## System Dynamics (22.554 \& 24.509)

## II. Mathematical Background

## Introduction

Three math-related topics that are required background material for this course are:

1. Solution of differential and difference equations,
2. Matrix notation, use/manipulation of matrix equations, functions of a square matrix, and
3. Laplace transform techniques.

We will discuss items 1 and 2 in this section of notes and defer the review of Laplace transforms for the section on transform methods. The intent in this section is to review these subjects only briefly, highlighting the most important features in each area. A student not familiar with parts of this review should consult appropriate advanced math texts for further study. For example, Advanced Engineering Mathematics by Kreyszig is one good choice. Ogata's text, Modern Control Engineering, is also an excellent resource here.

## Differential Equations

In general, multivariate continuous dynamic systems are described by a system of differential equations. Very shortly we will address the solution of matrix differential equations. For now let's just refresh our memory with the solution of a few simple example problems containing only one unknown.

A general $\mathrm{n}^{\text {th }}$-order, linear, variable coefficient, non-homogeneous differential equation can be written as

$$
\begin{equation*}
\frac{d^{n}}{d t^{n}} y(t)+a_{n-1}(t) \frac{d^{n-1}}{d t^{n-1}} y(t)+\quad \ldots \quad+a_{0}(t) y(t)=g(t) \tag{2.1}
\end{equation*}
$$

For continuous $a_{i}(t)$ and $g(t)$ over some interval, there is a unique solution to the linear differential equation which satisfies the initial conditions,

$$
\begin{equation*}
\mathrm{y}(0)=\mathrm{b}_{0},\left.\quad \frac{\mathrm{dy}}{\mathrm{dt}}\right|_{0}=\mathrm{b}_{1},\left.\quad \ldots \quad \frac{\mathrm{~d}^{\mathrm{n}-1} \mathrm{y}}{\mathrm{dt}^{\mathrm{n}-1}}\right|_{0}=\mathrm{b}_{\mathrm{n}-1} \tag{2.2}
\end{equation*}
$$

For a unique solution, an $\mathrm{n}^{\text {th }}$-order initial value problem (IVP) must have n initial conditions.
A general solution for $\mathrm{y}(\mathrm{t})$ can be constructed as a linear combination of a homogeneous and particular solution to eqn. (2.1), or

$$
\begin{equation*}
\mathrm{y}(\mathrm{t})=\mathrm{y}_{\mathrm{h}}(\mathrm{t})+\mathrm{y}_{\mathrm{p}}(\mathrm{t}) \tag{2.3}
\end{equation*}
$$

where the homogeneous solution is the solution to eqn. (2.1) with $g(t)=0$. The unique solution is obtained from the general solution after application of the initial conditions.

Analytical solutions to problems of this type are difficult (or impossible) for non-stationary systems (i.e. variable coefficients). However, for linear differential equations with constant coefficients, a general technique for constructing the homogeneous solution can be given as follows. Given the homogeneous equation

$$
\begin{equation*}
\frac{d^{n}}{d t^{n}} z(t)+a_{n-1} \frac{d^{n-1}}{d t^{n-1}} z(t)+\quad \ldots \quad+a_{0} z(t)=0 \tag{2.4}
\end{equation*}
$$

assume a solution of the form

$$
\begin{equation*}
\mathrm{z}(\mathrm{t})=\mathrm{e}^{\lambda \mathrm{t}} \tag{2.5}
\end{equation*}
$$

for some constant $\lambda$. Now, substituting eqn. (2.5) into eqn. (2.4) gives

$$
\begin{equation*}
\lambda^{n} \mathrm{e}^{\lambda t}+\mathrm{a}_{\mathrm{n}-1} \lambda^{\mathrm{n}-1} \mathrm{e}^{\lambda \mathrm{t}}+\ldots \quad+\mathrm{a}_{0} \mathrm{e}^{\lambda \mathrm{t}}=0 \tag{2.6}
\end{equation*}
$$

or $\quad \lambda^{n}+a_{n-1} \lambda^{n-1}+\ldots \quad+a_{0}=0$
where the latter expression is termed the characteristic equation. A characteristic value is any root of the characteristic polynomial [i.e. the left hand side (LHS) of eqn. (2.6)]. For an $n^{\text {th }}$ order equation, there will be n roots (not necessarily distinct).
Letting $\mathrm{Z}_{\mathrm{i}}(\mathrm{t})=\mathrm{e}^{\lambda_{\mathrm{i}} \mathrm{t}}$, we can construct a general solution to eqn. (2.4) as a linear combination of $\mathrm{z}_{\mathrm{i}}(\mathrm{t})$ (for distinct roots),

$$
\begin{equation*}
\mathrm{z}(\mathrm{t})=\sum_{\mathrm{i}} \mathrm{c}_{\mathrm{i}} \mathrm{z}_{\mathrm{i}}(\mathrm{t})=\sum_{\mathrm{i}} \mathrm{c}_{\mathrm{i}} \mathrm{e}^{\lambda_{\mathrm{i}} \mathrm{t}} \tag{2.7}
\end{equation*}
$$

If the roots are not all distinct, one usually uses the reduction of order method (sometimes referred to as the variation of parameters method) to obtain additional homogeneous solutions. In this case, one assumes $\mathrm{z}_{\mathrm{i}+1}(\mathrm{t})=\mathrm{u}(\mathrm{t}) \mathrm{z}_{\mathrm{i}}(\mathrm{t})$, where $\mathrm{u}(\mathrm{t})$ can be determined by substitution into the original differential equation. For linear constant coefficient systems, $u(t)=t^{m}$ works well, where $m$ is the smallest integer that leads to linearly independent solutions.

For the particular solution, the method of undetermined coefficients is commonly used when $\mathrm{g}(\mathrm{t})$ is a simple analytical function. In this case, one usually assumes that $\boldsymbol{y}_{\boldsymbol{p}}(t)$ is of the same form as the forcing function $g(t)$ and all its linearly independent derivatives. This assumed solution is put back into the original non-homogeneous equation and then the unknown coefficients can be determined explicitly by equating coefficients of like terms. Note that, if the forcing function is of the same form as the homogeneous solution, then multiplication by $\mathrm{t}^{\mathrm{m}}$ (as above) should lead to a linearly independent particular solution to the defining ODE.
Examples 2.1 and 2.2 illustrate this basic solution procedure for the case of a simple $2^{\text {nd }}$-order linear stationary system.

## Example 2.1 Solution of Simple Continuous IVP

## Problem Statement:

Solve $\frac{\mathrm{d}^{2}}{\mathrm{dt}^{2}} \mathrm{y}(\mathrm{t})+\omega^{2} \mathrm{y}(\mathrm{t})=\mathrm{C} \sin \left(\omega_{0} \mathrm{t}\right)$ with initial conditions $\left.\mathrm{y}(\mathrm{t})\right|_{0}=0$ and $\left.\frac{\mathrm{dy}(\mathrm{t})}{\mathrm{dt}}\right|_{0}=0$.

## Problem Solution:

General Solution: The general form of the solution can be written as $y(t)=y_{h}(t)+y_{p}(t)$.
Homogeneous Solution: Letting $\mathrm{y}(\mathrm{t})=\mathrm{e}^{\lambda \mathrm{t}}$ in the defining equation with the RHS set to zero gives $\lambda^{2}+\omega^{2}=0$, or $\lambda_{1,2}= \pm i \omega$. Thus the homogeneous solution can be written as

$$
y_{h}(t)=A_{1} e^{i \omega t}+A_{2} e^{-\mathrm{i} \omega \mathrm{t}}=\mathrm{A}_{3} \sin (\omega \mathrm{t})+\mathrm{A}_{4} \cos (\omega \mathrm{t})
$$

Particular Solution: Assume $\mathrm{y}_{\mathrm{p}}(\mathrm{t})=\mathrm{k}_{1} \sin \left(\omega_{0} \mathrm{t}\right)+\mathrm{k}_{2} \cos \left(\omega_{0} \mathrm{t}\right)$. Substitution of this into the original equation gives

$$
-k_{1} \omega_{0}^{2} \sin \left(\omega_{0} t\right)-k_{2} \omega_{0}^{2} \cos \left(\omega_{0} t\right)+k_{1} \omega^{2} \sin \left(\omega_{0} t\right)+k_{2} \omega^{2} \cos \left(\omega_{0} t\right)=C \sin \left(\omega_{0} t\right)
$$

and equating coefficients of similar terms gives

$$
\mathrm{k}_{1}=\frac{\mathrm{C}}{\omega^{2}-\omega_{0}^{2}} \quad \text { and } \quad \mathrm{k}_{2}=0
$$

