(s) F(s) \quad \text{where} \quad G(s) = \frac{1}{ms^2 + k}$$

Now for any input forcing function,  $f(t)$ , we can determine  $F(s)$  and  $X(s)$ , and then using the inverse transform, we can get  $x(t)$ . Two specific cases follow:

##### Case 1

Let the forcing function be a unit impulse, or  $f(t) = \delta(t)$ . This gives  $F(s) = 1$ , and  $x(t)$  can be given as

$$x(t) = L^{-1} \{X(s)\} = L^{-1} \{G(s)F(s)\} = L^{-1} \{G(s)\}$$

$$\text{with} \quad G(s) = \frac{1}{ms^2 + k} = \frac{1/m}{s^2 + (\sqrt{k/m})^2} = \frac{1/m}{s^2 + 2\alpha s + (\beta^2 + \alpha^2)} = \frac{1}{m} \sqrt{\frac{m}{k}} \frac{\sqrt{k/m}}{s^2 + (\sqrt{k/m})^2}$$

where, in the last equality, we have set  $\alpha = 0$  and  $\beta = \sqrt{k/m}$ . Comparing this form to the Laplace transform pairs given in Table 4.1, we see that the inverse transform is given by

$$x(t) = \frac{1}{m} \sqrt{\frac{m}{k}} \sin\left(\sqrt{\frac{m}{k}} t\right)$$

which is the desired time domain solution for this problem. Note here that the time domain solution for a unit impulse input is simply the inverse transform of  $G(s)$ , which is denoted as  $L^{-1}\{G(s)\}$ . This is referred to as the **impulse response** of the system.

### Case 2

As a second illustration, let the input function be a unit step function,  $f(t) = u(t)$  [here  $u(t)$  denotes the unit step]. In this case,  $F(s) = 1/s$  and the time domain solution is written as

$$x(t) = L^{-1}[X(s)] = L^{-1}\left[G(s) \frac{1}{s}\right]$$

Thus, writing  $X(s)$  as

$$X(s) = \frac{1}{s} \left( \frac{1}{ms^2 + k} \right) = \frac{A}{s} + \frac{Bs + C}{ms^2 + k}$$

we can determine the coefficient associated with the linear factor as

$$A = \left. \frac{1}{ms^2 + k} \right|_{s=0} = \frac{1}{k}$$

Now, with  $A$  known, clearing fractions gives

$$1 = \left( \frac{1}{k} \right) (ms^2 + k) + Bs^2 + Cs$$

$$\text{or} \quad 1 = \left( \frac{m}{k} + B \right) s^2 + Cs + 1$$

Finally equating coefficients of like terms, we have  $C = 0$  and  $B = -m/k$ . Therefore,  $X(s)$  becomes

$$X(s) = \frac{1}{k} \frac{1}{s} - \frac{m}{k} \frac{1}{m} \frac{s}{s^2 + (\sqrt{k/m})^2} = \frac{1}{k} \left[ \frac{1}{s} - \frac{s}{s^2 + (\sqrt{k/m})^2} \right]$$

With  $\alpha = 0$  and  $\beta = \sqrt{k/m}$  in the standard form for a quadratic factor, we get the following result

$$x(t) = \frac{1}{k} \left[ 1 - \cos\left(\sqrt{\frac{m}{k}} t\right) \right]$$

-----

### The Convolution Integral

We have seen that, quite often, the time-domain solution can be given as the inverse Laplace transform of a product of Laplace transforms,

$$X(s) = G(s)F(s) \quad (4.21)$$

$$\text{and } x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{G(s)F(s)\} \quad (4.22)$$

In general, one can write the *inverse transform of a product of transforms* using the **Convolution Integral**, which is given by

$$\mathcal{L}^{-1}\{X_1(s)X_2(s)\} = \int_0^t x_1(t-\tau)x_2(\tau) d\tau \quad (4.23)$$

where  $x_1(t)$  and  $x_2(t)$  are causal time functions. [Recall that we defined a causal time function as one which satisfies the relation  $f(t) = 0$  for  $t < 0$ ].

The **proof** of the relationship in eqn. (4.23) is instructive. Let's assume that  $f(t)$  is given by

$$f(t) = \int_0^\infty x_1(t-\tau)x_2(\tau) d\tau$$

Now taking the Laplace transform of  $f(t)$  gives

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} \left[ \int_0^\infty x_1(t-\tau)x_2(\tau) d\tau \right] dt = \int_0^\infty \left[ \int_0^\infty e^{-st} x_1(t-\tau) dt \right] x_2(\tau) d\tau$$

But, because we are working with causal time functions,  $x_1(t-\tau) = 0$  for  $t < \tau$ . Therefore, we have

$$F(s) = \int_0^\infty \left[ \int_\tau^\infty e^{-st} x_1(t-\tau) dt \right] x_2(\tau) d\tau$$

Now letting  $p = t - \tau$  and  $dp = dt$ , gives

$$\begin{aligned} F(s) &= \int_0^\infty \left[ \int_0^\infty e^{-s(p+\tau)} x_1(p) dp \right] x_2(\tau) d\tau = \int_0^\infty \left[ \int_0^\infty e^{-sp} x_1(p) dp \right] e^{-s\tau} x_2(\tau) d\tau \\ &= \left[ \int_0^\infty x_1(p) e^{-sp} dp \right] \left[ \int_0^\infty x_2(\tau) e^{-s\tau} d\tau \right] = X_1(s)X_2(s) \end{aligned}$$

Thus,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{X_1(s)X_2(s)\} = \int_0^{t \text{ (or } \infty)} x_1(t-\tau)x_2(\tau) d\tau \quad (\text{as originally proposed})$$

Note that if we let  $X_1(s) \rightarrow G(s)$  and  $X_2(s) \rightarrow F(s)$ , then  $x_1(t)$  is called the impulse response of the system [since  $F(s) = 1$  for  $f(t) = \delta(t)$ ]. The impulse response is often denoted as  $h(t)$ . Now, for any forcing function,  $f(t)$ , the desired response,  $y(t)$ , becomes

$$y(t) = \int_0^t h(t-\tau)f(\tau) d\tau \quad \text{where} \quad h(t) = \mathcal{L}^{-1}\{G(s)\} \quad (4.24)$$

### Laplace Transforms and the State Equations

To be consistent with our state variable approach for the analysis of dynamic systems, we now extend the treatment of Laplace transforms to include matrix equations. Since the Laplace transform is a linear operator, one can show easily that the Laplace transform of a vector or matrix is simply the Laplace transform of the individual components. In addition, all the operator relations (ie.  $\mathcal{L}\left[\frac{d^n \underline{x}}{dt^n}\right]$ ,  $\mathcal{L}\left[\int_0^t \underline{x}(\tau)d\tau\right]$ , etc.) are still valid.

Therefore, to transform the standard linear time invariant state space model, we simply take the Laplace transform of each term,

$$\mathcal{L}\left\{\frac{d}{dt}\underline{x}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t)\right\}$$

$$s\underline{X}(s) - \underline{x}(0) = \underline{A}\underline{X}(s) + \underline{B}\underline{U}(s)$$

or  $(s\underline{I} - \underline{A})\underline{X}(s) = \underline{x}(0) + \underline{B}\underline{U}(s)$

and the final form of the state equations can be written as

$$\underline{X}(s) = (s\underline{I} - \underline{A})^{-1}[\underline{x}(0) + \underline{B}\underline{U}(s)] \quad (4.25)$$

By analogy with the scalar case, we propose the following relationships:

scalar case  $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = \mathcal{L}^{-1}\left[(s-a)^{-1}\right] = e^{at} \quad (4.26)$

matrix case  $\mathcal{L}^{-1}\left[(s\underline{I} - \underline{A})^{-1}\right] = e^{\underline{A}t} \quad (4.27)$

As a proof of the last relationship, we rewrite eqn. (4.25) as

$$\underline{X}(s) = (s\underline{I} - \underline{A})^{-1} \underline{x}_0 + (s\underline{I} - \underline{A})^{-1} \underline{B}\underline{U}(s)$$

and take its inverse transform using the relationship in eqn. (4.27), giving

$$\underline{x}(t) = \mathcal{L}^{-1}\{\underline{X}(s)\} = e^{\underline{A}t} \underline{x}_0 + \int_0^t e^{\underline{A}(t-\tau)} \underline{B}\underline{u}(\tau)d\tau$$

where the last term results from the Convolution Integral. But we know that this result is correct from our previous time-domain analysis. Thus, our hypothesis that  $\mathcal{L}^{-1}\left[(s\underline{I} - \underline{A})^{-1}\right] = e^{\underline{A}t}$  must be correct!

The simple illustration in Example 4.2 shows how to apply this relationship for  $e^{\underline{A}t}$  to a 2<sup>nd</sup> order system. Higher order systems simply give larger matrices. Note that the system matrix in this case is identical to that used in Example 2.11, where we used the confluent form of Sylvester's theorem to generate  $e^{\underline{A}t}$  for this problem. Both methods require a bit of algebra, but the Laplace transform technique does not need to treat repeated roots as a special case. Either method can be used, as desired.

**Example 4.2 Laplace Transform Solution for Matrix Exponential****Problem Statement:**

Determine  $e^{\underline{\underline{A}}t}$  for  $\underline{\underline{A}} = \begin{bmatrix} 6 & 1 \\ -4 & 2 \end{bmatrix}$  using the Laplace transform representation.

**Problem Solution:**

We know that  $L^{-1}\left\{\left(s\underline{\underline{I}} - \underline{\underline{A}}\right)^{-1}\right\} = e^{\underline{\underline{A}}t}$ . Therefore, let's first compute  $\left(s\underline{\underline{I}} - \underline{\underline{A}}\right)^{-1}$  as follows:

$$s\underline{\underline{I}} - \underline{\underline{A}} = \begin{bmatrix} s-6 & -1 \\ 4 & s-2 \end{bmatrix}$$

$$\det(s\underline{\underline{I}} - \underline{\underline{A}}) = (s-6)(s-2) + 4 = s^2 - 8s + 16 = (s-4)^2$$

$$\left(s\underline{\underline{I}} - \underline{\underline{A}}\right)^{-1} = \frac{1}{(s-4)^2} \begin{bmatrix} s-2 & -4 \\ 1 & s-6 \end{bmatrix}^T = \begin{bmatrix} \frac{s-2}{(s-4)^2} & \frac{1}{(s-4)^2} \\ \frac{-4}{(s-4)^2} & \frac{s-6}{(s-4)^2} \end{bmatrix}$$

Therefore,

$$e^{\underline{\underline{A}}t} = L^{-1}\left\{\left(s\underline{\underline{I}} - \underline{\underline{A}}\right)^{-1}\right\} = L^{-1}\left\{\begin{bmatrix} \frac{s-4}{(s-4)^2} + \frac{2}{(s-4)^2} & \frac{1}{(s-4)^2} \\ \frac{-4}{(s-4)^2} & \frac{s-4}{(s-4)^2} - \frac{2}{(s-4)^2} \end{bmatrix}\right\}$$

$$\text{but } L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} \quad \text{and} \quad L^{-1}\left\{\frac{1}{(s+a)^2}\right\} = te^{-at}$$

Thus, the final form for the matrix exponential can be written as

$$e^{\underline{\underline{A}}t} = \begin{bmatrix} e^{4t} + 2te^{4t} & te^{4t} \\ -4te^{4t} & e^{4t} - 2te^{4t} \end{bmatrix} = e^{4t} \begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix}$$

which is the same result that was obtained in Example 2.11.

