System Dynamics (22.554 & 24.509)

I. Introduction to the Analysis of Dynamic Systems

Introduction

The intent of this course is to introduce the analytical and computational aspects of system dynamics. Emphasis will be placed on the mathematical foundation necessary for the analysis of any dynamic system. The word 'dynamic' implies that time-dependent processes will be the subject of main interest. In most engineering design studies, one performs a large part of the preliminary design based on the steady state characteristics and performance of the system. However, all systems are time dependent in some way and, in many cases, the performance of the system under transient conditions can dictate the major elements of the design.

Complex engineering systems for the supply and delivery of energy in a form useful to the consumer are examples of important dynamic systems. Whether the primary source of energy is solar, nuclear, hydropower, fossil fuel, etc., the specific system design is dominated by the dynamic response of the system. In a nuclear reactor core, for example, the final design of the core configuration and control systems are dictated by the dynamic behavior of the system under normal operational transients (startup and shutdown, load following, xenon transients, etc.). In addition, the balance-of-plant design, with all its backup safety systems, is driven by the overall system response under off-normal transient conditions.

In solar energy applications (and for other energy systems), the situation is the same. For solar driven systems, the primary energy source clearly represents a strong time-dependent input to the system. Weather conditions, battery storage systems, and the required load delivered to the consumer are additional dynamic components that must be considered in designing a complete and useful system. Clearly, the ability to model and control the time-dependent behavior of any system is important. This course will emphasize the fundamental tools required for analysis of the time-dependent behavior of such systems.

The applications used to illustrate the subjects covered as a part of this course will focus on simple mechanical, electrical, thermal, and fluid systems. A good background in these areas is common to most graduate engineering students. Thus, these engineering subjects will serve as a good base from which to draw relatively simple examples of important dynamic phenomena. Other areas of specialization, such as nuclear reactor core dynamics, chemical reactor kinetics, process simulation and control, the dynamics of battery storage systems, photovoltaic devices, etc., will be left for future specific study depending on the student's interest. This course, however, should provide a good basis for additional research and specialization in any of these areas.

Our final introductory note concerns the subject of control system design. In many courses of this type, the design and control of dynamic systems is the main thrust of the course. This is not true here. Practical design and control concepts are extremely important, but the design of specific control systems is not the major emphasis of this course. Instead, we are interested primarily in developing the mathematical techniques used for the simulation, analysis, and control of dynamic systems. The actual design of control systems for particular applications will be left for future, more specialized courses. However, examples of control systems are indeed

highlighted, not from the perspective of controller design, but more from the view that systems controlled via feedback mechanisms represent an important class of dynamic systems -- a class of problems that can be solved readily with the methods discussed as part of this course.

This section of notes introduces the basic terminology and classification scheme for dynamic systems and it overviews the goals and subject content for this course.

A note about Matlab and Simulink: Matlab and Simulink are used as the primary problemsolving tools in this course. The base Matlab package is used quite extensively in many courses, and the student is expected to have some familiarity with using Matlab's base capabilities. In this course we will also use Simulink and Matlab's Control Toolbox. The Simulink package provides an easy-to-use graphical interface for performing modeling and simulation studies and the Control Toolbox has a number of specialized functions for analysis of linear time-invariant (LTI) systems. These resources, combined with the powerful mathematical and visualization capabilities of the base Matlab package, give us a wide range of computational capabilities. We will take advantage of these computational resources in many of the examples and cases studies presented within these notes and during the class lectures...

Some Terminology

As a starting point in the study of dynamic systems, one should develop a good understanding of the terminology used to describe dynamic behavior. In this subsection (and the next), we will identify several new terms used to describe and classify dynamic systems.

As implied by the term *dynamic system*, we will be studying the time behavior of the physical system under investigation. Most physical processes can be described by balance equations of the form,

rate of change = production rate - loss rate of quantity - of quantity

For *continuous systems*, the rate of change component gives rise to a first-order differential equation. For *discrete systems*, where events occur at discrete time intervals (such as in economic and biological systems), the balance equation is written in terms of discrete difference equations. A simple example that contrasts the difference between continuous and discrete time systems is given in Example 1.1. In this course, the primary emphasis will be on the study of continuous systems. For completeness, however, brief comments showing the analogy between continuous and discrete phenomena will be given at several points in our study.

In many undergraduate courses on dynamic systems, *single input-single output* (SISO) systems are investigated. For continuous systems, a single differential equation is used to describe the relationships between the input quantity, u(t), and the output quantity, y(t). We will generalize this input-output relationship to include several system inputs and outputs. Systems with more than one primary input/output quantity are referred to as multivariate or multivariable systems, and these systems are described by a set of differential equations. For such systems, the input and output quantities are represented as vector quantities, $\underline{u}(t)$ and $\underline{y}(t)$, respectively. In this case the system is said to have *multiple inputs and multiple outputs* (i.e. a MIMO system).