Therefore, $\mathrm{y}(\mathrm{t})=\mathrm{A}_{3} \sin (\omega \mathrm{t})+\mathrm{A}_{4} \cos (\omega \mathrm{t})+\frac{\mathrm{C}}{\omega^{2}-\omega_{0}^{2}} \sin \left(\omega_{0} \mathrm{t}\right)$.
Unique Solution: Applying the initial conditions gives

$$
\begin{array}{ll}
\left.\mathrm{y}(\mathrm{t})\right|_{0}=0=\mathrm{A}_{3}(0)+\mathrm{A}_{4}(1)+\frac{\mathrm{C}}{\omega^{2}-\omega_{0}^{2}}(0) & \therefore \mathrm{A}_{4}=0 \\
\left.\frac{\mathrm{~d}}{\mathrm{dt}} \mathrm{y}(\mathrm{t})\right|_{0}=0=\left.\left[\mathrm{A}_{3} \omega \cos (\omega \mathrm{t})+\frac{\mathrm{C} \omega_{0}}{\omega^{2}-\omega_{0}^{2}} \cos \left(\omega_{0} t\right)\right]\right|_{0} & \therefore \mathrm{~A}_{3}=\frac{-\omega_{0}}{\omega}\left[\frac{C}{\omega^{2}-\omega_{0}^{2}}\right]
\end{array}
$$

Finally, one has

$$
\mathrm{y}(\mathrm{t})=\frac{\mathrm{C}}{\omega^{2}-\omega_{0}^{2}}\left[\sin \left(\omega_{0} \mathrm{t}\right)-\frac{\omega_{0}}{\omega} \sin (\omega \mathrm{t})\right]
$$

## Example 2.2 Another Simple Continuous IVP (modification of Ex. 2.1)

Problem Statement:
Solve the same problem as Ex. 2.1 but let the forcing function be $g(t)=C \sin (\omega t)$.
Problem Solution:
Particular Solution: Note that this problem has the same homogeneous solution as Ex. 2.1. However, now the forcing term is of the same form as the homogeneous solution. Thus, since all the terms in the particular solution must be linearly independent from the homogeneous or complementary solution, one should choose the following expression for the particular solution:

$$
y_{p}(t)=k_{1} t \sin (\omega t)+k_{2} t \cos (\omega t)
$$

Substitution of this back into the original equation gives $k_{1}=0 \quad$ and $\quad k_{2}=-\frac{C}{2 \omega}$
General Solution: Therefore, our assumed $\mathrm{y}_{\mathrm{p}}(\mathrm{t})$ is correct and the general solution becomes

$$
y(t)=A_{1} \sin (\omega t)+A_{2} \cos (\omega t)-\frac{C}{2 \omega} t \cos (\omega t)
$$

Unique Solution: Application of the initial conditions (from Ex. 2.1) gives the unique solution,

$$
y(t)=\frac{C}{2 \omega}\left[\frac{1}{\omega} \sin (\omega t)-t \cos (\omega t)\right]
$$

## Difference Equations

In contrast with continuous systems, discrete dynamic behavior is often characterized by discrete difference equations. A general $n^{\text {th }}$-order, linear, variable coefficient, non-homogeneous difference equation can be written as

$$
\begin{equation*}
y(k+n)+a_{n-1}(k) y(k+n-1)+\quad \ldots \quad+a_{0}(k) y(k)=g(k) \tag{2.8}
\end{equation*}
$$

Analogous to the continuous form, the general solution for $\mathrm{y}(\mathrm{k})$ can be constructed as a linear combination of a homogeneous and particular solution [where the homogeneous component is the solution to eqn. (2.8) with $\mathrm{g}(\mathrm{k})=0$ ], or

$$
\begin{equation*}
\mathrm{y}(\mathrm{k})=\mathrm{y}_{\mathrm{h}}(\mathrm{k})+\mathrm{y}_{\mathrm{p}}(\mathrm{k}) \tag{2.9}
\end{equation*}
$$

Again, analogous to the continuous case, linear difference equations with constant coefficients have relatively simple homogeneous solutions. Given the homogeneous equation

$$
\begin{equation*}
\mathrm{z}(\mathrm{k}+\mathrm{n})+\mathrm{a}_{\mathrm{n}-1} \mathrm{z}(\mathrm{k}+\mathrm{n}-1)+\quad \ldots \quad+\mathrm{a}_{0} \mathrm{z}(\mathrm{k})=0 \tag{2.10}
\end{equation*}
$$

assume a solution of the form

$$
\begin{equation*}
\mathrm{z}(\mathrm{k})=\lambda^{\mathrm{k}} \tag{2.11}
\end{equation*}
$$

for some constant $\lambda$. Now, performing the appropriate substitutions gives

$$
\begin{array}{ll} 
& \lambda^{k+n}+a_{n-1} \lambda^{k+n-1}+\quad \ldots \quad+a_{0} \lambda^{k}=0 \\
\text { or } \quad & \lambda^{n}+a_{n-1} \lambda^{n-1}+\quad \ldots \quad+a_{0}=0 \tag{2.12}
\end{array}
$$

where the characteristic equation is identical to that obtained for the continuous system [see eqn. (2.6)].

Finally, letting $z_{i}(k)=\lambda_{i}^{k}$, we can construct a general solution to eqn. (2.10) with a linear combination of $z_{i}(k)$ (for distinct roots), giving

$$
\begin{equation*}
\mathrm{z}(\mathrm{k})=\sum_{\mathrm{i}} \mathrm{c}_{\mathrm{i}} \mathrm{z}_{\mathrm{i}}(\mathrm{k})=\sum_{\mathrm{i}} \mathrm{c}_{\mathrm{i}} \lambda_{\mathrm{i}}^{\mathrm{k}} \tag{2.13}
\end{equation*}
$$

The situation for repeated roots and for the determination of particular solutions is similar to that for continuous systems. Examples 2.3 and 2.4 illustrate the basic solution scheme for this type of problem ( $2^{\text {nd }}$-order linear stationary discrete systems).

## Example 2.3 Solution of Simple Discrete IVP

## Problem Statement:

Solve $y(k+2)-2 a y(k+1)+a^{2} y(k)=b^{k}$ with initial conditions $y(0)=0$ and $y(1)=0$.
Problem Solution:
General Solution: The general form of the solution can be written as $y(k)=y_{h}(k)+y_{p}(k)$.
Homogeneous Solution: Letting $\mathrm{y}(\mathrm{k})=\lambda^{\mathrm{k}}$ in the defining equation with the RHS set to zero gives $\lambda^{2}-2 \mathrm{a} \lambda+\mathrm{a}^{2}=0$ or $\lambda_{1,2}=\mathrm{a}$ (which gives repeated roots). For this situation we can find a second homogeneous solution using the reduction of order method, $y_{2}(k)=u(k) y_{1}(k)$. Trying $\mathrm{u}(\mathrm{k})=\mathrm{k}$ and substituting $\mathrm{y}_{2}(\mathrm{k})=\mathrm{ka}^{\mathrm{k}}$ into the original homogeneous equation gives

$$
\begin{aligned}
& (k+2) a^{k+2}-2 a(k+1) a^{k+1}+a^{2} k a^{k}=0 \\
& k a^{k+2}+2 a^{k+2}-2 k a^{k+2}-2 a^{k+2}+k a^{k+2}=0
\end{aligned}
$$

which shows the desired balance relationship. Therefore, $y_{h}(k)=C_{1} a^{k}+C_{2} \mathrm{ka}^{\mathrm{k}}$.
Particular Solution: Assume $\mathrm{y}_{\mathrm{p}}(\mathrm{k})=\mathrm{C}_{3} \mathrm{r}^{\mathrm{k}}$. Then, upon substitution, we have

$$
\mathrm{C}_{3} \mathrm{r}^{\mathrm{k}+2}-2 \mathrm{aC}_{3} \mathrm{r}^{\mathrm{k}+1}+\mathrm{a}^{2} \mathrm{C}_{3} \mathrm{r}^{\mathrm{k}}=\mathrm{br}^{\mathrm{k}} \quad \therefore \mathrm{C}_{3}=\frac{\mathrm{b}}{(\mathrm{r}-\mathrm{a})^{2}}
$$

General Solution: Combining the homogeneous and particular solutions gives

$$
\mathrm{y}(\mathrm{k})=\left(\mathrm{C}_{1}+\mathrm{C}_{2} \mathrm{k}\right) \mathrm{a}^{\mathrm{k}}+\frac{\mathrm{br}^{\mathrm{k}}}{(\mathrm{r}-\mathrm{a})^{2}}
$$

Unique Solution: Applying the initial conditions gives

$$
\begin{array}{ll}
y(0)=0=C_{1}+\frac{b}{(r-a)^{2}} & \therefore C_{1}=\frac{-b}{(r-a)^{2}} \\
y(1)=0=\frac{-b a}{(r-a)^{2}}+C_{2} a+\frac{b r}{(r-a)^{2}} & \therefore C_{2}=-\frac{b}{a} \frac{(r-a)}{(r-a)^{2}}=-\frac{b}{a} \frac{1}{(r-a)}
\end{array}
$$

Finally, one has

$$
\mathrm{y}(\mathrm{k})=\frac{\mathrm{b}}{(\mathrm{r}-\mathrm{a})^{2}}\left[-\left(1+\frac{\mathrm{k}(\mathrm{r}-\mathrm{a})}{\mathrm{a}}\right) \mathrm{a}^{\mathrm{k}}+\mathrm{r}^{\mathrm{k}}\right]
$$

## Example 2.4 Another Simple Discrete IVP (modification of Ex. 2.3).

## Problem Statement:

Solve the same problem as Ex. 2.3 but let the forcing function be $g(k)=b a^{k}$.