-----



### System Transfer Function Matrix

Now define the **System Transfer Function** as the "*ratio*" of the Laplace transform of the output to the Laplace transform of the input when the initial conditions are zero. This last condition can always be satisfied by writing the linear state equations in terms of perturbations around the initial state -- then the initial conditions on  $\delta \underline{x}(t)$  [or  $\underline{x}(t)$ , for convenience] automatically vanish). Thus, given the linear constant coefficient system

$$\frac{d}{dt} \underline{x} = \underline{A} \underline{x}(t) + \underline{B} \underline{u}(t) \quad \text{with } \underline{x}(0) = 0 \quad (4.28)$$

$$\underline{y}(t) = \underline{C} \underline{x}(t) + \underline{D} \underline{u}(t) \quad (4.29)$$

the **System Transfer Function Matrix**,  $\underline{G}(s)$ , is defined through the relation

$$\underline{Y}(s) = \underline{G}(s) \underline{U}(s) \quad (4.30)$$

where

$$\begin{aligned} \underline{Y}(s) &= \underline{C} \underline{X}(s) + \underline{D} \underline{U}(s) \\ &= \underline{C} \left[ (s\underline{I} - \underline{A})^{-1} \underline{x}(0) + (s\underline{I} - \underline{A})^{-1} \underline{B} \underline{U}(s) \right] + \underline{D} \underline{U}(s) \\ &= \left[ \underline{C} (s\underline{I} - \underline{A})^{-1} \underline{B} + \underline{D} \right] \underline{U}(s) \quad \text{with } \underline{x}(0) = 0 \end{aligned}$$

Therefore, the transfer function matrix can be written as

$$\underline{G}(s) = \underline{C} (s\underline{I} - \underline{A})^{-1} \underline{B} + \underline{D} \quad (4.31)$$

Using the convolution integral and the fact that  $L\{e^{\underline{A}t}\} = (s\underline{I} - \underline{A})^{-1}$ , then

$$\underline{y}(t) = \underline{C} \left[ \int_0^t e^{\underline{A}(t-\tau)} \underline{B} \underline{u}(\tau) d\tau \right] + \underline{D} \underline{u}(t) \quad (4.32)$$

Note also that in component form, eqn. (4.30) becomes

$$Y_i(s) = \sum_j G_{ij}(s) U_j(s) \quad (4.33)$$

where the  $ij^{\text{th}}$  element,  $G_{ij}(s)$ , is the transfer function relating the  $j^{\text{th}}$  input to the  $i^{\text{th}}$  output.

### Block Diagram Representations

Large dynamic systems can usually be broken into several blocks that have relatively simple single-input-single-output (SISO) relationships. These blocks can be manipulated to obtain the overall transfer function. **Block diagram algebra** is identical for the scalar and matrix cases except, for the matrix representation, one must always be careful to maintain the order of operation. To be as general as possible, we will illustrate the basic ideas using matrix notation.

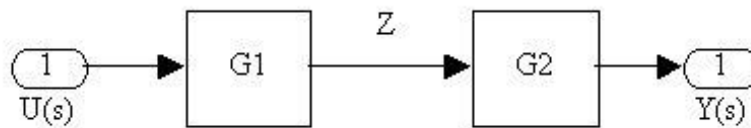
The goal is to generate an equivalent overall system transfer function matrix for various common block diagram configurations.

### Overall System



$$\underline{Y}(s) = \underline{G}(s)\underline{U}(s)$$

### Series Connections



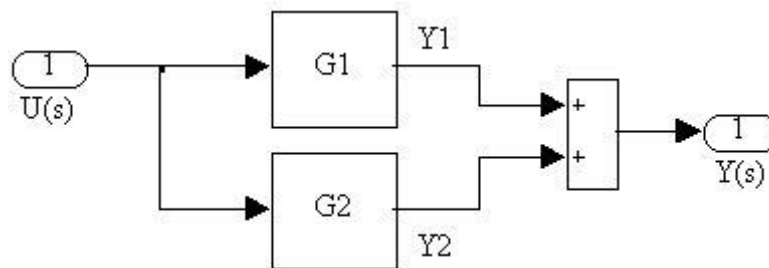
$$\underline{Y}(s) = \underline{G}_2(s)\underline{Z}(s) = \underline{G}_2(s)\underline{G}_1(s)\underline{U}(s)$$

Therefore

$$\underline{G}(s) = \underline{G}_2(s)\underline{G}_1(s) \quad (\text{for series connection}) \quad (4.34)$$

where the order of the matrices is important.

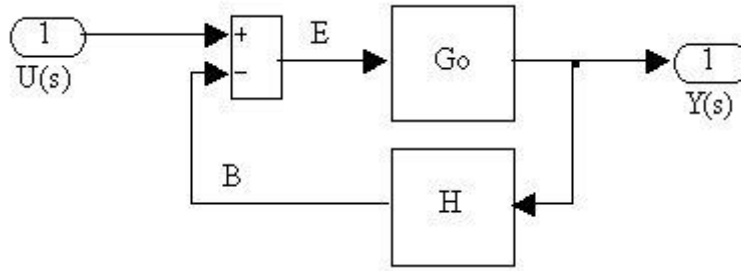
### Parallel Connections



$$\begin{aligned} \underline{Y}(s) &= \underline{Y}_1(s) + \underline{Y}_2(s) \\ &= \underline{G}_1(s)\underline{U}(s) + \underline{G}_2(s)\underline{U}(s) = [\underline{G}_1(s) + \underline{G}_2(s)]\underline{U}(s) \end{aligned}$$

Therefore

$$\underline{G}(s) = \underline{G}_1(s) + \underline{G}_2(s) \quad (\text{for parallel connections}) \quad (4.35)$$

Closed-Loop Systems (Feedback Connections)

$$\begin{aligned}\underline{Y}(s) &= \underline{G}_o(s)\underline{E}(s) = \underline{G}_o(s)[\underline{U}(s) - \underline{B}(s)] \\ &= \underline{G}_o(s)[\underline{U}(s) - \underline{H}(s)\underline{Y}(s)]\end{aligned}$$

$$[\underline{I} + \underline{G}_o(s)\underline{H}(s)]\underline{Y}(s) = \underline{G}_o(s)\underline{U}(s)$$

or  $\underline{Y}(s) = [\underline{I} + \underline{G}_o(s)\underline{H}(s)]^{-1} \underline{G}_o(s)\underline{U}(s)$

Therefore

$$\underline{G}(s) = [\underline{I} + \underline{G}_o(s)\underline{H}(s)]^{-1} \underline{G}_o(s) \quad (\text{for feedback connections}) \quad (4.36)$$

Note that, in the scalar case, the feedback connection relationship becomes

$$G(s) = \frac{G_o(s)}{1 + G_o(s)H(s)} \quad (4.37)$$

Block diagram representations are quite useful. By inspection they tell us a lot about designing systems to give some specified dynamic response. Given some system behavior described by  $G_o(s)$ , we see that the input-output relationship can be altered by inserting either series, parallel, or feedback components. The new output response would be modified as indicated above. Thus, one has considerable flexibility in design.

As an example of a particular dynamic system, consider the block diagram illustrated earlier in this section, where this time we represent the input-output relationship in each block with a transfer function,  $H_i(s)$  (see sketch on the next page). In this system, each block represents the relationship between a single scalar input and output. The intermediate signals are denoted as  $X_1, X_2$ , etc. The overall system transfer function can be developed as follows:

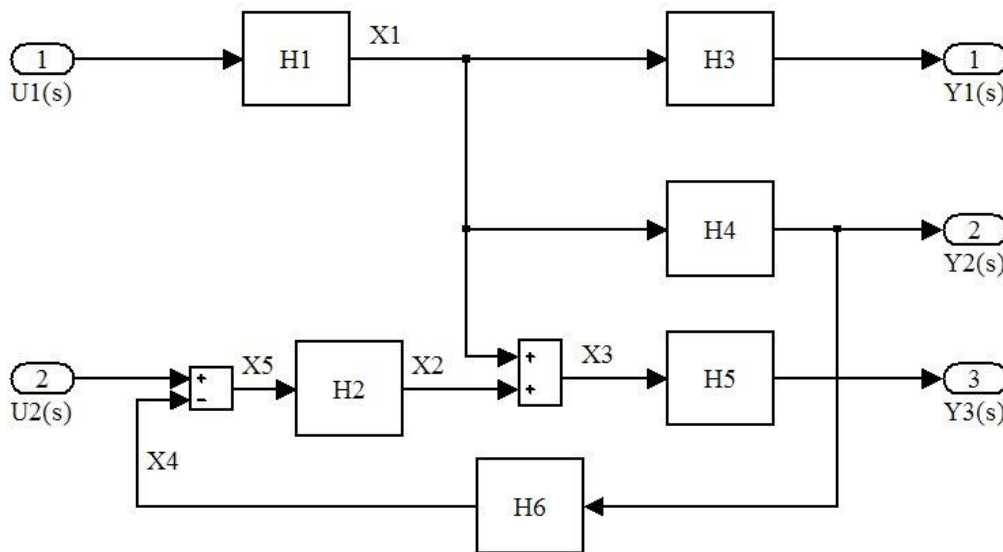
$$Y_1 = H_3X_1 = H_3H_1U_1 \quad \text{or} \quad G_{11} = H_3H_1$$

$$Y_2 = H_4X_1 = H_4H_1U_1 \quad \text{or} \quad G_{21} = H_4H_1$$

$$Y_3 = H_5X_3 = H_5(X_1 + X_2) = H_5H_1U_1 + H_5X_2$$

and  $X_2 = H_2X_5 = H_2(U_2 - X_4) = H_2U_2 - H_2H_6Y_2$

$$\begin{aligned}Y_3 &= H_5H_1U_1 + H_5H_2U_2 - H_5H_2H_6(H_4H_1U_1) \\ &= H_1H_5(1 - H_2H_4H_6)U_1 + H_2H_5U_2\end{aligned}$$



Therefore

$$G_{31} = H_1 H_5 (1 - H_2 H_4 H_6) \quad \text{and} \quad G_{32} = H_2 H_5$$

Putting these into matrix form gives the overall system transfer function matrix,  $\underline{\underline{G}}$ ,

$$\underline{\underline{G}} = \begin{bmatrix} G_{11} & 0 \\ G_{21} & 0 \\ G_{31} & G_{32} \end{bmatrix}$$

Note that there is no direct link to  $Y_1$  and  $Y_2$  from  $U_2$ .

