A 3rd order SISO plant is described by the following transfer function

$$\frac{Y(s)}{U(s)} = \frac{8}{(s+1)^2(s+2)}$$

(a) Determine the 3rd order ODE that describes the plant dynamics (assume zero ICs)

$$\frac{Y(s)}{U(s)} = \frac{8}{(s+1)^2(s+2)} = \frac{8}{s^3+4s^2+5s+2}$$

$$\begin{array}{r} s+1 \\ s+1 \\ \hline s^2+2s+1 \\ s+2 \\ \hline s^3+2s^2+s \\ 2s^2+4s+2 \\ \hline s^3+4s^2+5s+2 \end{array}$$

$$\therefore \left(s^3+4s^2+5s+2\right)Y(s) = 8\,U(s)$$

Taking the inverse L.T.

$$\boxed{y''' + 4y'' + 5y' + 2y = 8u}$$

ans

(b) Determine the state space representation of the plant if y(t) is the desired output (ie find the A, B, C, and D matrices for this system). Assume that y(t) is measurable by direct means.

let $x_1 = y \qquad x_2 = y' \qquad x_3 = y''$

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = x_3 \qquad \frac{dx_3}{dt} = -4x_3 - 5x_2 - 2x_1 + 8u$$

$$\boxed{\therefore \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]\,u}$$

ans

(c) Is the plant state controllable? Address this qualitatively first by analyzing the structure of the A and B matrices. Discuss your observations and make a decision on controllability based on the observed structure. Now formally determine the controllability status by determining the rank of the controllability matrix — do this via hand calc.

Qualitative

The input u affects state 3 directly. But, eqn 2 shows that x_3 affects x_2 and eqn. 1 indicates that state 1 is a function of x_2. Thus, all three states are affected, either directly or indirectly, by state 3. Therefore, this plant is probably state controllable.

Quantitative

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$$

The controllability matrix is

$$M = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -5 & -4 \\ 8 & 18 & 11 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 8 \\ -32 \\ 88 \end{bmatrix} \qquad AB = \begin{bmatrix} 0 \\ 8 \\ -32 \end{bmatrix}$$

$$\therefore \quad M = \begin{bmatrix} 0 & 0 & 8 \\ 0 & 8 & -32 \\ 8 & -32 & 88 \end{bmatrix}$$

Note that this matrix is already in row echelon form and we can immediately see that each row is linearly independent

$\therefore$ rank$(M) = 3$ $\quad$ $\therefore$ system is state controllable

(Note) If the above statement is not obvious, then simply interchange rows 1 and 3 to give

$$\begin{bmatrix} 8 & -32 & 88 \\ 0 & 8 & -32 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow$$

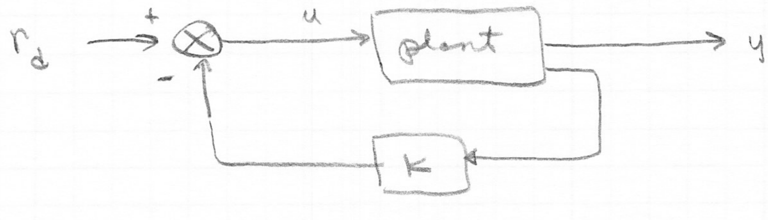
now clearly this is in row echelon form with rank 3 (upper triangular matrix)

$\det \neq 0$

(d) Assuming state controllability (from Part c) and the control law, $u = r_d - Kx$, determine the state gain matrix, K, that gives the following closed loop poles (again, do this via hand calc).