## Problem Solution:

Particular Solution: Note that the homogeneous solution is unchanged from Ex. 2.3. This time if we try $y_{p}(k)=C_{3} \mathrm{ka}^{\mathrm{k}}$ (as in the usual case), we see that this form is also contained as part of the homogeneous solution (because of the repeated roots). Thus, for this problem, let's try

$$
y_{p}(\mathrm{k})=\mathrm{C}_{3} \mathrm{k}^{2} \mathrm{a}^{\mathrm{k}}
$$

General Solution: Upon substitution into the original equation, one has $\mathrm{C}_{3}=\mathrm{b} / 2 \mathrm{a}^{2}$ and the general solution becomes

$$
\mathrm{y}(\mathrm{k})=\left(\mathrm{C}_{1}+\mathrm{C}_{2} \mathrm{k}\right) \mathrm{a}^{\mathrm{k}}+\frac{\mathrm{b}}{2 \mathrm{a}^{2}} \mathrm{k}^{2} \mathrm{a}^{\mathrm{k}}
$$

A unique solution for this problem can be found by applying appropriate initial conditions as above.

## Review of Matrix Algebra and Matrix Calculus

As mentioned several times thus far, our approach to studying dynamic systems will emphasize the state variable methodology. This technique uses a concise matrix notation to describe the important mathematical relationships for a particular system. Thus, the ability to perform standard matrix algebra and matrix calculus operations is important. This subsection reviews much of the notation and standard matrix operations that will be required in our description of dynamic systems. It also introduces some new material for subsequent use that deals with functions of a square matrix. This will be useful for determining analytical solutions to loworder linear stationary systems.

## Elementary Operations

Given: $\quad \underline{\underline{A}}=\left[\mathrm{a}_{\mathrm{ij}}\right] \quad \underline{\underline{B}}=\left[\mathrm{b}_{\mathrm{ij}}\right] \quad \underline{\underline{C}}=\left[\mathrm{c}_{\mathrm{ij}}\right] \quad \underline{\underline{D}}=\left[\mathrm{d}_{\mathrm{ij}}\right]$
addition:

$$
\begin{align*}
& \underline{\underline{A}}+\underline{\underline{B}}=\underline{\underline{C}} \quad \Rightarrow \quad c_{i j}=a_{i j}+b_{i j}  \tag{2.14}\\
& \alpha \underline{\underline{A}}=\underline{\underline{C}} \quad \Rightarrow \quad c_{i j}=\alpha a_{i j}  \tag{2.15}\\
& \underline{\underline{A B}}=\underline{\underline{C}} \quad \Rightarrow \quad c_{i j}=\sum_{k} a_{i k} b_{k j}  \tag{2.16}\\
& \underline{\underline{B}} \underline{\underline{A}}=\underline{\underline{D}} \quad \Rightarrow \quad d_{i j}=\sum_{k} b_{i k} a_{k j}
\end{align*}
$$

scalar multiplication:
matrix multiplication:
where, in general, $\mathrm{c}_{\mathrm{ij}} \neq \mathrm{d}_{\mathrm{ij}}$ (unless $\underline{\underline{\mathrm{A}}}$ and $\underline{\underline{B}}$ are symmetric).
matrix times a vector: $\quad \underline{y}=\underline{\underline{A}} \underline{x} \quad \Rightarrow \quad y_{i}=\sum_{j} a_{i j} x_{j}$
inner product: $\quad \underline{y}^{T} \underline{x}=\alpha \quad \Rightarrow \quad \alpha=\sum_{i} y_{i} x_{i}$
transpose: $\quad$ if $\underline{\underline{A}}=\left[a_{i j}\right] \quad$ then $\quad \underline{\underline{A}}^{T}=\left[a_{j i}\right]$
Also note that

$$
\begin{equation*}
(\underline{\underline{\mathrm{A}}} \underline{\underline{\underline{B}}})^{\mathrm{T}}=\underline{\underline{B}}^{\mathrm{T}} \underline{\underline{A}}^{\mathrm{T}} \tag{2.19}
\end{equation*}
$$

## Determinants

Given matrix $\underline{\underline{A}}=\left[\mathrm{a}_{\mathrm{ij}}\right]$, the determinant of $\underline{\underline{A}}$ is denoted as $\operatorname{det} \underline{\underline{A}}$ or $|\mathrm{A}|$

$$
2 \times 2 \text { matrix: } \quad \underline{\underline{A}}=\left[\begin{array}{ll}
a_{11} & a_{12}  \tag{2.21}\\
a_{21} & a_{22}
\end{array}\right] \quad \operatorname{det} \underline{\underline{A}}=a_{11} a_{22}-a_{21} a_{12}
$$

$\mathrm{n} \times \mathrm{n}$ matrix: use Laplace's Expansion

$$
\begin{array}{rlr}
\operatorname{det} \underset{\underline{A}}{=} & =\sum_{j} \mathrm{a}_{\mathrm{ij}} \mathrm{c}_{\mathrm{ij}} &  \tag{2.22}\\
\text { for any } \mathrm{i} \\
& =\sum_{\mathrm{i}} \mathrm{a}_{\mathrm{ij}} \mathrm{c}_{\mathrm{ij}} & \text { for any } \mathrm{j}
\end{array}
$$

where the cofactor, $\mathrm{c}_{\mathrm{ij}}$, of the ij element of the original matrix is given as

$$
\begin{equation*}
\mathrm{c}_{\mathrm{ij}}=(-1)^{\mathrm{i}+\mathrm{j}} \mathrm{M}_{\mathrm{ij}} \tag{2.23}
\end{equation*}
$$

and the minor, $\mathrm{M}_{\mathrm{ij}}$, of the $\mathrm{a}_{\mathrm{ij}}$ element in the $\underset{=}{\mathrm{A}}$ matrix is the determinant of the matrix formed by deleting the $\mathrm{i}^{\text {th }}$ row and the $\mathrm{j}^{\text {th }}$ column from the original matrix. As an example, consider a general $3 \times 3$ matrix,

$$
\underline{=}=\left[\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right]
$$

Expanding along row one, we have

$$
\begin{aligned}
& \mathbf{M}_{11}=\left|\begin{array}{ll}
a_{22} & a_{23} \\
a_{32} & a_{33}
\end{array}\right|=a_{22} a_{33}-a_{23} a_{32} \\
& \mathbf{M}_{12}=\left|\begin{array}{ll}
a_{21} & a_{23} \\
a_{31} & a_{33}
\end{array}\right|=a_{21} a_{33}-a_{23} a_{31} \\
& \mathbf{M}_{13}=\left|\begin{array}{ll}
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{array}\right|=a_{21} a_{32}-a_{22} a_{31}
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{det} \underline{\underline{A}} & =\sum_{j} a_{1 j} c_{1 j}=\sum_{j} a_{1 j}(-1)^{1+j} M_{1 j} \\
& =a_{11}\left(a_{22} a_{33}-a_{23} a_{32}\right)-a_{12}\left(a_{21} a_{33}-a_{23} a_{31}\right)+a_{13}\left(a_{21} a_{32}-a_{22} a_{31}\right)
\end{aligned}
$$

Note also that if $\underset{\underline{A}}{ }$ is in triangular form (i.e. all the elements below or above the diagonal are zero), then the determinant of $\underline{\underline{A}}$ is simply the product of the diagonal terms. If the lower elements are zero, the system is said to be upper triangular; if the upper elements are zero, the matrix is lower triangular. If the diagonal elements are also zero, the system is said to be strictly upper/lower triangular. Example 2.5 illustrates how elementary row operations can be used to put a matrix into upper triangular form. One can then find its determinant by multiplication of the diagonal elements. Note that interchanging rows of a matrix changes the sign of the determinant. Laplace's expansion is also used for this simple $3 \times 3$ example.
A second example for finding the determinant of a $3 \times 3$ matrix is also given in Example 2.6. The matrix used here is considered in several other examples to find the matrix inverse, the eigenvalues and eigenvectors, and for computing the matrix exponential (see below). These tasks are needed quite frequently when performing analytical analyses for low-order systems.

## Example 2.5 Determinant of a 3x3 Matrix

## Problem Statement:

Given $\underline{\underline{A}}=\left[\begin{array}{ccc}2 & 1 & 9 \\ -2 & 3 & -1 \\ 4 & 2 & 1\end{array}\right]$, find $\operatorname{det} \underline{\underline{A}}$.