Given the characteristics of a system or some process of interest (say we know  $H_3$ ,  $H_4$ , and  $H_5$  in the above example), one can modify the overall system (original process plus the controllers and feedback loop) by appropriate adjustment of  $H_1$ ,  $H_2$ , and  $H_6$ . In this course, we will not focus on the details of controller design so as to yield a desired overall transfer function. This is more appropriate in a course in automatic controls or in the design of dynamic systems. However, it should be clear that the block diagram/transfer function approach does lead to a straightforward means for designing systems to behave in a specified way. We will see some examples of this in later sections.

### Frequency Response Methods (for continuous systems)

We have looked at Laplace transform methods and we have shown how transfer functions play an important role in describing and analyzing dynamic systems. In particular, in the previous subsections, we have derived the following relationships:

| Scalar Case       | Matrix Case   |        |
|-------------------|---|--------|
| $Y(s) = G(s)U(s)$ | $\underline{Y}(s) = \underline{G}(s)\underline{U}(s)$ | (4.38) |

|     |   |   |        |
|-----|---|---|--------|
| and | $y(t) = \int_0^t h(t-\tau)u(\tau)d\tau$ | $\underline{y}(t) = \int_0^t \underline{H}(t-\tau)\underline{U}(\tau)d\tau$ | (4.39) |
|-----|---|---|--------|

|       |                         |   |        |
|-------|-------------------------|---|--------|
| where | $h(t) = L^{-1}\{G(s)\}$ | $\underline{H}(t) = L^{-1}\{\underline{G}(s)\}$ | (4.40) |
|-------|-------------------------|---|--------|

If the transfer function is known, one first determines the impulse response as expressed in eqn. (4.40). Then, for any input function, the desired response can be obtained via the convolution integral given in eqn. (4.39). Alternatively, one could directly determine the inverse transform of eqn. (4.38) using the partial fraction expansion or theory of residues methods.

Another especially important input function, that helps characterize dynamic systems, is the sinusoid. The output response of a LTI system resulting from a sinusoidal variation in a selected system input is termed *the frequency response of the system*. As shown below, *the output response is a sinusoid of different magnitude, and it is shifted by some phase angle,  $\psi$ , relative to the input function*. The ratio of the amplitude of the output to the amplitude of the input and the phase angle of the output response relative to the input completely define the frequency response of the system. That is, given

$$u(t) = b \sin(\omega t) \quad (4.41)$$

the output of a system described by transfer function,  $G(s)$ , can be written as

$$y(t) = b |G(j\omega)| \sin(\omega t + \psi) \quad (4.42)$$

To justify this latter statement, consider a single-input single-output (SISO) system where the input is a simple sinusoid. For this case, the Laplace transform of the input function,  $u(t) = b \sin(\omega t)$ , is given by

$$U(s) = \frac{b\omega}{s^2 + \omega^2}$$

and the Laplace transform of the desired output is simply the product of  $G(s)$  and  $U(s)$ , or

$$Y(s) = G(s) \frac{b\omega}{s^2 + \omega^2}$$

Since  $G(s)$  can usually be written as  $Z(s)/P(s)$  which has  $n$  system poles (i.e. the  $n$  roots of  $P(s) = 0$ ), we can use the method of residues to formally determine the output time function. To see this, consider the following development:

$$y(t) = L^{-1}\{Y(s)\} = \sum_{\text{all poles}} \left( \text{residues of } G(s) \frac{b\omega}{(s-j\omega)(s+j\omega)} e^{st} \right)$$

and expanding this gives

$$y(t) = \left. \frac{G(s)b\omega e^{st}}{s + j\omega} \right|_{s=j\omega} + \left. \frac{G(s)b\omega e^{st}}{s - j\omega} \right|_{s=-j\omega} + \left. \frac{b\omega G(s)(s-s_1)e^{st}}{s^2 + \omega^2} \right|_{s=s_1} + \left. \frac{b\omega G(s)(s-s_2)e^{st}}{s^2 + \omega^2} \right|_{s=s_2} + \dots + \left. \frac{b\omega G(s)(s-s_n)e^{st}}{s^2 + \omega^2} \right|_{s=s_n}$$

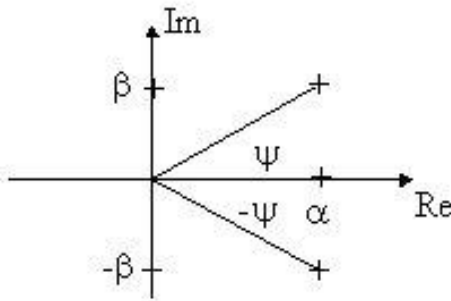
where  $s_i = s_1, s_2, s_3, \dots, s_n = \text{poles of } G(s)$ .

Now, for a stable system, all the  $s_i$ 's have negative real parts. Therefore, after a sufficient time, all the terms containing  $e^{s_i t}$  will have vanished and only the first two terms will remain.

Therefore, **after some transient time**, the above expression reduces to

$$y(t) = b \left\{ \frac{G(j\omega)\omega e^{j\omega t}}{2j\omega} - \frac{G(-j\omega)\omega e^{-j\omega t}}{2j\omega} \right\} = b \left\{ \frac{G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}}{2j} \right\}$$

Before reducing this last expression further, we note that  $G(j\omega)$  and  $G(-j\omega)$  are simply complex numbers. In fact,  $G(j\omega)$  and  $G(-j\omega)$  are complex conjugates. That is, if we denote  $G(j\omega) = \alpha + j\beta$ , then  $G(-j\omega) = \alpha - j\beta$ . Also, if one plots these in the complex plane, it becomes obvious that any complex number can be written in terms of a magnitude and angle, or



$$\begin{aligned} G(j\omega) &= \alpha + j\beta \\ &= |G(j\omega)|[\cos(\psi) + j\sin(\psi)] \\ &= |G(j\omega)|e^{j\psi} = |G(j\omega)|\angle\psi \end{aligned} \quad (4.43)$$

$$\text{where } |G(j\omega)| = \sqrt{\alpha^2 + \beta^2} \quad \text{and} \quad \psi = \tan^{-1}(\beta/\alpha) \quad (4.44)$$

Thus, we see that

$$G(j\omega) = |G(j\omega)|e^{j\psi} \quad \text{and} \quad G(-j\omega) = |G(-j\omega)|e^{-j\psi} = |G(j\omega)|e^{-j\psi}$$

Now, with this representation for the  $G(j\omega)$  and  $G(-j\omega)$  terms, we can write  $y(t)$  as

$$y(t) = b |G(j\omega)| \left\{ \frac{e^{j(\omega t + \psi)} - e^{-j(\omega t + \psi)}}{2j} \right\} y(t) = b |G(j\omega)| \sin(\omega t + \psi)$$

Therefore, the frequency response of a system is of the same form as the input function; the magnitude is multiplied by  $|G(j\omega)|$  and the sinusoid is shifted by phase angle  $\psi$ . These two quantities,  $|G(j\omega)|$  and  $\psi$ , define completely the response of a stable system to a sinusoidal input. A key point here is that these quantities can be obtained easily from knowledge of the system transfer function  $G(s)$ ,

$$|G(j\omega)| = \sqrt{\text{Re}\{G(j\omega)\}^2 + \text{Im}\{G(j\omega)\}^2} \quad (4.45)$$

$$\psi = \tan^{-1} \left( \frac{\text{Im}\{G(j\omega)\}}{\text{Re}\{G(j\omega)\}} \right) \quad (4.46)$$

An important characteristic of frequency response analysis and testing is that a simple link between the theoretical transfer function and the experimental frequency response exists. Frequency response testing allows one to easily determine the system transfer function. If we know the transfer function of a complex dynamic system, then we can determine the system's time response due to any input.

To compute the frequency response, one simply evaluates  $|G(j\omega)|$  and  $\psi$  at a large number of frequencies and plots the data. For computer calculation, one can manipulate the matrices as follows,

$$\underline{\underline{G}}(s) = \underline{\underline{C}}(s\underline{\underline{I}} - \underline{\underline{A}})^{-1} \underline{\underline{B}} + \underline{\underline{D}} \quad (4.47)$$

$$\text{or } \underline{\underline{G}}(j\omega) = \underline{\underline{C}}(j\omega\underline{\underline{I}} - \underline{\underline{A}})^{-1} \underline{\underline{B}} + \underline{\underline{D}} \quad (4.48)$$

where a conceptual calculational outline would be:

a. input the  $\underline{\underline{A}}$ ,  $\underline{\underline{B}}$ ,  $\underline{\underline{C}}$ , and  $\underline{\underline{D}}$  matrices

┌ loop over specified values of  $\omega$

b. form  $(j\omega\underline{\underline{I}} - \underline{\underline{A}})$  and take its inverse

c. do matrix manipulation in eqn. (4.48)

d. obtain  $\text{Re}\{G(j\omega)\}$  and  $\text{Im}\{G(j\omega)\}$ , and compute  $|G(j\omega)|$  and  $\psi(\omega)$

e. plot results (several possibilities here -- see below)

Note that if we know the analytic form for  $G_{ij}(s) = Z(s)/P(s)$  for a specific case, then evaluation of the frequency response at many frequencies can be done rapidly by evaluating the polynomials for selected values of  $s = j\omega$ . Also, as you might expect, Matlab has a series of built-in functions for efficiently evaluating and plotting the frequency response of any LTI system (see discussion in the next subsection).