$$\mu_{1,2} = -2 \pm j2\sqrt{3} \qquad \mu_3 = -6$$

The block diagram for this system is



$$\therefore \quad \frac{dx}{dt} = Ax + Bu$$

$$\frac{dx}{dt} = Ax + Br_d - BKx$$

$$\frac{dx}{dt} = (A - BK)x + Br_d$$

The dynamics of the closed loop system is defined via the eigenvalues of this matrix

with the poles given, the desired characteristic eqn becomes

$$(s-\mu_1)(s-\mu_2)(s-\mu_3) = (s^2+4s+16)(s+6)$$

$$= s^3 + 10s^2 + 40s + 96$$

we want the characteristic eqn for the closed loop system to look like this

$$\begin{array}{r}
s+2 - j2\sqrt{3} \\
s+2 + j2\sqrt{3} \\
\hline
s^2 + (2-j2\sqrt{3})s \\
(2+j2\sqrt{3})s + 4 + 12 \\
\hline
s^2 + 4s + 16 \\
s + 6 \\
\hline
s^3 + 4s^2 + 16s \\
6s^2 + 24s + 96 \\
\hline
s^3 + 10s^2 + 40s + 96
\end{array}$$

Now, the actual characteristic eqn is given by

$$\det(sI-(A-BK)) = \det(sI - A + BK)$$

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 2 & 5 & s+4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8k_1 & 8k_2 & 8k_3 \end{bmatrix}} = sI - A + Bk$$

$$\therefore \det(sI-A+BK) = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 8k_1+2 & 8k_2+5 & s+8k_3+4 \end{vmatrix}$$

expanding along row #1

$$= s\begin{vmatrix} s & -1 \\ 8k_2+5 & s+8k_3+4 \end{vmatrix} + (-1)(-1)\begin{vmatrix} 0 & -1 \\ 8k_1+2 & s+8k_3+4 \end{vmatrix}$$

$$= s\left[s^2+(8k_3+4)s+8k_2+5\right] + 8k_1+2$$

characteristic eqn for closed loop system ⇒

$$= \boxed{s^3+(8k_3+4)s^2+(8k_2+5)s+8k_1+2}$$ actual

and this must be identical to the desired characteristic eqn ⇒

$$= \boxed{s^3+10s^2+40s+96}$$ desired

$\therefore$ $8k_3+4=10$ $\quad$ $8k_2+5=40$ $\quad$ $8k_1+2=96$

$8k_3=6$ $\quad$ $8k_2=35$ $\quad$ $8k_1=94$

$\boxed{k_3=\frac{3}{4}}$ $\quad$ $\boxed{k_2=\frac{35}{8}}$ $\quad$ $\boxed{k_1=\frac{47}{4}}$

$\boxed{K=[11.75 \;\; 4.375 \;\; 0.75]}$ $\therefore$ $\boxed{K = \begin{bmatrix} \frac{47}{4} & \frac{35}{8} & \frac{3}{4} \end{bmatrix}}$ ans

needed state gains

(e) Estimate the rise time, settling time, and max. overshoot that you would expect from this system and then compare to the actual Matlab simulation.

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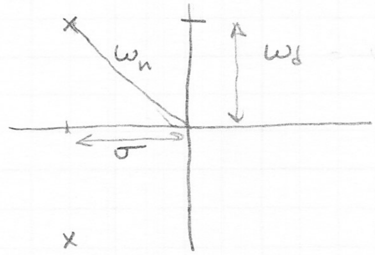
the dominant poles are $P_{1,2} = -2 \pm j2\sqrt{3}$

$\therefore$ from Design Aids sheet

$$P_{1,2} = -\sigma \pm j\omega_d$$

where $\sigma = \zeta\omega_n = 2$ $\qquad \omega_d = \omega_n\sqrt{1-\zeta^2} = 2\sqrt{3}$



but $\omega_n = \sqrt{\sigma^2 + \omega_d^2} = \sqrt{4+12} = 4$ rad/sec

$$\therefore \zeta = \frac{\sigma}{\omega_n} = \frac{2}{4} = \frac{1}{2}$$

now $t_r \approx \dfrac{1.8}{\omega_n} = \dfrac{1.8}{4} \approx 0.45$ s

$t_s \approx \dfrac{4.6}{\sigma} = \dfrac{4.6}{2} \approx 2.3$ s

$M_p \approx 15\%$

---

from Matlab's LTIVIEW GUI (see plot)

$t_r \approx 0.5$ s

$t_s \approx 2.3$ s

$M_p \approx 12\%$

not bad!