Since we will be dealing with several differential equations, it is convenient to develop a formal matrix notation for a concise description of the mathematical representation of the phenomena of

interest. The formal matrix methods treatment of dynamic system analysis is termed the *State Variable or State Space Approach*. We will see that most dynamic systems can be cast into a standard state variable form. Much of our work for the first part of the semester will be to identify the specific notation, to determine how to put any system into the standard state variable form, and to discuss various schemes for solution of the state equations.

Example 1.1 Continuous Versus Discrete Modeling

1st Order Continuous Time System

For continuous time systems that display exponential growth or decay, the basic growth/decay law can be stated as *"The rate of change is proportional to the amount present."* This can be written mathematically as

 $\frac{d}{dt}x(t) = ax(t)$ where $a = \begin{cases} > 0 & \text{growth constant} \\ < 0 & \text{decay constant} \end{cases}$

The solution for this system is given as,

 $x(t) = e^{at}x_o$ where x_o = initial amount present

1st Order Discrete Time System

For discrete time systems that display geometric growth or decay, the basic growth/decay law can be stated as "*The value at interval k+1 is proportional to the value at k.*" Mathematically, this can be written as

 $x_{k+1} = ax_k$ where $a = \begin{cases} > 0 & \text{growth constant} \\ < 0 & \text{decay constant} \end{cases}$

and k is the discrete time index. This discrete time variable is sometimes written as $t_k = kT$, where T is the sampling period. The solution to this difference equation can be obtained by assuming a solution of the form $x_k = c^k$, where c is some constant to be determined. In this case, it is easy to show that c = a and the solution becomes a simple geometric series,

$$X_{k} = a^{k}$$

A graphical illustration showing exponential/geometric growth for these two situations is shown below:



Classification of Dynamic Systems

We will find it convenient to group systems with certain characteristics into special classes and then identify mathematical techniques that apply to certain classes of problems. In particular, most systems have five characteristics that clearly identify the fundamental nature of the system. These five attributes are:

- A. Linear or Nonlinear
- B. Constant or Variable Coefficients
- C. Forced or Unforced
- D. Lumped or Distributed Parameter
- E. Deterministic or Stochastic

Let's briefly define each of these terms (see Table 1.1 for some mathematical examples and Fig. 1.1 [from the textbook by Takahashi et. al.] for a visual illustration of this classification system):

- A.1 *Linear* -- Dependent variables and their derivatives are raised only to the first power (products of dependent variables are not allowed).
- A.2 *Nonlinear* -- System which must be described by dependent variables or their derivatives raised to powers other than the first. Systems which contain products of dependent variables and/or derivatives are also nonlinear.
- B.1 Constant Coefficient -- System coefficients are not functions of the independent variable.
- B.2 Variable Coefficient -- System coefficients are functions of the independent variable.
- C.1 *Forced* -- System contains non-homogeneous terms (i.e. terms that do not contain the dependent variable) or explicit independent forcing functions.
- C.2 *Unforced* -- System is described by homogeneous equations (i.e. all terms contain the dependent variable). No independent forcing functions exist in the system model.
- D.1 *Lumped Parameter* -- System described by ordinary difference or differential equations with time as the independent variable.
- D.2 *Distributed Parameter* -- System described by partial differential equations with two or more independent variables (both space and time, for example).
- E.1 *Deterministic* -- System is driven by a forcing function which can be described explicitly (i.e. step or sinusoidal change, etc.).
- E.2 *Stochastic* -- System forcing function is random and a probabilistic treatment is required.

In this course we will learn how to simulate and analyze systems that fall into any of these classes. However, some techniques (such as analytical solutions, transform methods, sensitivity methods, etc.) are designed to be applicable only for certain classes of problems. Because of this, much of our emphasis will be on deterministic systems with the following attributes:

linear, constant coefficient, $\left\{ \begin{array}{c} \text{forced} \\ \text{unforced} \end{array} \right\}$, and lumped parameter

However, we will be sure not to exclude other systems. In particular,

- 1. Nonlinear systems will be linearized around some reference point or the full nonlinear behavior will be modeled with numerical methods,
- 2. Variable coefficient systems will only be treated numerically (since many of the analytical methods of analysis do not apply),
- 3. Distributed parameter systems will be converted to lumped systems (via finite difference methods, thermal network models, etc.), and
- 4. The study of stochastic systems will be treated as a separate topic (as time permits).

Thus, we will cover the complete range of possibilities. It just happens that there is a wider variety of analysis techniques available for linear, constant coefficient, forced/unforced, lumped parameter systems. Therefore, we will spend more time on this class of deterministic systems.