## Problem Solution:

Performing some elementary row operations, one has
$2 \times$ row 2 added to row $3 \quad\left[\begin{array}{ccc}2 & 1 & 9 \\ -2 & 3 & -1 \\ 0 & 8 & -1\end{array}\right]$
row 1 added to row $2 \quad\left[\begin{array}{ccc}2 & 1 & 9 \\ 0 & 4 & 8 \\ 0 & 8 & -1\end{array}\right]$
$-2 \times$ row 2 added to row $3 \quad\left[\begin{array}{ccc}2 & 1 & 9 \\ 0 & 4 & 8 \\ 0 & 0 & -17\end{array}\right]$
$\therefore \quad \operatorname{det} \underline{\underline{A}}=(2)(4)(-17)=-136$
Using Laplace's expansion as a check, one has by expanding along row 1 of the original matrix,

$$
\begin{aligned}
\operatorname{det} \underline{\underline{A}} & =2\left|\begin{array}{cc}
3 & -1 \\
2 & 1
\end{array}\right|-1\left|\begin{array}{cc}
-2 & -1 \\
4 & 1
\end{array}\right|+9\left|\begin{array}{cc}
-2 & 3 \\
4 & 2
\end{array}\right| \\
& =2(3+2)-1(-2+4)+9(-4-12)=10-2-144=-136
\end{aligned}
$$

## Example 2.6 Another Example of a 3x3 Determinant

## Problem Statement:

Given $\underline{\underline{A}}=\left[\begin{array}{ccc}0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3\end{array}\right]$, find $\operatorname{det} \underline{\underline{A}}$.

## Problem Solution:

Since there is only one nonzero element in column 1, let's expand down this column, giving

$$
\operatorname{det} \underline{\underline{A}}=3\left|\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right|=3
$$

Thus, the determinant here is easily found to be 3 .

## Matrix Inverse

The inverse matrix is defined by the relation $\underline{\underline{A}}^{-1} \underline{\underline{A}}=\underline{\underline{I}}$, where $\underline{\underline{I}}$ is the identity matrix and $\underline{\underline{A}}^{-1}$ is called the inverse of $\underline{\underline{A}}$. A simple formula can be written in terms of the adjoint matrix and determinant, or

$$
\begin{equation*}
\underline{\underline{A}}^{-1}=\frac{\operatorname{adj} \underline{\underline{\underline{A}}}}{\operatorname{det} \underline{\underline{A}}}=\frac{\underline{\underline{C}}^{T}}{\operatorname{det} \underline{\underline{A}}} \tag{2.24}
\end{equation*}
$$

where the adjoint matrix is formed by replacing each element of a matrix by its cofactor and then taking the transpose [i.e. $\operatorname{adj} \underline{\underline{A}}=\underline{\underline{C}}^{\mathrm{T}}$, where $\underline{\underline{\mathrm{C}}}$ is the cofactor matrix with elements given by eqn. (2.23)]. Example 2.7 illustrates the use of eqn. (2.24) to find the inverse of a $3 \times 3$ matrix.

Note that if $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are square matrices, then

$$
\begin{equation*}
(\underline{\underline{\mathrm{AB}}})^{-1}=\underline{\underline{B}}^{-1} \underline{\underline{A}}^{-1} \tag{2.25}
\end{equation*}
$$

Proof:

$$
\begin{aligned}
& \underline{\underline{C}}=(\underline{\underline{A B}})^{-1} \\
& \underline{\underline{A B}} \underline{\underline{\underline{C}}}=(\underline{\underline{A B}})(\underline{\underline{A B}} \underline{\underline{B}})^{-1}=\underline{\underline{I}} \\
& \underline{\underline{B C}}=\underline{\underline{A^{-1}}} \\
& \underline{\underline{C}}=\underline{\underline{B}}^{-1} \underline{\underline{A}}^{-1}
\end{aligned}
$$

## Matrix Equations

With the formal definition of a matrix inverse, one can now write a general formal relation for the solution to a set of simultaneous algebraic equations. Given the matrix equation, $\underline{\underline{A}} \underline{x}=\underline{b}$, the solution vector $\underline{x}$ can be written as

$$
\begin{equation*}
\underline{x}=\underline{\underline{A}}^{-1} \underline{b} \tag{2.26}
\end{equation*}
$$

Two important classes of problems arise, depending on the value of $\underline{b}$ :

1. If $\underline{b} \neq \underline{0}$, the system is said to be non-homogeneous and there is a non-trivial solution only if $\underline{\underline{A}}^{-1}$ exists. This implies that the det $\underline{\underline{A}} \neq 0$ and that the rows and columns of $\underline{\underline{A}}$ are linearly independent.
2. If $\underline{b}=\underline{0}$, the system is said to be homogeneous and there is a non- trivial solution only if $\underline{\underline{A}}$ is singular. A singular matrix is one whose determinant is zero. This means that the rows and columns of $\underline{\underline{A}}$ are linearly dependent.

## Example 2.7 Finding Inverses via the Adjoint Matrix

## Problem Statement:

Given $\underline{\underline{A}}=\left[\begin{array}{ccc}0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3\end{array}\right]$, find $\underline{\underline{A}}^{-1}$.

## Problem Solution:

Using eqn. (2.24) to find the inverse matrix, we will need to know the determinant and the adjoint of the given matrix. From Ex. 2.6, $\operatorname{det} \underline{\underline{A}}=3$ and the cofactor of the original matrix is given by (recall that the adjoint matrix is the transpose of the cofactor matrix),

$$
\underline{\underline{C}}=\left[\begin{array}{ccc}
-1 & 3 & 0 \\
3 & 0 & 3 \\
1 & 0 & 0
\end{array}\right]
$$

Therefore,

$$
\underline{\underline{A}}^{-1}=\frac{\underline{\underline{C}}^{\mathrm{T}}}{\operatorname{det} \underline{\underline{A}}}=\frac{1}{3}\left[\begin{array}{ccc}
-1 & 3 & 1 \\
3 & 0 & 0 \\
0 & 3 & 0
\end{array}\right]
$$

and a quick check on $\underline{\underline{A}}^{-1} \underline{\underline{A}}=\underline{\underline{I}}$ gives

$$
\frac{1}{3}\left[\begin{array}{ccc}
-1 & 3 & 1 \\
3 & 0 & 0 \\
0 & 3 & 0
\end{array}\right]\left[\begin{array}{ccc}
0 & 1 & 0 \\
0 & 0 & 1 \\
3 & 1 & -3
\end{array}\right]=\frac{1}{3}\left[\begin{array}{lll}
3 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 3
\end{array}\right]=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]=\underline{\underline{I}}
$$

which shows that the above manipulations have been done correctly!

## Eigenvalues and Eigenvectors

From the above brief review concerning the existence and uniqueness of solutions to algebraic equations, one can define the so-called eigenvalue problem. If we let $\underline{b}=\lambda \underline{x}$, then one has

$$
\begin{equation*}
\underline{\underline{A x}} \underline{\underline{x}}=\lambda \underline{x} \tag{2.27}
\end{equation*}
$$

$\lambda$ is an eigenvalue of an $n \times n$ matrix $\underline{\underline{A}}$ if there is a non-zero vector $\underline{x}$ such that eqn. (2.27) is valid. In this case, $\underline{x}$ is said to be an eigenvector of the matrix $\underline{\underline{A}}$ corresponding to eigenvalue $\lambda$. Equation (2.27) is often rewritten as

$$
\begin{equation*}
(\underline{\underline{A}}-\lambda \underline{\underline{I}}) \underline{x}=\underline{0} \tag{2.28}
\end{equation*}
$$

In this form, it becomes apparent that for a non-trivial solution, we must require that

$$
\begin{equation*}
\operatorname{det}(\underline{\underline{A}}-\lambda \underline{\underline{I}})=0 \tag{2.29}
\end{equation*}
$$

This expression is called the characteristic equation. The solution (or roots) of the characteristic polynomial gives $n$ values of $\lambda$ (not necessarily distinct). Once the $n$ eigenvalues are known, the $\mathrm{i}^{\text {th }}$ eigenvector can be determined from eqn. (2.28) by substituting $\lambda=\lambda_{\mathrm{i}}$. There will be one eigenvector corresponding to each eigenvalue. A specific illustration is given in Example 2.8.

We will find that the eigenvalues and eigenvectors of a matrix (in our case, the system matrix) are extremely important for describing the dynamic behavior of time-dependent systems. If one knows all the eigenvalues and eigenvectors of a linear, stationary, lumped parameter system, then the system's complete time domain and frequency domain behavior can be simulated analytically, the stability characteristics of the system can be determined, the sensitivity of the system to changes in input parameters can be approximated, and so on. In fact, one can describe almost every aspect of this class of dynamic systems with the eigenvalues and eigenvectors of the system matrix (the system matrix will be defined precisely in the next section).
On the subject of eigenvalues and eigenvectors, one should note that similar matrices have identical eigenvalues. Two matrices, $\underset{\underline{A}}{ }$ and $\underline{\underline{B}}$, are said to be similar if they satisfy the similarity transformation

$$
\begin{equation*}
\underline{\underline{\mathrm{B}}}=\underline{\underline{\mathrm{C}}}^{-1} \underline{\underline{\mathrm{AC}}} \tag{2.30}
\end{equation*}
$$