Also calc setpoint gain and redo the simulation

$$N_r = \frac{-1}{C(A-BK)^{-1}B}$$

(use hand values and Matlab values)
↑ calcs. for both in Matlab

matrices from hand calcs

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$$

$$C = [\,1 \quad 0 \quad 0\,] \quad K = \left[\frac{47}{4} \quad \frac{35}{8} \quad \frac{3}{4}\right]$$

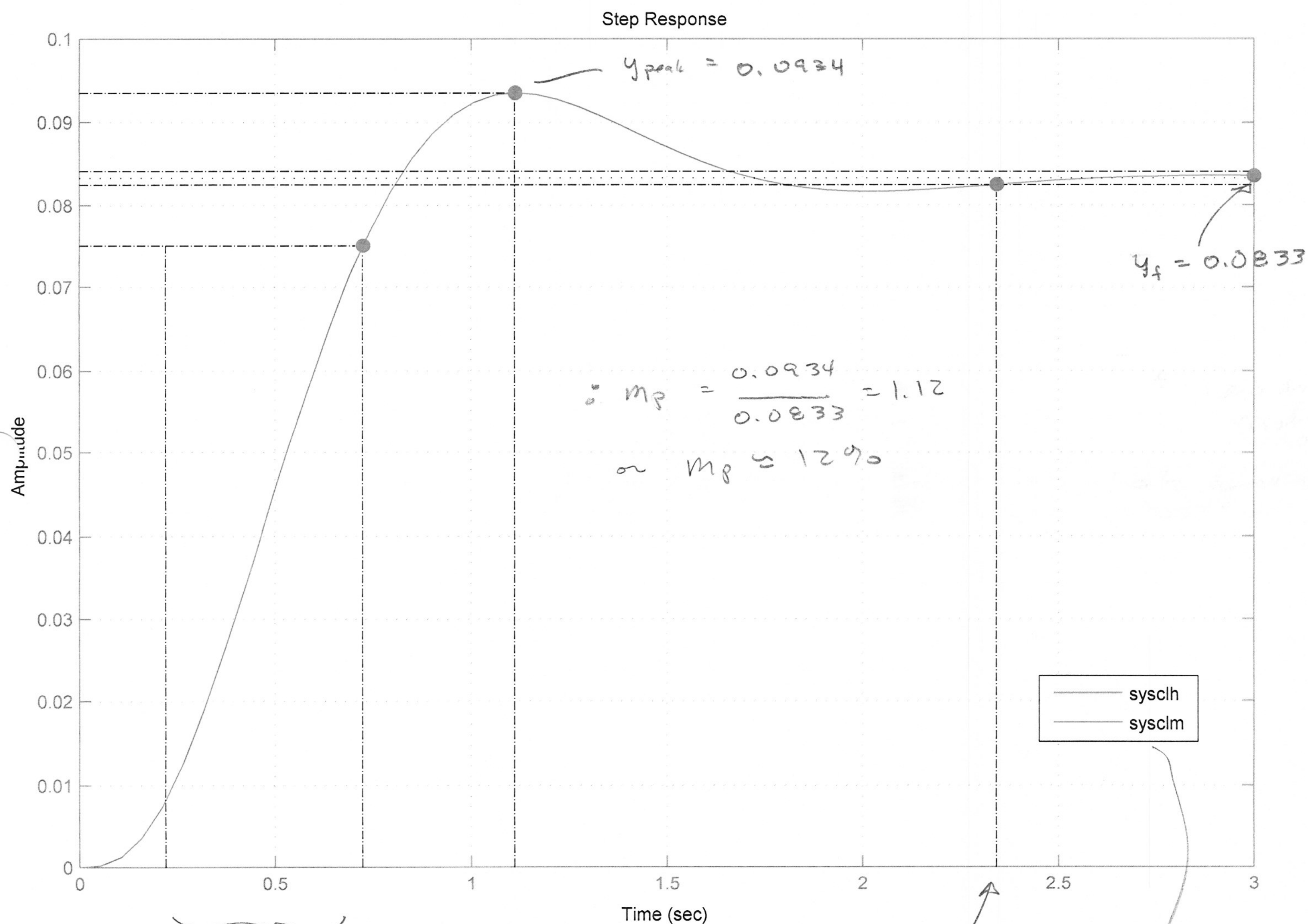
with N_r, the new input matrix for the closed loop system becomes

$$\frac{dx}{dt} = (A-BK)\,x + \underbrace{BN_r}_{\text{new input matrix}}\, r_d$$