### Frequency Response Plotting Techniques

There are several standard plotting techniques for obtaining meaningful frequency response curves. The three most common methods are:

**Bode plots** -- separate plots of amplitude and phase versus frequency

**Polar (Nyquist) plots** -- curves of the Im part versus the Re part of  $G(j\omega)$  for each  $\omega$

**Nichols plots** -- plots of gain (log-magnitude) versus phase angle for desired frequencies

Let's elaborate a little concerning Bode plots. First, we note that a logarithmic plot is usually used for the amplitude curve since wide ranges of amplitude and frequency are encountered and because certain graphical manipulations are facilitated on a logarithmic plot. Also, it is common to define the magnitude in terms of the decibel, where

$$|G(j\omega)| \text{ in db} = 20 \log_{10} |G(j\omega)| \text{ in absolute units} \quad (4.49)$$

We can illustrate the contribution of the individual terms within  $G(s)$  via example. Consider, for example, the following transfer function,

$$G(s) = \frac{s + a}{s(s + b)}$$

where  $G(s) \Rightarrow G(j\omega) = |G(j\omega)| e^{j\psi}$

Now writing the individual components in a similar way gives

$$s + a \Rightarrow j\omega + a = |j\omega + a| e^{j\psi_1}$$

$$\frac{1}{s} \Rightarrow \frac{1}{j\omega} = \frac{1}{|j\omega|} e^{j\psi_2}$$

and  $\frac{1}{s + b} \Rightarrow \frac{1}{j\omega + b} = \frac{1}{|j\omega + b|} e^{j\psi_3}$

Using these expressions gives

$$G(j\omega) = \frac{(j\omega + a)}{(j\omega)(j\omega + b)} \quad \text{or} \quad |G(j\omega)| e^{j\psi} = \frac{|j\omega + a|}{|j\omega| |j\omega + b|} e^{j(\psi_1 + \psi_2 + \psi_3)}$$

where  $\psi = \psi_1 + \psi_2 + \psi_3$

and  $|G(j\omega)| = \frac{|j\omega + a|}{|j\omega| |j\omega + b|}$

or  $20 \log_{10} |G(j\omega)| = 20 \log_{10} |j\omega + a| - 20 \log_{10} |j\omega| - 20 \log_{10} |j\omega + b|$

Thus, one sees that the composite phase angle is the **sum** of the phase angles of the individual factors in the overall transfer function. Also, the composite logarithmic amplitude is the **sum** of the logarithmic amplitudes of the individual factors. These two features are useful in constructing approximate Bode plots.

If we now concentrate on a single term, one can easily determine an approximate profile. For example, the term  $1/(s+b)$  becomes

$$\frac{1}{j\omega + b} \left( \frac{-j\omega + b}{-j\omega + b} \right) = \frac{b - j\omega}{b^2 + \omega^2}$$

Therefore, the real and imaginary parts can be separated as



$$\operatorname{Re}\left(\frac{1}{j\omega + b}\right) = \frac{b}{b^2 + \omega^2} \quad \text{and} \quad \operatorname{Im}\left(\frac{1}{j\omega + b}\right) = \frac{-\omega}{b^2 + \omega^2}$$

and the magnitude and phase become

$$\operatorname{mag} = (\operatorname{Re}^2 + \operatorname{Im}^2)^{1/2} \quad \text{and} \quad \text{angle} = \tan^{-1}\left(\frac{\operatorname{Im}}{\operatorname{Re}}\right)$$

$$\left|\frac{1}{j\omega + b}\right| = \left(\frac{1}{b^2 + \omega^2}\right)^{1/2} \quad \text{and} \quad \psi_3 = \tan^{-1}\left(\frac{-\omega}{b}\right)$$

Looking now at the asymptotic values, we have

$$\text{at low frequency} \quad 20 \log \left| \frac{1}{j\omega + b} \right| \Rightarrow -20 \log b$$

$$\psi_3 \Rightarrow \tan^{-1}(0) = 0$$

$$\text{at high frequency} \quad 20 \log \left| \frac{1}{j\omega + b} \right| \Rightarrow -20 \log \omega$$

$$\psi_3 \Rightarrow \tan^{-1}(-\infty) = -90^\circ$$

Similarly, for the  $s+a$  term, we have

$$\operatorname{Re}(j\omega + a) = a \quad \text{and} \quad \operatorname{Im}(j\omega + a) = \omega$$

and the magnitude and phase become

$$|j\omega + a| = (a^2 + \omega^2)^{1/2} \quad \text{and} \quad \psi_1 = \tan^{-1}\left(\frac{\omega}{a}\right)$$

Again, looking now at the asymptotic values, we have

$$\text{at low frequency} \quad 20 \log |\omega + a| \Rightarrow 20 \log a$$

$$\psi_1 \Rightarrow \tan^{-1}(0) = 0$$

$$\text{at high frequency} \quad 20 \log |\omega + a| \Rightarrow 20 \log \omega$$

$$\psi_1 \Rightarrow \tan^{-1}(\infty) = 90^\circ$$

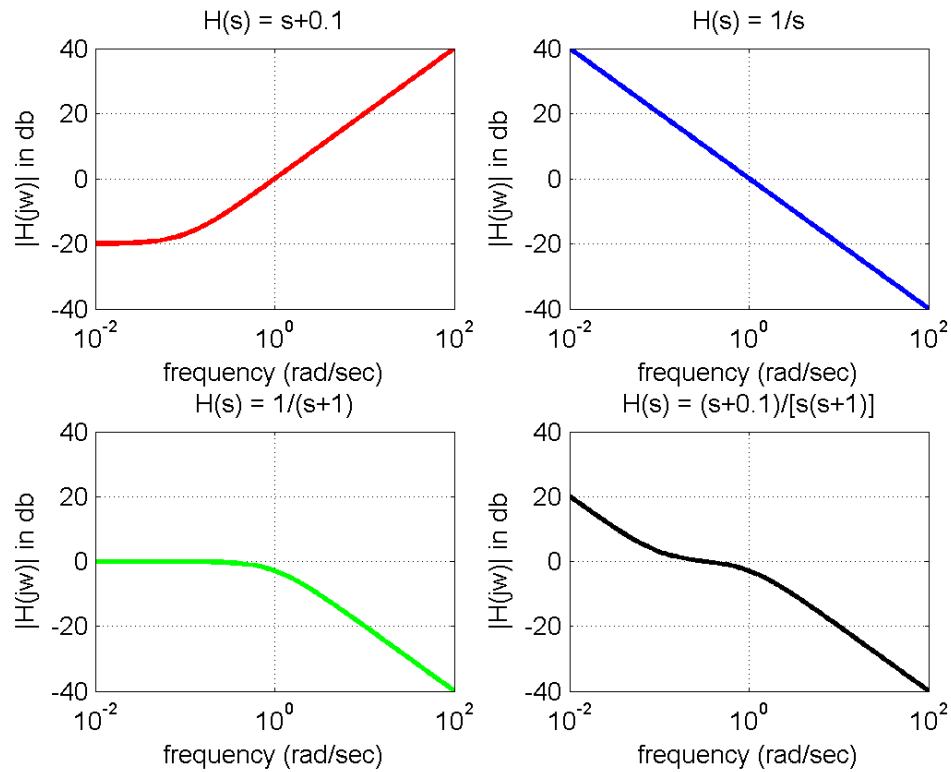
The high and low frequency curves intersect at the "break" or "corner" frequency ( $\omega = b$  in the first example, and  $\omega = a$  in the second case). Similar asymptotic values can be generated for any other factors and, as indicated in the above example, simply adding the curves gives a rough estimate of the overall frequency response. Notice from this simple illustration that, at high frequency, the slope of the magnitude plot should be the composite of +20 db/decade for each zero and -20 db/decade for each pole of the transfer function. Similarly, at high frequency, each zero should contribute a phase angle of +90° and each pole should add -90° to the total.

As a specific example, the individual components and composite Bode plots for the transfer function  $H(s) = (s+0.1)/[s(s+1)]$  are given in Figs. 4.1a and 4.1b (gain and phase plots, respectively). Although approximate Bode plots can be constructed rather easily, the best and most accurate way to generate any frequency response plot is to use the computer! The plots given here were generated with Matlab's **bode** command, as illustrated in the example m-file **bodedemo1.m** listed in Table 4.2. An alternate approach that uses the automatic plotting capability within some of the Matlab functions is illustrated in file **bodedemo2.m** in Table 4.3. The student is encouraged to spend some time getting familiar with these plots and the associated Matlab code, since there is a lot of good information that can be obtained from these curves. The reader is also referred to any good controls or dynamic systems text for further discussion of these points.

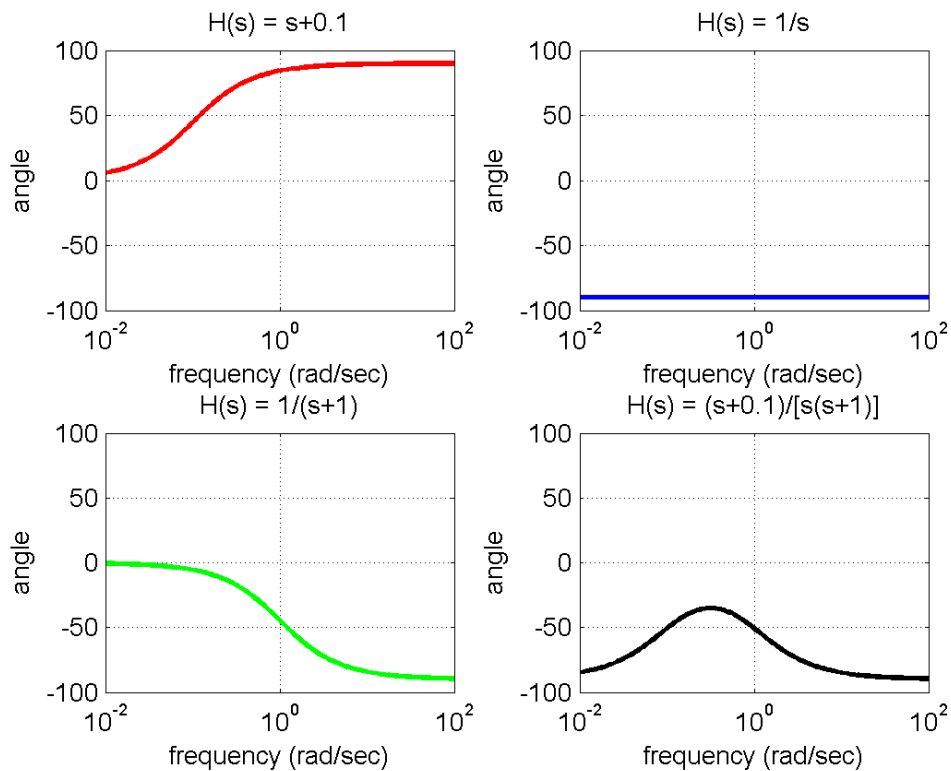
Finally, one should note that Nichols and Nyquist plots for the same transfer function as displayed in Figs. 4.1a and 4.1b are given in Figs. 4.2 and 4.3, respectively. The procedure for generating these plots is also listed in the m-file given in Table 4.2, with an alternate approach given in Table 4.3 (note that the output plots from **bodedemo2.m** are not shown here -- you will need to actually run the code to see these...).