Mathematical Model	Classification of System
$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = 5\mathbf{x}(t)$	linear, constant coefficient, unforced, lumped
$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = 5\mathbf{x}^2(t)$	nonlinear, constant coefficient, unforced, lumped
$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = 5\mathbf{x}(t) + 4t$	linear, constant coefficient, forced, lumped
$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = (9t+1)\mathbf{x}^2(t) + 5t$	nonlinear, variable coefficient, forced, lumped
$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{e}^{\mathbf{x}(t)} + 5\mathbf{x}(t) + \mathbf{g}(t)$	nonlinear, constant coefficient, forced, lumped
$\frac{\partial}{\partial t} \mathbf{x}(z,t) = \frac{\partial^2}{\partial z^2} \mathbf{x}(z,t) + 2\mathbf{x}(z,t)$	linear, constant coefficient, unforced, distributed
$\frac{d}{dt}x(t) + \frac{d}{dt}y(t) = 8x(t) - 6y(t)$ $\frac{d}{dt}y(t) = 3y(t)$	linear, constant coefficient, unforced, lumped [note that there are two dependent terms, x(t) and y(t)]

 Table 1.1 Examples of several deterministic dynamic systems.



Fig. 1.1 Some classifications for mathematical modeling.

Scope of Course

This course will develop the mathematical foundation necessary for the analysis of dynamic systems using the so-called *State Variable* or *State Space Approach*. The complexity of real systems can vary from simple to complex, and we will develop methods to handle most of these situations. The course will be broken into four major sections:

- A. Introduction and Mathematical Preliminaries
- B. Time and Frequency Domain Simulation Methods
- C. Mathematical Modeling of Engineering Systems
- D. Introduction to Advanced Topics (as time permits)

A more detailed course syllabus which expands upon each of these areas is given in Table 1.2.

Table 1.2 Detailed course syllabus for Dynamic Systems.

- I. Introduction
 - A. Terminology
 - B. Classification of Systems
- **II.** Mathematical Preliminaries
 - A. Difference and Differential Eqns.
 - B. Matrix Methods
- **III.** Transient Analysis
 - A. General State Variable Formulation
 - B. Linear State Eqns.
 - C. Conversion to State Form
 - D. Analytical Solution of Forced and Unforced Linear Systems
 - E. Numerical Solutions/Simulations
- IV. Transform Methods
 - A. Laplace Transforms for Continuous Systems
 - B. Solution of Differential Eqns.
 - C. Transfer Functions
 - D. Frequency Response Methods (continuous systems)
 - E. Introduction to Z-Transforms for Discrete Systems
- V. Modeling and Simulation with Matlab
 - A. Time and Frequency Domain Simulations
 - B. Model Conversion Capabilities
 - C. Model Building with Matlab and Simulink
- VI. Mathematical Modeling of Engineering Systems (Case Studies)
 - A. Simple Mechanical Systems
 - B. Simple Fluid Systems
 - C. Simple Thermal Systems
 - D. Tube and Shell Heat Exchanger
 - E. Light Tracking Servo System
 - F. Permanent Magnet DC Motor

VII. Introduction to Control System Design and Simulation

- A. Classical and Modern State-Space Control
- B. Time Domain Simulation of Controlled Systems
- $C. \ \ A \ Detailed \ Example \ \ The \ Inverted \ Pendulum$

VIII. Introduction to Stability Analysis (if time permits)

- A. Linear Stability
- B. Nonlinear Stability
- IX. Sensitivity Analysis of Linear Systems (if time permits)
 - A. Transient Response Sensitivity
 - B. Frequency Response Sensitivity
 - C. Pole Sensitivity via Perturbation Theory
- X. Additional Advanced Topics (as time permits)

To see the relationships among these various subjects, consider the following simple block diagram,



where the system is completely described by a set of system parameters $\underline{\alpha}$, the state vector \underline{x} , rate of change of \underline{x} (denoted by $\underline{\dot{x}}$), and J which represents some performance index to be optimized. The functional relationships denoted by the *general system* box can be written mathematically as

 $F(y, \underline{x}, \underline{\dot{x}}, \underline{u}, \underline{v}, \underline{\alpha}, J, t) = 0$

where F represents some formal functional dependence on all the system variables. The inputs can be deterministic or random (denoted by $\underline{u}(t)$ and $\underline{v}(t)$, respectively), and the system outputs are contained in the $\underline{y}(t)$ vector.

Simulation methods deal with the solution of a set of equations for the system that describe the relationship between the system inputs and outputs. We will address both time and frequency domain simulation techniques, and hopefully gain considerable insight into the analysis of dynamic systems. However, our main goal here will be to determine $\underline{y}(t)$ given $\underline{u}(t)$ and a deterministic mathematical model of the system (in either the classical transfer function or state-space differential equation form).

The next major step in understanding dynamic systems is to be able to construct the mathematical model for real systems. We will concentrate on very simple engineering systems and on basic modeling concepts. The goal here will be to develop some simple models from first

principles and then illustrate the analysis of real systems by applying the simulation methods discussed previously.

Techniques beyond the ability to model systems and then simulate their performance fall into the *Advanced Topics* category. Topics such as advanced control system design, stability analysis, parameter sensitivity, optimal control, stochastic processes, and system identification are examples of some important advanced topics. Expertise in these areas strengthens the ability of the engineer to make decisions concerning the design of dynamic systems. We will discuss as many of these areas as time permits. In all, this course should give the student a good inventory of tools for the analysis and design of time-dependent systems.