→ Also do the same calcs for the Matlab generated system matrices

→ all calcs and plots done in Matlab for this part...

see $\underbrace{\text{sf}}_{\text{state feedback}}$_2.m from LTIVIEW with settling time set to 1%



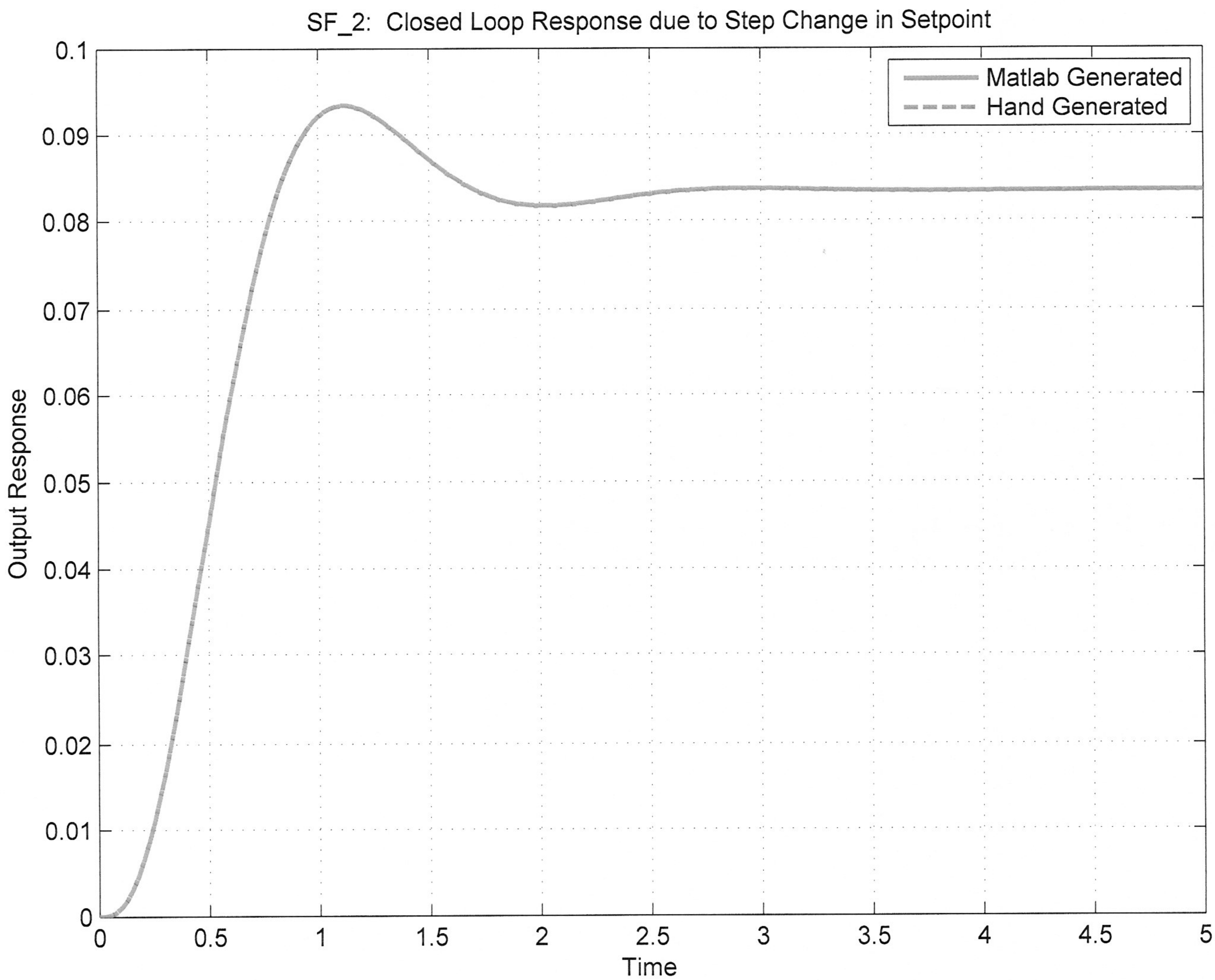
$\tau_r \simeq 0.5$ s

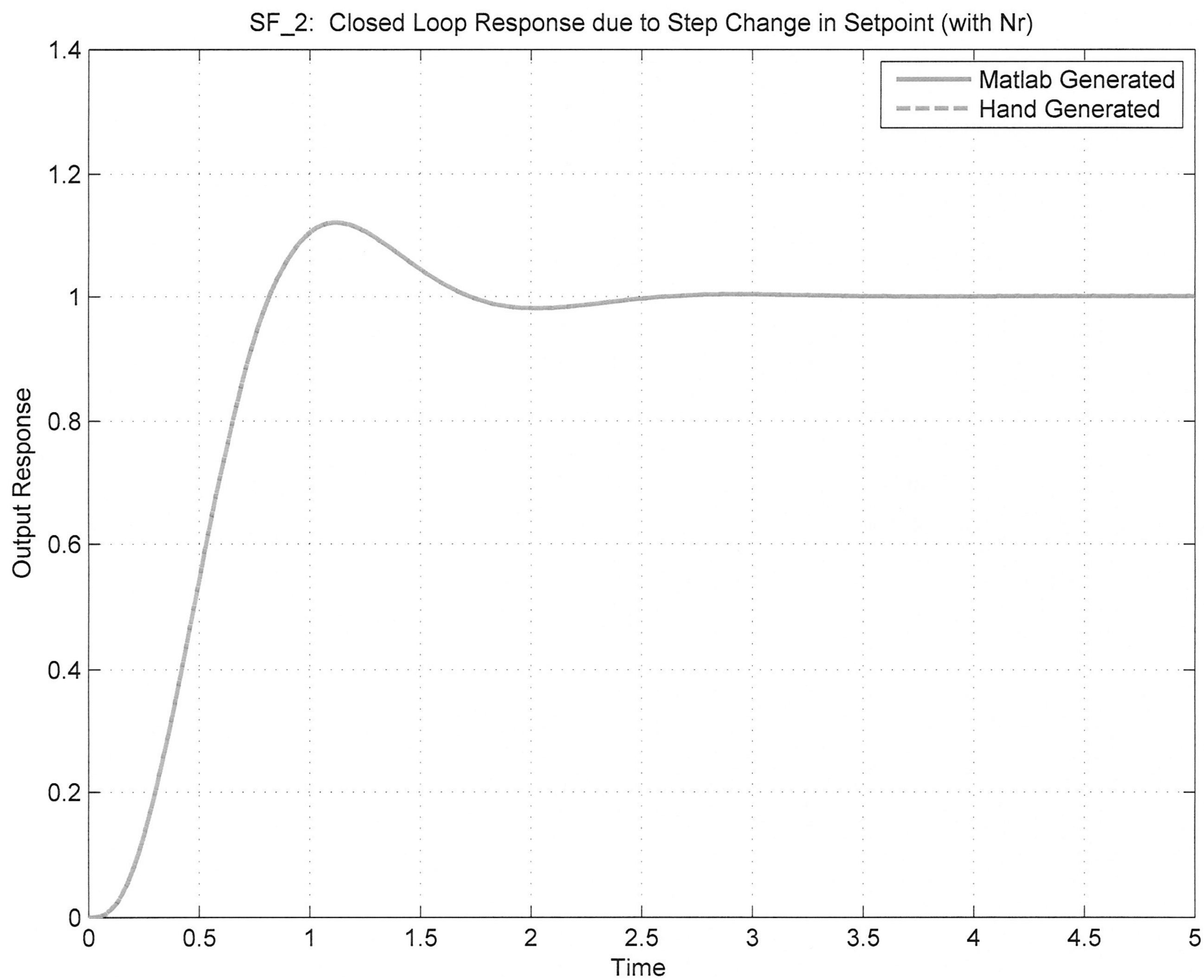
$t_s \simeq 2.3$ s

h - hand

m - matlab

exactly the same


SF_2: Closed Loop Response due to Step Change in Setpoint
Matlab Generated
Hand Generated
Output Response
Time
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



SF_2: Closed Loop Response due to Step Change in Setpoint (with Nr)
Matlab Generated
Hand Generated
Output Response
Time
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0.5
1.5
2
2.5
3
3.5
4
4.5
5

```
>> sf_2
   *** Results from SF_2 ***

Original Plant Transfer Function

Transfer function:
          8
---------------------
s^3 + 4 s^2 + 5 s + 2

Plant in State Space Form (Matlab Generated)
A =
   -4.0000   -2.5000   -1.0000
    2.0000         0         0
         0    1.0000         0
B =
     2
     0
     0
C =
     0     0     2
D =
     0
The Controllability Matrix
M =
     2    -8    22
     0     4   -16
     0     0     4