Overall, Figs. 4.1-4.3 illustrate the three standard representations for frequency response plots for a typical system. You are referred to any good text on dynamic systems for further examples on the generation and use of frequency response profiles (also see Example 4.3 below). Also, one should note that Matlab has a whole series of functions for determining the frequency response for LTI systems. Only the **bode** and **logspace** commands were used in **bodedemo1.m**. The student should also check out the **margin**, **ngrid**, **nichols**, and **nyquist** commands, for example (some of these are used in the file **bodedemo2.m**). These functions, and many others that are formally part of the *Signal Processing Toolbox* and the *Control System Toolbox*, can be very useful in a variety of situations.

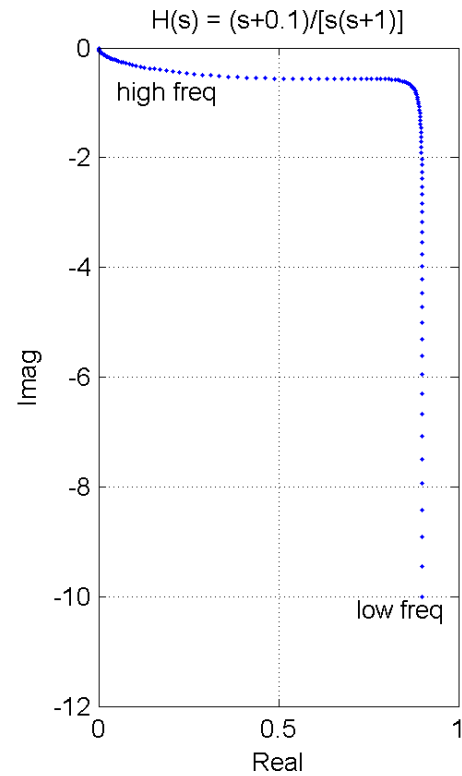
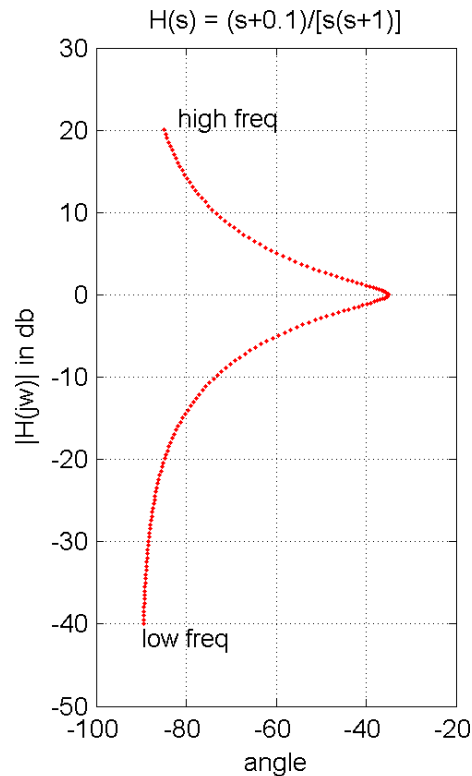
The frequency domain and time domain representations of systems are equivalent formulations; they simply look at the same information with a different perspective. Example 4.3 demonstrates this equivalency, and it also tries to shed some additional light on the interpretation of the Bode representation of the frequency response. This example uses the same RLC series circuit from Example 3.6. The first part of the example uses the transfer function form of the **step** command in Matlab to show the time domain solution to a 10 volt step change in the input. The results are identical with those from Example 3.6 (as expected). The second part of the example shows the Bode plots for the two cases,  $R = 100$  ohm and  $R = 10$  ohm, and it also simulates the system output for sinusoidal inputs at two different frequencies. The values of the magnitude and phase from the Bode plots are fully consistent with the time domain plots at these frequencies. Thus, the plots contain the same information; they simply have a different representation!



**Fig. 4.1a. Bode plots - log-magnitude versus frequency.**



**Fig. 4.1b. Bode plots - phase versus frequency.**



**Fig. 4.2. Nichols plot (gain vs phase). Fig. 4.3 Nyquist plot (Im vs Re).**

**Table 4.2 Listing of Matlab m-file bodedemo1.m**

```
%
% BODEDEMO1.M - This demo gives an example of some frequency response plots
%               for H(s) = (s+0.1)/[s(s+1)] = b(s)/a(s)
%
% This file uses some Matlab built-in functions to plot the frequency response
% for the above system (and its individual components) - Bode, Nyquist, and
% Nichols plots.
%
% A related file, BODEDEMO2.M, does the same thing, but we let Matlab produce its
% own default plots (eg. no left hand arguments are used in the Bode commands).
% Once you understand the process, this approach is clearly the easier path
% to follow! The current file, although more work, gives a better idea of the
% details.
%
% File prepared by J. R. White, UMass-Lowell (last update: Feb. 2020)
%
%
% clear all, close all, nfig = 0;
%
% set frequency range (logarithmic spacing)
% w = logspace(-2,2,160);
%
% data for the s+1 factor
% b = [1 .1]; a = 1; sys1 = tf(b,a);
% [mag1,phase1] = bode(sys1,w);
% mag1 = 20*log10(mag1); % needed to convert to db
%
```

```

% data for the 1/s factor
b = 1; a = [1 0]; sys2 = tf(b,a);
[mag2,phase2] = bode(sys2,w);
mag2 = 20*log10(mag2); % needed to convert to db

% data for the 1/(s+1) factor
b = 1; a = [1 1]; sys3 = tf(b,a);
[mag3,phase3] = bode(sys3,w);
mag3 = 20*log10(mag3); % needed to convert to db

% composite magnitude and phase
mag = mag1+mag2+mag3; phase = phase1+phase2+phase3;

% Note: The magnitude and phase of the transfer function are stored in 3-D arrays
% eg. mag(#outputs, # inputs, # frequency values)
% where this example has only a single input and single output
%
% Bode magnitude plots
tm1(:,1) = mag1(1,1,:); tm2(:,1) = mag2(1,1,:);
tm3(:,1) = mag3(1,1,:); tm(:,1) = mag(1,1,:);
nfig = nfig+1; figure(nfig)
subplot(2,2,1),semilogx(w,tm1,'r','LineWidth',2),grid
title('H(s) = s+0.1')
xlabel('frequency (rad/sec)'),ylabel('|H(jw)| in db')
v = axis; v(3)=-40; v(4)=40; axis(v);
subplot(2,2,2),semilogx(w,tm2,'b','LineWidth',2),grid
title('H(s) = 1/s')
xlabel('frequency (rad/sec)'),ylabel('|H(jw)| in db'),axis(v);
subplot(2,2,3),semilogx(w,tm3,'g','LineWidth',2),grid
title('H(s) = 1/(s+1)')
xlabel('frequency (rad/sec)'),ylabel('|H(jw)| in db'),axis(v);
subplot(2,2,4),semilogx(w,tm,'k','LineWidth',2),grid
title('H(s) = (s+0.1)/[s(s+1)]')
xlabel('frequency (rad/sec)'),ylabel('|H(jw)| in db'),axis(v);

% Bode phase plots
tp1(:,1) = phase1(1,1,:); tp2(:,1) = phase2(1,1,:);
tp3(:,1) = phase3(1,1,:); tp(:,1) = phase(1,1,:);
nfig = nfig+1; figure(nfig)
subplot(2,2,1),semilogx(w,tp1,'r','LineWidth',2),grid
title('H(s) = s+0.1')
xlabel('frequency (rad/sec)'),ylabel('angle')
v = axis; v(3)=-100; v(4)=100; axis(v);
subplot(2,2,2),semilogx(w,tp2,'b','LineWidth',2),grid
title('H(s) = 1/s')
xlabel('frequency (rad/sec)'),ylabel('angle'),axis(v);
subplot(2,2,3),semilogx(w,tp3,'g','LineWidth',2),grid
title('H(s) = 1/(s+1)')
xlabel('frequency (rad/sec)'),ylabel('angle'),axis(v);
subplot(2,2,4),semilogx(w,tp,'k','LineWidth',2),grid
title('H(s) = (s+0.1)/[s(s+1)]')
xlabel('frequency (rad/sec)'),ylabel('angle'),axis(v);

% Nichols plots (log-mag vs angle)
nfig = nfig+1; figure(nfig)
subplot(1,2,1),plot(tp,tm,'r','LineWidth',2),grid
title('H(s) = (s+0.1)/[s(s+1)]')
xlabel('angle'),ylabel('|H(jw)| in db')
gtext('low freq'),gtext('high freq')

% Nyquist plots (Im vs Re)
b = [1 .1]; a = [1 1 0];
h = freqs(b,a,w); hr = real(h); hi = imag(h);
subplot(1,2,2),plot(hr,hi,'b','LineWidth',2),grid
title('H(s) = (s+0.1)/[s(s+1)]')
xlabel('Real'),ylabel('Imag')
gtext('low freq'),gtext('high freq')

% end of example