A particularly useful similarity transformation involves the diagonalization of a matrix. In fact, one can show that for any square matrix with distinct eigenvalues, we have

$$
\begin{equation*}
\underline{\underline{\mathrm{D}}}=\underline{\underline{\mathrm{M}}}^{-1} \underline{\underline{\mathrm{~A}}} \underline{\underline{\mathrm{M}}} \tag{2.31a}
\end{equation*}
$$

or $\quad \underline{\underline{\mathrm{A}}}=\underline{\underline{M}} \underline{\underline{\underline{D}}} \underline{\underline{M}}^{-1}$
where $\underline{\underline{\mathrm{M}}}$ is defined as the modal matrix whose n columns are the n linearly independent eigenvectors of $\underline{\underline{A}}$ and the matrix $\underline{\underline{D}}$ is simply a diagonal matrix containing the $n$ distinct eigenvalues of $\underline{\underline{A}}$ as the diagonal terms. These matrices are usually written as

$$
\underline{\underline{D}}=\left[\begin{array}{llllll}
\lambda_{1} & & & &  \tag{2.32}\\
& \lambda_{2} & & & 0 & \\
& & \cdot & & \\
& 0 & & \cdot & & \\
& & & & \lambda_{n}
\end{array}\right] \text { and } \underline{\underline{M}}=\left[\begin{array}{llll}
\underline{e}_{1} & \underline{\mathrm{e}}_{2} & \ldots & \underline{\mathrm{e}}_{\mathrm{n}}
\end{array}\right]
$$

where $\underline{e}_{i}$ is the $i^{\text {th }}$ eigenvector which corresponds to eigenvalue $\lambda_{i}$. A quick proof of eqn. (2.31a) can be given as follows:

$$
\underline{\underline{\mathrm{A}}} \underline{\underline{\mathrm{M}}}=\underline{\underline{\mathrm{A}}}\left[\begin{array}{llll}
\underline{\mathrm{e}}_{1} & \underline{\mathrm{e}}_{2} & \cdots & \underline{\mathrm{e}}_{\mathrm{n}}
\end{array}\right]=\left[\begin{array}{llll}
\lambda_{1} & \underline{\mathrm{e}}_{1} & \lambda_{2} & \underline{\mathrm{e}}_{2}
\end{array} \cdots \lambda_{\mathrm{n}} \underline{\mathrm{e}}_{\mathrm{n}}\right]=\underline{\underline{\mathrm{MD}}}
$$

and pre-multiplying by $\underline{\underline{M}}^{-1}$ gives eqn. (2.31a).

## Example 2.8 Finding the Eigenvalues and Eigenvectors of a 3x3 Matrix

Problem Statement:
Given $\underset{\underline{A}}{=}=\left[\begin{array}{ccc}0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3\end{array}\right]$, find the corresponding eigenvalues and eigenvectors.

## Problem Solution:

The characteristic equation is given by

$$
|\underline{\underline{A}}-\lambda I|=\left|\begin{array}{ccc}
-\lambda & 1 & 0 \\
0 & -\lambda & 1 \\
3 & 1 & -3-\lambda
\end{array}\right|=0
$$

Expanding the determinant along row 1 using Laplace's expansion gives

$$
|\underline{\underline{A}}-\lambda \underline{\underline{I}}|=-\lambda\left|\begin{array}{cc}
-\lambda & 1 \\
1 & -3-\lambda
\end{array}\right|-1\left|\begin{array}{cc}
0 & 1 \\
3 & -3-\lambda
\end{array}\right|=-\lambda[(-\lambda)(-3-\lambda)-1]-(-3)=0
$$

and, upon simplification and multiplication by -1 , we have

$$
\lambda^{3}+3 \lambda^{2}-\lambda-3=0
$$

By inspection, $\lambda_{1}=1$ is a root to this $3^{\text {rd }}$ order polynomial and $\lambda-1$ is a linear factor. Upon division of the cubic polynomial with this linear factor, we are left with

$$
\lambda^{2}+4 \lambda+3=(\lambda+3)(\lambda+1)=0
$$

Thus, the three eigenvalues for the given $3^{\text {rd }}$ order system are

$$
\lambda_{1}=1, \quad \lambda_{2}=-1, \quad \text { and } \quad \lambda_{2}=-3
$$

Now, the eigenvector associated with the $\mathrm{i}^{\text {th }}$ eigenvalue can be determined by solving the matrix equations with the specific eigenvalue inserted into the equation. For example, for $\lambda_{1}=1$, we have

$$
\left[\begin{array}{ccc}
-1 & 1 & 0 \\
0 & -1 & 1 \\
3 & 1 & -4
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]_{1}=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]
$$

which gives three equations

$$
-x_{1}+x_{2}=0, \quad-x_{2}+x_{3}=0, \quad \text { and } \quad 3 x_{1}+x_{2}-4 x_{3}=0
$$

The first two equations say that $\mathrm{x}_{1}=\mathrm{x}_{2}=\mathrm{x}_{3}$ and the third equation is automatically satisfied when this is true (i.e. it is simply a linear combination of the other equations). Since there is an arbitrary normalization associated with the eigenvectors (because of the homogeneous nature of the equation), we select the elements to be unity and the desired eigenvector is $\underline{x}_{1}^{\mathrm{T}}=\left[\begin{array}{lll}1 & 1 & 1\end{array}\right]$.

For $\lambda_{2}=-1$, we have

$$
\left[\begin{array}{ccc}
1 & 1 & 0 \\
0 & 1 & 1 \\
3 & 1 & -2
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]_{2}=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]
$$

which gives

$$
\mathrm{x}_{1}+\mathrm{x}_{2}=0, \quad \mathrm{x}_{2}+\mathrm{x}_{3}=0, \quad \text { and } \quad 3 \mathrm{x}_{1}+\mathrm{x}_{2}-2 \mathrm{x}_{3}=0
$$

and this says that $x_{1}=x_{3}=-x_{2}$. Again, arbitrarily selecting $x_{1}=1$ gives $\underline{x}_{2}^{T}=\left[\begin{array}{lll}1 & -1 & 1\end{array}\right]$. Following the same procedure, one can easily show that the eigenvector corresponding to $\lambda_{3}=-3$ is $\underline{x}_{3}^{\mathrm{T}}=\left[\begin{array}{lll}1 & -3 & 9\end{array}\right]$, where element 1 has been arbitrarily normalized to unity.

The complete set of linearly independent eigenvectors for this problem can be grouped into a matrix referred to as the modal matrix, $\underline{\underline{\mathrm{M}}}$. Thus, for this problem the modal matrix is

$$
\underline{\underline{\mathrm{M}}}=\left[\begin{array}{lll}
\underline{\mathrm{x}}_{1} & \underline{\mathrm{x}}_{2} & \underline{\mathrm{x}}_{3}
\end{array}\right]=\left[\begin{array}{ccc}
1 & 1 & 1 \\
1 & -1 & -3 \\
1 & 1 & 9
\end{array}\right]
$$

which completes this problem.

## Matrix Calculus

Given: $\quad \underline{\underline{A}}(\mathrm{t})=\left[\mathrm{a}_{\mathrm{ij}}(\mathrm{t})\right]$
Differentiation with respect to the time variable:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{dt}} \underset{\underline{A}}{ }(\mathrm{t})=\left[\frac{\mathrm{d}}{\mathrm{dt}} \mathrm{a}_{\mathrm{ij}}(\mathrm{t})\right] \tag{2.33}
\end{equation*}
$$

Integration with respect to time:

$$
\begin{equation*}
\int \underline{\underline{\mathrm{A}}}(\mathrm{t}) \mathrm{dt}=\left[\int \mathrm{a}_{\mathrm{ij}}(\mathrm{t}) \mathrm{dt}\right] \tag{2.34}
\end{equation*}
$$

Product Rules: For the matrices $\underline{\underline{U}}(\mathrm{t})$ and $\underline{\underline{V}}(\mathrm{t})$, the product rules are analogous to their scalar counterparts, with particular attention to the order of operations:

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{dt}}(\underline{\underline{\mathrm{UV}}} \underline{\underline{V}})=\underline{\underline{\mathrm{U}}} \frac{\mathrm{~d}}{\mathrm{dt}} \underline{\underline{V}}+\left(\frac{\mathrm{d}}{\mathrm{dt}} \underline{\underline{U}}\right) \underline{\underline{V}}  \tag{2.35}\\
& \int \underline{\underline{U}} \frac{\mathrm{~d}}{\mathrm{dt}} \underline{\underline{V d t}}=\left.\underline{\underline{U}} \underline{\underline{V}}\right|_{\text {limits }}-\int\left(\frac{\mathrm{d}}{\mathrm{dt}} \underline{\underline{U}}\right) \underline{\underline{V} d t} \tag{2.36}
\end{align*}
$$

Differentiation of the Inverse Matrix:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{dt}} \underline{\underline{A}}^{-1}(\mathrm{t})=-\underline{\underline{A}}^{-1}(\mathrm{t})\left(\frac{\mathrm{d}}{\mathrm{dt}} \underline{\underline{A}}(\mathrm{t})\right) \underline{\underline{A}}^{-1}(\mathrm{t}) \tag{2.37}
\end{equation*}
$$