```

**Table 4.3 Listing of Matlab m-file bodedemo2.m**

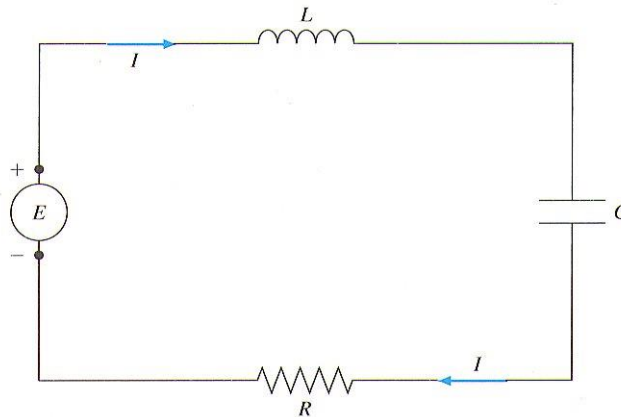
```

%
% BODEDEMO2.M - This demo gives an example of some frequency response plots
%               for  $H(s) = (s+0.1)/[s(s+1)] = b(s)/a(s)$ 
%
% This file uses Matlab's Bode, Nyquist, and Nichols commands to plot the
% frequency response for the above system (and its individual components).
%
% A related file, BODEDEMO1.M, does the same thing, but all the plots are done
% manually (i.e. explicitly by the user). In this file, we let Matlab produce its
% own default plots (no left hand arguments are used in the Bode, Nyquist, and
% Nichols commands). Once you understand the process, this is clearly the easier
% path to follow!
%
% File prepared by J. R. White, UMass-Lowell (last update: Feb. 2020)
%
%
%   clear all, close all, nfig = 0;
%
%   set frequency range (logarithmic spacing)
%   w = logspace(-2,2,160);
%
%   Bode plot for the s+1 factor
%   b = [1 0.1]; a = 1; sys1 = tf(b,a);
%   nfig = nfig+1; figure(nfig)
%   subplot(2,2,1),bode(sys1,'r',w),grid,title('H(s) = s+0.1')
%
%   Bode plot for the 1/s factor
%   b = 1; a = [1 0]; sys2 = tf(b,a);
%   subplot(2,2,2),bode(sys2,'b',w),grid,title('H(s) = 1/s')
%
%   Bode plot for the 1/(s+1) factor
%   b = 1; a = [1 1]; sys3 = tf(b,a);
%   subplot(2,2,3),bode(sys3,'g',w),grid,title('H(s) = 1/(s+1)')
%
%   Bode plot for the composite function:  $H(s) = (s+1)/[s(s+1)] = b(s)/a(s)$ 
%   b = [1 0.1]; a = [1 1 0]; sys = tf(b,a);
%   subplot(2,2,4),bode(sys,'k',w),grid,title('H(s) = (s+0.1)/[s(s+1)]')
%
%   set all the curves to have a larger linewidth
%   h = findobj(gcf,'LineStyle','-'); nh = length(h);
%   for i = 1:nh; set(h(i),'LineWidth',2); end
%
%   the individual factors and the composite could also be plotted on a single plot
%   nfig = nfig+1; figure(nfig)
%   bode(sys1,'r',sys2,'b',sys3,'g',sys,'k'),grid
%   h1 = findobj(gcf,'Color','r'); nh1 = length(h1);
%   for i = 1:nh1; set(h1(i),'LineWidth',2); end
%   h2 = findobj(gcf,'Color','b'); nh2 = length(h2);
%   for i = 1:nh2; set(h2(i),'LineWidth',2); end
%   h3 = findobj(gcf,'Color',[0 0.65 0]); nh3 = length(h3); % would not find 'g'???
%   for i = 1:nh3; set(h3(i),'LineWidth',2); end
%   h4 = findobj(gcf,'Color','k'); nh4 = length(h4);
%   for i = 1:nh4; set(h4(i),'LineWidth',2); end
%   legend([h1(1),h2(1),h3(1),h4(1)], 's+0.1','1/s','1/(s+1)','(s+0.1)/[s(s+1)]', ...
%         'Location','East')
%
%   Nichols and Nyquist plots
%   nfig = nfig+1; figure(nfig)
%   subplot(1,2,1),nichols(sys,'r.',w),grid % Nichols -- log-mag vs angle
%   subplot(1,2,2),nyquist(sys,'b.',w),grid % Nyquist -- Im vs Re
%
%   end of example
%

```

**Example 4.3 Frequency Domain Representation of the RLC Circuit Example**

Consider the simple series RLC circuit shown below:



In Example 3.6 we showed that the balance equation for this system can be written as

$$L \frac{d^2}{dt^2} I(t) + R \frac{d}{dt} I(t) + \frac{1}{C} I(t) = \frac{d}{dt} E(t)$$

With zero initial conditions, taking the Laplace transform of both sides gives

$$\left( Ls^2 + Rs + \frac{1}{C} \right) I(s) = sE(s)$$

Thus, the transfer function,  $G(s)$ , for this system is

$$G(s) = \frac{I(s)}{E(s)} = \frac{s}{Ls^2 + Rs + \frac{1}{C}}$$

This representation of the series RLC circuit can be used to perform time domain simulations of the system for different inputs. In particular, using the same values for  $R$ ,  $L$ , and  $C$  from Example 3.6, the step response of the system can be obtained in Matlab using the transfer function form of the **step** command. For a 10 volt step change, this is given by a series of commands that resemble

```
sys = tf(num,den); Y = step(sys,T); Y = 10*Y;
```

where **Y** and **T** are the output response vector and input time vector, respectively (same as in Example 3.6). The **num** and **den** variables represent the numerator and denominator polynomials of the transfer function given in Matlab syntax, and the **tf** command creates a transfer function LTI object that can be manipulated by a variety of Matlab functions. For this problem we have

```
num = [1 0]; and den = [L R 1/C];
```

which are vectors containing the coefficients of the polynomials in decreasing powers of  $s$ . Additional forms for the **tf**, **step**, **impulse**, etc. commands are described in the Matlab help files.

Figures 4.4a and 4.4b show the time domain solutions for two cases:  $R = 100$  ohm and  $10$  ohm. These solutions are identical with those obtained in Example 3.6 -- for  $R = 100$  ohm we see an overdamped time domain profile and for  $R = 10$  ohm, underdamped behavior is observed.

The frequency response can also be easily obtained from the transfer function. In this case the **bode** command is used directly to plot the magnitude and phase of the transfer function over the default frequency range. This is given by the single command: **bode(sys)**.

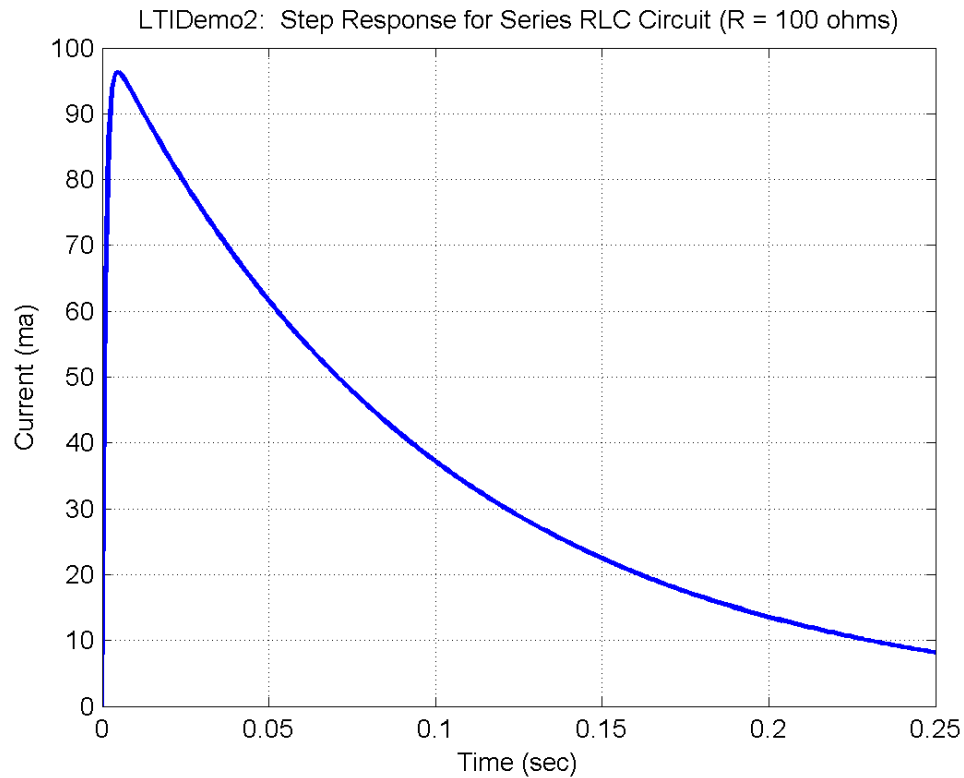
Since the frequency response data give information about the output response for a sinusoidal input at different frequencies, we have also computed the time response for two specific frequencies:  $\omega = 1$  rad/sec and  $\omega = 100$  rad/sec. The Bode and time domain plots are given in Figs. 4.5 and 4.6, respectively for the case with  $R = 100$  ohm. Figures 4.7 and 4.8 have similar plots for  $R = 10$  ohm. In the time domain plots, the input function,  $E(t)$ , is shown with a dashed line, and the output response,  $I(t)$ , is given as a solid line. Note also, for ease of plotting on the same scale, the output has been normalized by the magnitude of the transfer function at the specific frequency of interest. With this normalization, the magnitude of the system input and output should be identical, and the profiles should only differ due to the phase angle difference between input and output. Note that both Figs. 4.6 and 4.8 show this behavior after a short transient period has elapsed. The observed phase differences in these figures are also consistent with those given in the Bode plots in Figs. 4.5 and 4.7.

The normalization factors noted on the time domain plots can also be obtained directly from the Bode plots. For example, Fig. 4.8 indicates that the ratio of the output to the input is  $0.1$  at  $\omega = 100$  rad/sec for the  $R = 10$  ohm case. However, at this frequency, the Bode magnitude plot in Fig. 4.7 shows a value of just about  $-20$  db. Recalling the definition of a decibel, we have  $|G(j\omega)|$  in db  $= 20 \log_{10} |G(j\omega)|$  in absolute units. Therefore,  $-20$  db corresponds to  $|G(j\omega)| = 0.1$ , which is exactly as expected.

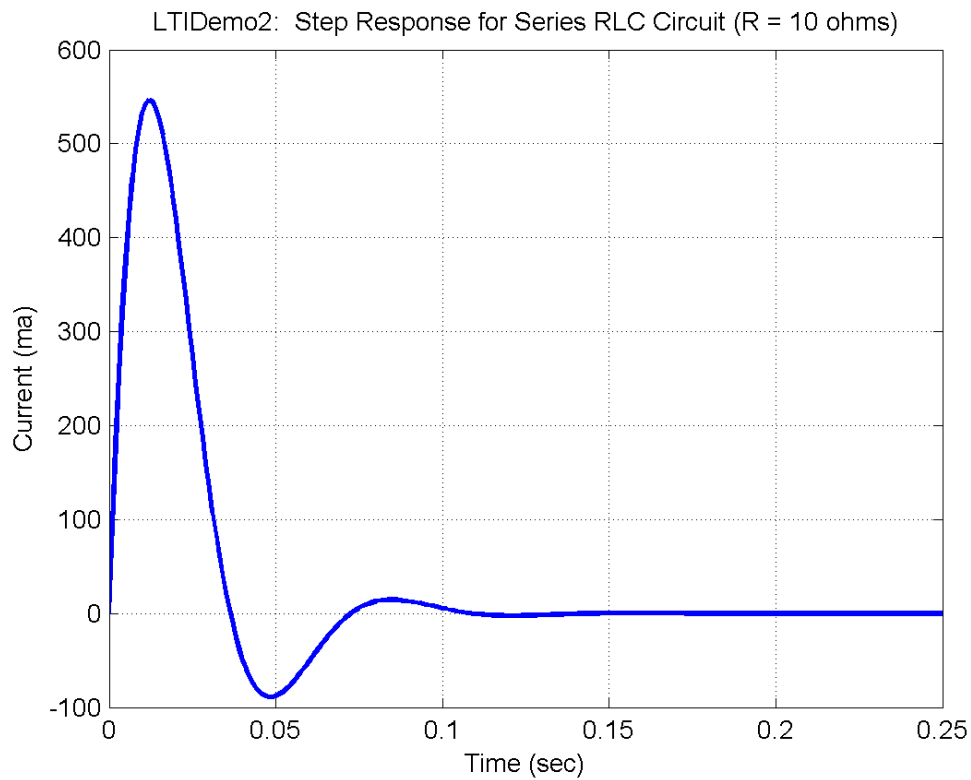
The Matlab m-file for this example is **ltidemo2.m** and this is listed in Table 4.4.

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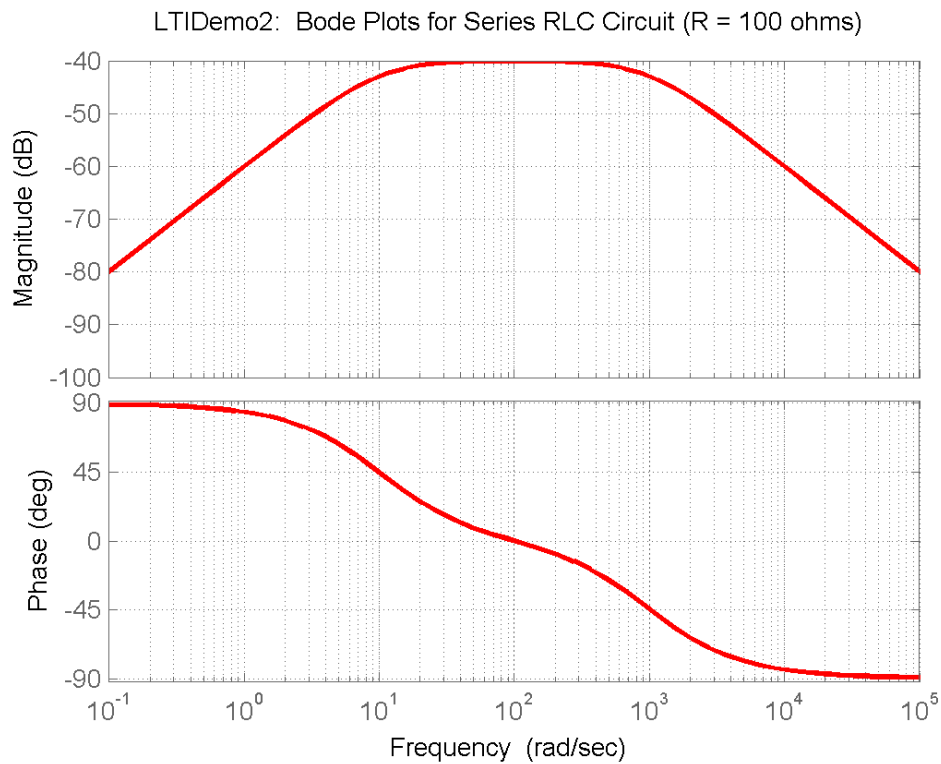




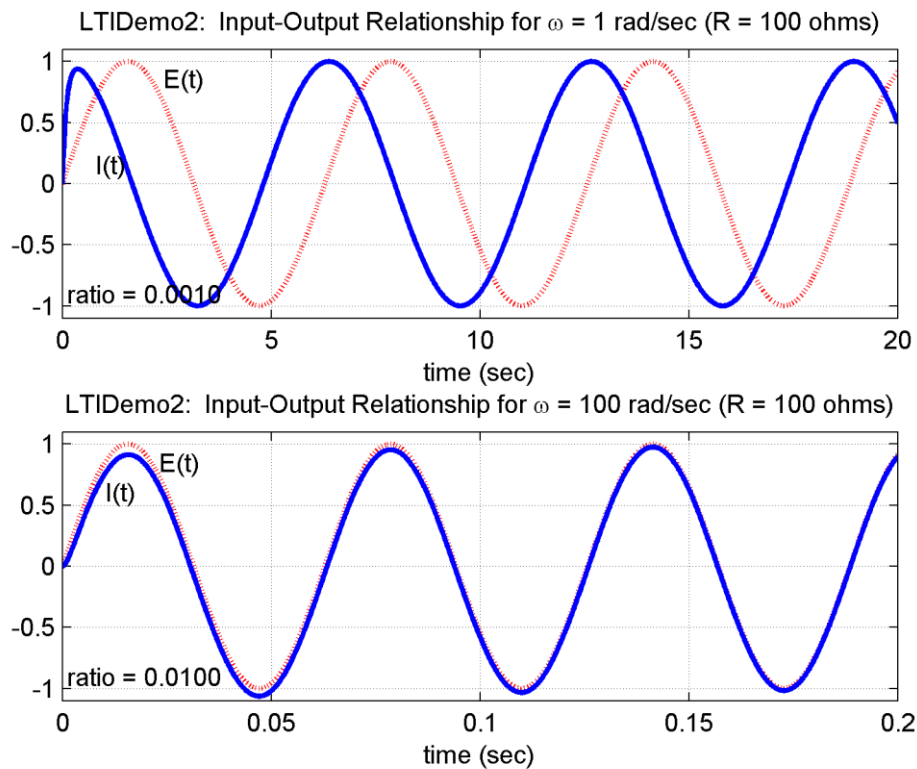
**Fig. 4.4a** Time response for 10 volt step change in input ( $R = 100$  ohm).



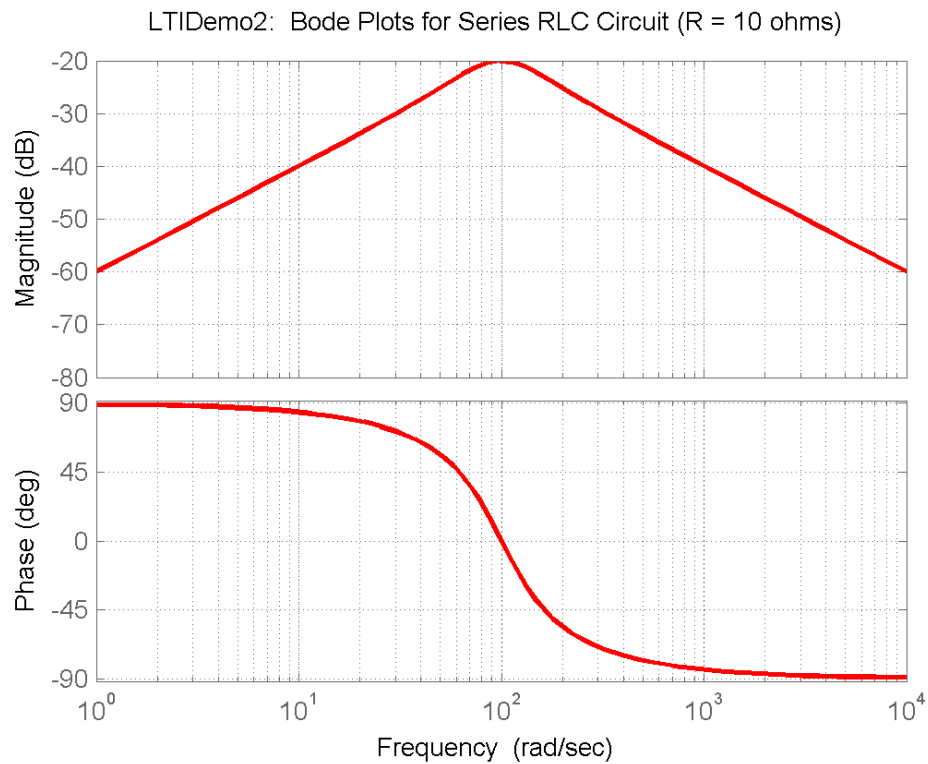
**Fig. 4.4b** Time response for 10 volt step change in input ( $R = 10$  ohm).



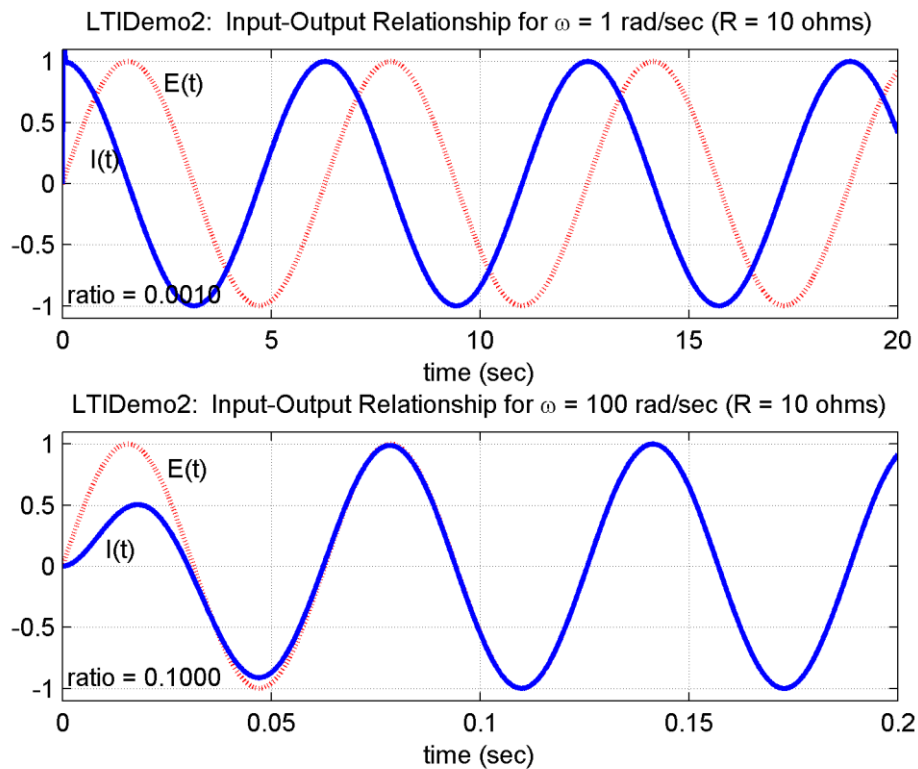
**Fig. 4.5** Bode plot for RLC series circuit (R = 100 ohm).