Proof: Starting with $\underline{\underline{I}}=\underline{\underline{A}}^{-1} \underline{\underline{A}}$, one has

$$
\frac{\mathrm{d}}{\mathrm{dt}}=\frac{\mathrm{d}}{\mathrm{dt}}\left({\underline{\underline{A^{-1}}}}_{\underline{\mathrm{A}}}^{\underline{=}}\right)=\underline{\underline{A}}^{-1} \frac{\mathrm{~d}}{\mathrm{dt}} \underline{\underline{A}}+\left(\frac{\mathrm{d}}{\mathrm{dt}} \underline{\underline{A}}^{-1}\right) \underline{\underline{A}}=\underline{\underline{0}}
$$

and solving this expression for $\frac{d}{d t} \stackrel{A}{=}^{-1}$ gives the desired result.
Differentiation of a Determinant (time independent): Recalling Laplace's Expansion

$$
\begin{array}{rlr}
\operatorname{det} \underset{\underline{A}}{=} & =\sum_{\mathrm{j}} \mathrm{a}_{\mathrm{ij}} \mathrm{c}_{\mathrm{ij}} & \text { for any } \mathrm{i} \\
& =\sum_{\mathrm{i}} \mathrm{a}_{\mathrm{ij}} \mathrm{c}_{\mathrm{ij}} & \text { for any } \mathrm{j}
\end{array}
$$

where $\mathrm{c}_{\mathrm{ij}}$ is the cofactor of element ij , one has

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{da}_{\mathrm{ij}}}(\operatorname{det} \underline{\underline{A}})=\mathrm{c}_{\mathrm{ij}} \tag{2.38}
\end{equation*}
$$

## Matrix Exponential

Shortly we will discover that a certain class of dynamic systems has analytic solutions that can be written in terms of the so-called matrix exponential. We formally define the matrix exponential, $\mathrm{e}^{\text {At }}$, in terms of an infinite Taylor series expansion,

$$
\begin{equation*}
e^{\underline{\underline{A}} t}=\sum_{k=0}^{\infty} \frac{\underline{\underline{A^{k}}} t^{k}}{k!}=\underline{\underline{I}}+\underline{\underline{A A}}+\frac{{\underline{\underline{A^{2}}}}^{2} t^{2}}{2!}+\frac{\underline{\underline{A^{3}} t^{3}}}{3!}+\ldots \tag{2.39}
\end{equation*}
$$

This is an analogous definition to its scalar counterpart,

$$
\mathrm{e}^{\mathrm{at}}=\sum_{\mathrm{k}=0}^{\infty} \frac{\mathrm{a}^{\mathrm{k}} \mathrm{t}^{\mathrm{k}}}{\mathrm{k}!}=1+\mathrm{at}+\frac{\mathrm{a}^{2} \mathrm{t}^{2}}{2!}+\frac{\mathrm{a}^{3} \mathrm{t}^{3}}{3!}+\quad \cdots
$$

We will look at several properties of this function in subsequent sections. For now, we are primarily interested in integrals and derivatives of $e^{{ }^{\underline{A} t}}$, where $\underline{\underline{A}}$ is a constant matrix. We shall see that these relationships are very similar to those for the well-known scalar exponential function. Specifically, for the matrix exponential, the time derivative and integral relationships are:

Derivative: $\quad \frac{d}{d t} e^{\underline{\underline{A} t}}=\underline{\underline{A}} e^{\underline{\underline{A} t}}=e^{\underline{\underline{A}} t} \stackrel{\underline{\underline{A}}}{ }$

Proof: Take the derivative of each term in eqn. (2.39), giving

$$
\begin{equation*}
\frac{d}{d t} e^{\underline{\underline{A}}}=\underline{\underline{A}}+\underline{\underline{A^{2}}} t+\frac{\underline{\underline{A^{3}}} t^{2}}{2!}+\frac{\underline{\underline{A^{4}}} t^{3}}{3!}+\cdots=\underline{\underline{A}}\left(\underline{\underline{I}}+\underline{\underline{A t}}+\frac{\underline{\underline{A^{2}} t^{2}}}{2!}+\cdots\right)=\underline{\underline{A}} e^{\underline{A t}} \tag{2.41}
\end{equation*}
$$

Integral: $\quad \int e^{\underline{\underline{\underline{A}} t}} d t=\underline{\underline{A}}^{-1} e^{\underline{\underline{A} t}}=e^{\underline{\underline{\underline{A}} t}} \underline{\underline{A}}^{-1}$
Proof: Let $\underset{\underline{F}}{=} \underline{\underline{A}}^{-1} \mathrm{e}^{\underline{\underline{A t}}}$, then

## Functions of Square Matrix

In addition to the infinite series expansion for $e^{\underline{\underline{A t}}}$ given in eqn. (2.39), one can find a closed form expansion for the matrix exponential. In fact, a general closed-form result can be obtained for any polynomial function of a square matrix. Sylvester's theorem is used for this purpose.

## Sylvester's Theorem (Distinct Roots)

If $\mathrm{P}(\underline{\underline{A}})$ is any polynomial of the square matrix $\underline{\underline{A}}$, and if $\lambda_{\mathrm{i}}$ represents one of the n distinct eigenvalues of $\underline{\underline{A}}$, then

$$
\begin{equation*}
P(\underline{\underline{A}})=(-1)^{\mathrm{n}+1} \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{\mathrm{P}\left(\lambda_{\mathrm{i}}\right) \operatorname{Adj}\left(\underline{\underline{\mathrm{A}}}-\lambda_{\mathrm{i}} \underline{I}\right)}{\prod_{\mathrm{j} \neq \mathrm{i}}^{\mathrm{I}}\left(\lambda_{\mathrm{i}}-\lambda_{\mathrm{j}}\right)} \tag{2.42}
\end{equation*}
$$

One can also demonstrate that

$$
\begin{equation*}
(-1)^{\mathrm{n}+1} \operatorname{Adj}\left(\underline{\underline{\mathrm{~A}}}-\lambda_{\mathrm{i}} \underline{\underline{I}}\right)=\prod_{\mathrm{j} \neq \mathrm{i}}\left(\underline{\underline{\mathrm{~A}}}-\lambda_{\mathrm{j}} \underline{\underline{I}}\right) \tag{2.43}
\end{equation*}
$$

Therefore, an equivalent expression to eqn. (2.42) is given by,

$$
\begin{equation*}
P(\underline{\underline{A}})=\sum_{i=1}^{n} P\left(\lambda_{i}\right) \prod_{\mathrm{j} \neq \mathrm{i}}^{\mathrm{n}} \frac{\left(\underline{\underline{\mathrm{~A}}-\lambda_{\mathrm{j}} \mathrm{I}}\right)}{\left(\lambda_{\mathrm{i}}-\lambda_{\mathrm{j}}\right)} \tag{2.44}
\end{equation*}
$$

and this latter relationship is often easier to apply since it does not require construction of the adjoint matrix to $\left(\underline{\underline{A}}-\lambda_{\mathrm{i}} \mathrm{I}\right)$.

An important application of Sylvester's theorem is in finding a closed-form solution for the matrix exponential. Example 2.9 illustrates this process for the case of a $2^{\text {nd }}$ order system with distinct roots. A $3 \times 3$ matrix example, which is a bit more work, is also given in Example 2.10.

## Sylvester's Theorem (Repeated Roots)

If the eigenvalues of the system matrix are not all distinct, then an alternate form of eqn. (2.42) is required. This is sometimes referred to as the confluent form of Sylvester's theorem. The theorem can be stated mathematically as follows:

If $\mathrm{P}(\underline{\underline{A}})$ is any polynomial of the square matrix $\underline{\underline{A}}$ and if the $\mathrm{i}^{\text {th }}$ eigenvalue, $\lambda_{\mathrm{i}}$, is repeated $\mathrm{s}_{\mathrm{i}}$ times, then

$$
\begin{equation*}
\mathrm{P}(\underline{\underline{\mathrm{~A}}})=\sum_{\substack{\text { ell distinct } \\ \text { (witenvalues, } \lambda_{i} \\ \text { with multiplicity } \mathrm{s}_{\mathrm{i}} \text { ) }}} \mathrm{T}\left(\lambda_{\mathrm{i}}\right) \tag{2.45}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{T}\left(\lambda_{\mathrm{i}}\right)=\left.\frac{(-1)^{\mathrm{n+1}}}{\left(\mathrm{~s}_{\mathrm{i}}-1\right)!}\left[\frac{\mathrm{d}^{\mathrm{s}_{\mathrm{i}}-1}}{\mathrm{~d} \lambda^{\mathrm{s}_{\mathrm{i}}-1}}\left(\frac{\mathrm{P}(\lambda) \operatorname{Adj}(\underline{\underline{\mathrm{A}}-\lambda \underline{\underline{I}})}}{\Delta_{\mathrm{i}}(\lambda)}\right)\right]\right|_{\lambda=\lambda_{\mathrm{i}}} \tag{2.46}
\end{equation*}
$$

with $\quad \Delta_{i}(\lambda)=\prod_{j \neq i}\left(\lambda-\lambda_{j}\right)^{s_{j}}$
and, if all the eigenvalues are equal, then $\Delta_{i=1}(\lambda)=1$.
This form is a little more cumbersome. However, one sees that, in the limit of no repeated roots (i.e. $\mathrm{s}_{\mathrm{i}}=1$ for all i ), the confluent form of Sylvester's theorem reduces to the standard form given in eqn. (2.42). Examples 2.11 and 2.12 illustrate the use of eqns. (2.45) - (2.47) for the case of a $2^{\text {nd }}$ order system with repeated roots and for a $3^{\text {rd }}$ order system with only two distinct roots.

## Example 2.9 Analytical Solution for 2x2 Matrix Exponential (Distinct Roots)

## Problem Statement:

Determine $e^{\underline{\underline{A t} t}}$ for $\underline{\underline{A}}=\left[\begin{array}{cc}-3 & 2 \\ 1 & -2\end{array}\right]$.

## Problem Solution:

First we find the eigenvalues

$$
|\underline{\underline{A}}-\lambda \underline{\underline{I}}|=\left|\begin{array}{cc}
-3-\lambda & 2 \\
1 & -2-\lambda
\end{array}\right|=(3+\lambda)(2+\lambda)-2=\lambda^{2}+5 \lambda+4=0
$$

or $\quad \lambda_{1}=-4$ and $\quad \lambda_{2}=-1 \quad$ (distinct roots)
Using the standard form of Sylvester's theorem, we have from eqn. (2.44)

$$
\begin{aligned}
& e^{\underline{\underline{A} t}}=\frac{e^{-4 t}}{-4-(-1)}\left[\begin{array}{cc}
-3-(-1) & 2 \\
1 & -2-(-1)
\end{array}\right]+\frac{e^{-t}}{-1-(-4)}\left[\begin{array}{cc}
-3-(-4) & 2 \\
1 & -2-(-4)
\end{array}\right] \\
& e^{e^{\frac{A}{} t}}=-\frac{e^{-4 t}}{3}\left[\begin{array}{cc}
-2 & 2 \\
1 & -1
\end{array}\right]+\frac{e^{-t}}{3}\left[\begin{array}{ll}
1 & 2 \\
1 & 2
\end{array}\right]=\frac{1}{3}\left[\begin{array}{cc}
e^{-t}+2 e^{-4 t} & 2 e^{-t}-2 e^{-4 t} \\
e^{-t}-e^{-4 t} & 2 e^{-t}+e^{-4 t}
\end{array}\right]
\end{aligned}
$$

## Example 2.10 A 3x3 Example for $e^{{ }^{\text {At }}}$ using Sylvester's Theorem

## Problem Statement:

Determine $\mathrm{e}^{\underline{A^{t}}}$ for $\underset{\underline{A}}{ }=\left[\begin{array}{ccc}0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & -3\end{array}\right]$.

## Problem Solution:

We have already found the eigenvalues for this matrix in Ex. 2.8,

$$
\lambda_{1}=1, \quad \lambda_{2}=-1, \quad \text { and } \quad \lambda_{2}=-3
$$

Thus, using eqn. (2.44), we have
and putting in numerical values gives

$$
\mathrm{e}^{\underline{\underline{A} t}}=\mathrm{e}^{\mathrm{t}} \frac{\left[\begin{array}{ccc}
1 & 1 & 0 \\
0 & 1 & 1 \\
3 & 1 & -2
\end{array}\right]}{2} \frac{\left[\begin{array}{lll}
3 & 1 & 0 \\
0 & 3 & 1 \\
3 & 1 & 0
\end{array}\right]}{4}+\mathrm{e}^{-\mathrm{t}} \frac{\left[\begin{array}{ccc}
-1 & 1 & 0 \\
0 & -1 & 1 \\
3 & 1 & -4
\end{array}\right]}{-2} \frac{\left[\begin{array}{lll}
3 & 1 & 0 \\
0 & 3 & 1 \\
3 & 1 & 0
\end{array}\right]}{2}+\mathrm{e}^{-3 \mathrm{t}} \frac{\left[\begin{array}{ccc}
-1 & 1 & 0 \\
0 & -1 & 1 \\
3 & 1 & -4
\end{array}\right]}{-4} \frac{\left[\begin{array}{ccc}
1 & 1 & 0 \\
0 & 1 & 1 \\
3 & 1 & -2
\end{array}\right]}{-2}
$$

or

$$
\mathrm{e}^{\underline{A t}}=\frac{\mathrm{e}^{\mathrm{t}}}{8}\left[\begin{array}{lll}
3 & 4 & 1 \\
3 & 4 & 1 \\
3 & 4 & 1
\end{array}\right]-\frac{2 \mathrm{e}^{-\mathrm{t}}}{8}\left[\begin{array}{ccc}
-3 & 2 & 1 \\
3 & -2 & -1 \\
-3 & 2 & 1
\end{array}\right]+\frac{\mathrm{e}^{-3 t}}{8}\left[\begin{array}{ccc}
-1 & 0 & 1 \\
3 & 0 & -3 \\
-9 & 0 & 9
\end{array}\right]
$$

Finally, putting the three terms together gives the desired result

$$
e^{\underline{A} t}=\frac{1}{8}\left[\begin{array}{ccc}
3 e^{t}+6 e^{-t}-e^{-3 t} & 4 e^{t}-4 e^{-t} & e^{t}-2 e^{-t}+e^{-3 t} \\
3 e^{t}-6 e^{-t}+3 e^{-3 t} & 4 e^{t}+4 e^{-t} & e^{t}+2 e^{-t}-3 e^{-3 t} \\
3 e^{t}+6 e^{-t}-9 e^{-3 t} & 4 e^{t}-4 e^{-t} & e^{t}-2 e^{-t}+9 e^{-3 t}
\end{array}\right]
$$

Note that $\left.e^{\underline{\underline{A} t}}\right|_{t=0}=\underline{\underline{I}}$, and this is always a quick and easy check to make. If this check is not valid in a particular case, then you know the matrix exponential result is wrong. However, if the check works, then you are not guaranteed that you have the correct answer. In the present case, a quick comparison to Matlab's symbolic solution for the given matrix validated the above solution.

## Example 2.11 Analytical Solution for 2x2 Matrix Exponential (Repeated Roots)

Problem Statement:
Determine $e^{\underline{\underline{\underline{A}}}}$ for $\underline{\underline{A}}=\left[\begin{array}{cc}6 & 1 \\ -4 & 2\end{array}\right]$.

## Problem Solution:

First compute the eigenvalues

$$
\left|\begin{array}{cc}
6-\lambda & 1 \\
-4 & 2-\lambda
\end{array}\right|=(6-\lambda)(2-\lambda)+4=\lambda^{2}-8 \lambda+16=0
$$

or $\lambda_{1}=\lambda_{2}=4$ (repeated roots). Since the roots are identical, the confluent form of Sylvester's theorem must be used. Proceeding in an orderly fashion, let's first compute the term $\operatorname{Adj}(\underline{\underline{\text { A }}}-\lambda \underline{\underline{I}})$. Recalling, the definition of an adjoint matrix, one has

$$
\operatorname{Adj}(\underline{\underline{\mathrm{A}}}-\lambda \underline{\underline{I}})=\left[\mathrm{c}_{\mathrm{ij}}\right]^{\mathrm{T}}=\left[\begin{array}{cc}
2-\lambda & 4 \\
-1 & 6-\lambda
\end{array}\right]^{\mathrm{T}}=\left[\begin{array}{cc}
2-\lambda & -1 \\
4 & 6-\lambda
\end{array}\right]
$$

Now, since $\Delta_{i=1}(\lambda)=1$ (for the case where all the roots are equal), we have

$$
\mathrm{P}(\lambda) \operatorname{Adj}(\underline{\underline{\mathrm{A}}}-\lambda \underline{\underline{I}})=\mathrm{e}^{\lambda t}\left[\begin{array}{cc}
2-\lambda & -1 \\
4 & 6-\lambda
\end{array}\right]
$$

Taking the first derivative of this expression with respect to $\lambda$, gives

$$
\frac{\mathrm{d}}{\mathrm{~d} \lambda}\left\{\mathrm{e}^{\lambda t}\left[\begin{array}{cc}
2-\lambda & -1 \\
4 & 6-\lambda
\end{array}\right]\right\}=\mathrm{e}^{\lambda t}\left[\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right]+\mathrm{te}^{\lambda t}\left[\begin{array}{cc}
2-\lambda & -1 \\
4 & 6-\lambda
\end{array}\right]
$$

Evaluating this expansion at $\lambda=\lambda_{1}=4$ and recognizing that the sum in eqn. (2.45) only has a single term, gives

$$
\mathrm{e}^{\text {At }}=\frac{(-1)^{3}}{1!}\left\{\mathrm{e}^{4 \mathrm{t}}\left[\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right]+\mathrm{te}^{4 \mathrm{t}}\left[\begin{array}{cc}
-2 & -1 \\
4 & 2
\end{array}\right]\right\}=\mathrm{e}^{4 \mathrm{t}}\left[\begin{array}{cc}
(1+2 \mathrm{t}) & \mathrm{t} \\
-4 \mathrm{t} & (1-2 \mathrm{t})
\end{array}\right]
$$

This is the desired result for this problem.