**Fig. 4.6** Input-output profiles for RLC series circuit (R = 100 ohm).



**Fig. 4.7 Bode plot for RLC series circuit ( $R = 10$  ohm).**



**Fig. 4.8 Input-output profiles for RLC series circuit ( $R = 10$  ohm).**

**Table 4.4 Listing of Matlab m-file ltidemo2.m**

```

%
% LTIDEMO2.M  MATLAB sample that demonstrates various analysis schemes for
%             Linear Time Invariant (LTI) systems.  DEMO2 in this series focuses
%             on frequency domain solutions and plotting techniques.
%             See LTIDEMO1.M for the time domain simulations.
%
% Two areas of focus:
%   A. Step response using transfer function form (for comparison with DEMO1)
%   B. Generation and Interpretation of Bode plots
%
% Simple RLC electrical circuit with applied voltage
%    $LI'' + RI' + I/C = E'$ 
%
% This can be put into standard Laplace transform form as follows:
%    $I(s) = G(s)E(s)$       with       $G(s) = s/(Ls^2 + Rs + 1/C)$ 
%
% File prepared by J. R. White, UMass-Lowell (last update: Feb. 2020)
%

clear all, close all, nfig = 0;

%
% define coefficients for specific problem
L = 0.1;           % inductance (henry)
Ca = 0.001;        % capacitance (farad)
RR = [100 10 2];   % resistance (ohms)
ir = menu('Choose R value in RLC circuit', ...
          'R = 100 ohms (over damped response)', ...
          'R = 10 ohms (under damped response)', ...
          'R = 2 ohms (slightly damped response)');
R = RR(ir);

%
% create the transfer function polynomials and LTI object for this system
us = 10; num = [1 0]; den = [L R 1/Ca]; sys = tf(num,den);

%
% Part A Step Response of LTI System (to compare to DEMO1)
%
% set integration time and time vector
to = 0; tf = 0.25; nt = 251; t = linspace(to,tf,nt);

%
% simulate and plot the time domain response for a step input
nfig = nfig+1; figure(nfig)
y = step(sys,t); y = us*y; % normalize output by input strength (us volts)
plot(t,1000*y,'b','LineWidth',2),grid
title(['LTIDemo2: Step Response for Series RLC Circuit (R = ', ...
       num2str(R),' ohms)'])
ylabel('Current (ma)'), xlabel('Time (sec)')

%
% Part B Interpretation of Frequency Response Plots (Bode)
%
% generate bode plot
nfig = nfig+1; figure(nfig), bode(sys,'r'),grid
h = findobj(gcf,'Color','red'); nh = length(h);
for i = 1:nh; set(h(i),'LineWidth',2); end
title(['LTIDemo2: Bode Plots for Series RLC Circuit (R = ', ...
       num2str(R),' ohms)'])

%
% now simulate and plot system response for two different frequencies
t1 = linspace(0,20,1001); w1 = 1;
u1 = sin(w1*t1); y1 = lsim(sys,u1,t1);
n1 = polyval(num,w1*1i); d1 = polyval(den,w1*1i); f1 = abs(n1/d1);
t2 = linspace(0,0.2,1001); w2 = 100;
u2 = sin(w2*t2); y2 = lsim(sys,u2,t2);
n2 = polyval(num,w2*1i); d2 = polyval(den,w2*1i); f2 = abs(n2/d2);

```

```

nfig = nfig+1;    figure(nfig)
subplot(2,1,1),plot(t1,u1,'r:',t1,y1/f1,'b-', 'LineWidth',2),grid
title(['LTIDemo2:  Input-Output Relationship for \omega = ',num2str(w1), ...
      ' rad/sec (R = ',num2str(R),' ohms)'])
v1 = axis;    v1(3) = -1.1;    v1(4) = 1.1;    axis(v1);
xlabel('time (sec)')
gtext('E(t)'),gtext('I(t)'),gtext(['ratio = ',num2str(f1,'%5.4f')])
subplot(2,1,2),plot(t2,u2,'r:',t2,y2/f2,'b-', 'LineWidth',2),grid
title(['LTIDemo2:  Input-Output Relationship for \omega = ',num2str(w2), ...
      ' rad/sec (R = ',num2str(R),' ohms)'])
v2 = axis;    v2(3) = -1.1;    v2(4) = 1.1;    axis(v2);
xlabel('time (sec)')
gtext('E(t)'),gtext('I(t)'),gtext(['ratio = ',num2str(f2,'%5.4f')])
%
% end of demo

```

### Discrete Systems

To give a complete treatment of transform methods, we need to say a few words on transforms for discrete systems before leaving this section of notes. As noted previously, however, we will not do much with discrete systems in this course -- other than to briefly show that the analysis of discrete systems can be done in a fashion quite analogous to continuous systems. As such, the student is advised to consult one of the many good dynamics texts that are available for a detailed discussion of this subject.

As part of our brief introduction for discrete systems, we note that one needs the Z-transform instead of Laplace transforms. The Z-transform is a mathematical operation that, when applied to a sequence of numbers, produces a function of the variable  $z$ . The formal definition is

$$Z\{f(k)\} = F(z) = \sum_{k=0}^{\infty} \frac{f(k)}{z^k} \quad (4.50)$$

The main point of focus here is that the application and manipulation of Z-transforms for discrete systems are completely analogous to our treatment of Laplace transforms for the continuous time problem.

The procedure for the analysis of discrete systems in the *frequency domain* can be outlined as follows:

1. Define the important transform relationships;  $f(k) \Leftrightarrow F(z)$
2. Perform inversion by partial fractions.
3. Obtain the transform solution of difference equations.
4. Extend method to include matrix state equations with definition of the system transfer function matrix.

For example, given the discrete system

$$\underline{x}(k+1) = \underline{A}\underline{x}(k) + \underline{B}\underline{u}(k) \quad (4.51)$$

$$\underline{y}(k) = \underline{C}\underline{x}(k) + \underline{D}\underline{u}(k) \quad (4.52)$$

Taking the Z-transform (letting  $\underline{x}(0) = 0$ ) gives

$$\underline{Y}(z) = \underline{C}[z\underline{I} - \underline{A}]^{-1} \underline{B}\underline{U}(z) + \underline{D}\underline{U}(z) \quad (4.53)$$

Now defining the transfer function matrix as

$$\underline{Y}(z) = \underline{H}(z)\underline{U}(z) \quad (4.54)$$

one has

$$\underline{H}(z) = \underline{C}[z\underline{I} - \underline{A}]^{-1} \underline{B} + \underline{D} \quad (4.55)$$

#### 5. Importance of impulse response

$$y(k) = \sum_{j=0}^k h(k-j)u(j) \quad (4.56)$$

where  $h(k)$  is the system impulse response which is given by the inverse Z-transform of  $H(z)$ .

#### 6. Use and manipulation of block diagrams.

#### 7. Etc.

In general, every operation for continuous time systems has an analogous counterpart in the treatment of discrete systems. One simply uses Laplace transforms for continuous systems and Z-transforms for discrete systems.

Some examples of important Z-transform pairs follow (see Table 4.1 for a more complete list):

#### Basic Definition

$$Z\{f(k)\} = F(z) = \sum_{k=0}^{\infty} \frac{f(k)}{z^k}$$

#### Unit Step

$$Z\{f(k) = 1\} = \sum_{k=0}^{\infty} (1)z^{-k} = 1 + z^{-1} + z^{-2} + \dots = \frac{z}{z-1} \quad (4.57)$$

#### Exponential

$$Z\{f(k) = e^{\lambda kT}\} = \sum_{k=0}^{\infty} e^{\lambda kT} z^{-k} = 1 + z^{-1}e^{\lambda T} + z^{-2}e^{2\lambda T} + \dots = \frac{z}{z - e^{\lambda T}} \quad (4.58)$$

#### Differentiation

$$Z\{f(k+1)\} = \sum_{k=0}^{\infty} f(k+1)z^{-k}$$

To simplify this, let  $m = k+1$ , or

$$Z\{f(k+1)\} = \sum_{m=1}^{\infty} f(m)z^{-m+1}$$

Now simply replace  $m$  with  $k$  (since these are just dummy indices for the summation), giving

$$Z\{f(k+1)\} = \sum_{k=1}^{\infty} f(k)z^{-k+1} = f(1) + f(2)z^{-1} + f(3)z^{-2} \quad \dots$$

Now adding and subtracting  $zf(0)$  gives

$$Z\{f(k+1)\} = \sum_{k=1}^{\infty} f(k)z^{-k+1} = zf(0) + f(1) + f(2)z^{-1} + f(3)z^{-2} \quad \dots \quad -zf(0)$$

And this reduces to

$$Z\{f(k+1)\} = z \left( \sum_{k=0}^{\infty} f(k)z^{-k} - f(0) \right) = z \left( f(0) + f(1)z^{-1} + \dots - f(0) \right) = zF(z) - zf(0)$$

$$\text{or} \quad Z\{f(k+1)\} = zF(z) - zf(0) \quad (4.59)$$

Now, with this expression for  $Z\{f(k+1)\}$ , we can formally derive eqn. (4.53). To see this, we take the Z-transform of eqn. (4.51), giving

$$Z\{\underline{x}(k+1) = \underline{A}\underline{x}(k) + \underline{B}\underline{u}(k)\} \Rightarrow z\underline{X}(z) = \underline{A}\underline{X}(z) + \underline{B}\underline{U}(z) \quad (\text{for zero ICs})$$

and, with a little algebra, we have

$$(z\underline{I} - \underline{A})\underline{X}(z) = \underline{B}\underline{U}(z) \quad \text{and} \quad \underline{X}(z) = (z\underline{I} - \underline{A})^{-1} \underline{B}\underline{U}(z)$$

With this expression and eqn. (4.52), we have

$$Z\{\underline{y}(k) = \underline{C}\underline{x}(k) + \underline{D}\underline{u}(k)\} \Rightarrow \underline{Y}(z) = \underline{C}\underline{X}(z) + \underline{D}\underline{U}(z)$$

$$\text{or} \quad \underline{Y}(z) = \left[ \underline{C}(z\underline{I} - \underline{A})^{-1} \underline{B} + \underline{D} \right] \underline{U}(z) = \underline{H}(z)\underline{U}(z)$$

which is the result shown in eqns. (4.53) – (4.55).

Well, one can certainly continue this basic process to develop a full set of common Z-transform pairs and functional relationships. However, we will not elaborate any further in these notes -- but, should the need arise, one can refer to Table 4.1 for several additional examples and/or consult an appropriate text on discrete systems for a more detailed treatment of this subject...