## Example 2.12 Analytical Solution for 3x3 Matrix Exponential (Repeated Roots)

Problem Statement:
Determine $e^{\underline{\underline{A} t}}$ for $\underline{\underline{A}}=\left[\begin{array}{ccc}0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2\end{array}\right]$.

## Problem Solution:

First we need to compute the eigenvalues of the given matrix,

$$
|\underline{\underline{A}}-\lambda \underline{\underline{I}}|=\left|\begin{array}{ccc}
-\lambda & 1 & 0 \\
0 & -\lambda & 1 \\
0 & -1 & -2-\lambda
\end{array}\right|=0
$$

Expanding the determinant down column 1 using Laplace's expansion gives

$$
|\underline{\underline{A}}-\lambda \underline{\underline{I}}|=-\lambda\left|\begin{array}{cc}
-\lambda & 1 \\
-1 & -2-\lambda
\end{array}\right|=-\lambda[(-\lambda)(-2-\lambda)+1]=-\lambda\left[\lambda^{2}+2 \lambda+1\right]=-\lambda(\lambda+1)^{2}=0
$$

Thus, we have $\lambda_{1}=0$ and $\lambda_{2,3}=-1$.
For repeated roots, we need to use the confluent form of Sylvester's theorem. For the current case we have two distinct eigenvalues, $\lambda_{1}=0$ and $\lambda_{2}=-1$, with multiplicities $\mathrm{s}_{1}=1$ and $\mathrm{s}_{2}=2$, respectively.

Now, from eqn. (2.45), we have

$$
\mathrm{e}^{\underline{\mathrm{A}^{t}}}=\mathrm{T}\left(\lambda_{1}\right)+\mathrm{T}\left(\lambda_{2}\right)
$$

where $\mathrm{T}\left(\lambda_{1}\right)=\frac{(-1)^{4}}{(1-1)!} \mathrm{e}^{\lambda_{1} \mathrm{t}} \frac{\operatorname{Adj}\left(\underline{\left.\underline{\mathrm{A}}-\lambda_{1} \underline{\underline{I}}\right)}\right)}{\left(\lambda_{1}-\lambda_{2}\right)^{2}}$

$$
\mathrm{T}\left(\lambda_{2}\right)=\frac{(-1)^{4}}{(2-1)!}\left[\left.\frac{\mathrm{d}}{\mathrm{~d} \lambda}\left\{\mathrm{e}^{\lambda \mathrm{t}} \frac{\operatorname{Adj}(\underline{\underline{\underline{\mathrm{~A}}}-\lambda \underline{\underline{I}})}}{\left(\lambda-\lambda_{1}\right)}\right\}\right|_{\lambda=\lambda_{2}}\right.
$$

Note that both these expressions require the calculation of $\operatorname{Adj}(\underline{\underline{A}}-\lambda \underline{\underline{I}})$. Doing this, noting that $\operatorname{Adj}(\underline{\underline{A}}-\lambda \underline{\underline{I}})$ is the transpose of the cofactor matrix, gives

$$
\operatorname{Adj}(\underline{\underline{\mathrm{A}}}-\lambda \underline{\underline{I}})=\left[\begin{array}{ccc}
\lambda^{2}+2 \lambda+1 & 0 & 0 \\
\lambda+2 & \lambda^{2}+2 \lambda & -\lambda \\
1 & \lambda & \lambda^{2}
\end{array}\right]^{\mathrm{T}}=\left[\begin{array}{ccc}
(\lambda+1)^{2} & \lambda+2 & 1 \\
0 & \lambda(\lambda+2) & \lambda \\
0 & -\lambda & \lambda^{2}
\end{array}\right]
$$

With this result, we can immediately evaluate $T\left(\lambda_{1}\right)$ as (with $\lambda_{1}=0$ )

$$
\mathrm{T}\left(\lambda_{1}\right)=\left[\begin{array}{lll}
1 & 2 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

However, for $\mathrm{T}\left(\lambda_{2}\right)$, we still have a bit of work to do. First let's form part of the expression,

$$
\mathrm{e}^{\lambda \mathrm{t}} \frac{\operatorname{Adj}(\underline{\underline{\mathrm{~A}}-\lambda \underline{\underline{I}}})}{\left(\lambda-\lambda_{1}\right)}=\mathrm{e}^{\lambda \mathrm{t}}\left[\begin{array}{ccc}
(\lambda+1)^{2} / \lambda & (\lambda+2) / \lambda & 1 / \lambda \\
0 & \lambda+2 & 1 \\
0 & -1 & \lambda
\end{array}\right]=\mathrm{e}^{\lambda \mathrm{t}} \underline{\underline{\mathrm{~B}}}
$$

where the $\underline{\underline{B}}$ matrix is simply defined by the above correspondence for convenience in our subsequent manipulations. In particular, we can write the derivative of the above expression as

$$
\frac{\mathrm{d}}{\mathrm{~d} \lambda}\left\{\mathrm{e}^{\lambda \mathrm{t}} \underline{\underline{B}}\right\}=\mathrm{e}^{\lambda \mathrm{t}} \frac{\mathrm{~d}}{\mathrm{~d} \lambda} \underline{=}+\mathrm{te}^{\lambda \mathrm{t}} \underline{\underline{B}}=\mathrm{e}^{\lambda \mathrm{t}}\left(\frac{\mathrm{~d}}{\mathrm{~d} \mathrm{\lambda}} \underline{\underline{B}}+\mathrm{tB} \underline{\underline{B}}\right)
$$

Now, computing $\mathrm{dB} / \mathrm{D} / \mathrm{d} \lambda$ gives

$$
\frac{\mathrm{d}}{\mathrm{~d} \lambda} \underline{=}=\left[\begin{array}{ccc}
-(\lambda+1)^{2} / \lambda^{2}+2(\lambda+1) / \lambda & -(\lambda+2) / \lambda^{2}+1 / \lambda & -1 / \lambda^{2} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

and evaluating this expression and the $\underline{\underline{B}}$ matrix at $\lambda=\lambda_{2}=-1$ gives

$$
\left.\frac{\mathrm{d}}{\mathrm{~d} \lambda} \mathrm{~B}\right|_{\lambda=\lambda_{2}=-1}=\left[\begin{array}{ccc}
0 & -2 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right] \quad \text { and }\left.\quad \underline{B}\right|_{\lambda=\lambda_{2}=-1}=\left[\begin{array}{ccc}
0 & -1 & -1 \\
0 & 1 & 1 \\
0 & -1 & -1
\end{array}\right]
$$

Thus, we can now write $\mathrm{T}\left(\lambda_{2}\right)$ as

$$
\mathrm{T}\left(\lambda_{2}\right)=\left.\mathrm{e}^{\lambda \mathrm{t}}\left(\frac{\mathrm{~d}}{\mathrm{~d} \lambda} \underline{=}+\mathrm{t} \underline{=}\right)\right|_{\lambda=\lambda_{2}=-1}=\mathrm{e}^{-\mathrm{t}}\left[\begin{array}{ccc}
0 & -(2+\mathrm{t}) & -(1+\mathrm{t}) \\
0 & (1+\mathrm{t}) & \mathrm{t} \\
0 & -\mathrm{t} & (1-\mathrm{t})
\end{array}\right]
$$

Finally, combining this with the expression for $\mathrm{T}\left(\boldsymbol{\lambda}_{1}\right)$ gives

$$
\mathrm{e}^{\text {At }}=T\left(\lambda_{1}\right)+\mathrm{T}\left(\lambda_{2}\right)=\left[\begin{array}{ccc}
1 & 2-(2+t) e^{-t} & 1-(1+t) \mathrm{e}^{-t} \\
0 & (1+\mathrm{t}) \mathrm{e}^{-t} & t e^{-t} \\
0 & -t e^{-t} & (1-t) \mathrm{e}^{-t}
\end{array}\right]
$$

Again, a quick check shows that $\left.e^{\underline{\underline{A} t}}\right|_{t=0}=I$ and a comparison with Matlab's symbolic solver confirms that all the above manipulations were indeed done correctly!

## Concluding Remarks

At this point, we have finished the preliminary review of some basic mathematical tools for dynamic systems analysis. We will now start addressing the representation, modeling, and simulation of dynamic systems. This course will continue to be mathematically oriented, since a good understanding of the mathematics usually implies a good command of the system analysis tools that will be highlighted throughout this course.

Except for the review of Laplace transforms in Section IV of these notes, any new mathematical concepts that may be required will be introduced as needed directly within the context of the subject under study. In this way, we can get right to the subject of dynamic systems, and simply incorporate any additional mathematics that may be needed as part of the normal development.

Summary Note: The capability to do matrix computations of the type illustrated in this section is built directly into Matlab and other similar programs and, in practice, automated routines like those in Matlab are used in day-to-day engineering applications as the need arises. However, the student should definitely be familiar with the fundamentals of these numerical algorithms (although the details are not always necessary). By assuring that you can do the above manipulations by hand for low-order systems, you will gain the confidence and experience necessary to intelligently and efficiently use the automated software. Thus, you should make sure you understand the examples in this section, and be able to perform similar manipulations on small systems as verification of the computer tools that simply automate the procedures.