Rank of Controllability Matrix
ans =
     3
Desired closed loop poles for state feedback controller
clp =
  -2.0000 + 3.4641i  -2.0000 - 3.4641i  -6.0000
State feedback gains needed to give desired poles (Matlab Generated)
Ks =
    3.0000    8.7500   23.5000
Calculated eigenvalues of system with state feedback
ans =
  -6.0000
  -2.0000 + 3.4641i
  -2.0000 - 3.4641i
Plant in State Space Form (Generated by hand)
AA =
     0     1     0
     0     0     1
    -2    -5    -4
BB =
     0
     0
     8
CC =
     1     0     0
DD =
     0
Feedback Gain Matrix (Generated by Hand)
KKs =
   11.7500    4.3750    0.7500
Eigenvalues of hand generated system with state feedback
ans =
  -2.0000 + 3.4641i
  -2.0000 - 3.4641i
  -6.0000
Setpoint gain (Matlab Generated)
Nrm =
   12.0000
Setpoint gain (Generated by Hand)
Nrh =
    12
```

Note The state equations and the gain matrix are different

However, the dynamics are equivalent:
- open loop eigenvalues are same
- closed loop eigenvalues are identical
- The step responses are the same etc. etc.

Recall that the definition of the state is NOT unique – thus, this is really not surprising...

```
%
%  SF_2.M   HW Problem for State Feedback Design  --  Simple 3x3 System
%
%  This is a Matlab equivalent of the simple 3x3 test problem done via hand
%  calculations.  In particular, there are five steps to be performed:
%     1.  Convert given transfer function into state form.
%     2.  Determine the rank of the controllability matrix.
%     3.  Find the state feedback gain matrix for a set of given closed loop poles.
%     4.  Find the response of the closed loop system due to a unit step change in
%         the setpoint.
%     5.  Compare the response of the system designed by hand to the one obtained via
%         Matlab manipulations.
%     6.  Calc a set point gain and redo comparisons...
%
%  File prepared by J. R. White, UMass-Lowell (April 2014)
%

   clear all,  close all,  nfig = 0;
   format compact
%
%  1. Convert given transfer function into state form.
   disp('   *** Results from SF_2 ***'),   disp('  ')
   disp('Original Plant Transfer Function'),  systf = tf(8,[1 4 5 2])
   disp('Plant in State Space Form (Matlab Generated)')
   sysss = ss(systf);  [A,B,C,D] = ssdata(sysss)
%
%  2. Determine the rank of the controllability matrix.
   disp('The Controllability Matrix'), M = ctrb(A,B)
   disp('Rank of Controllability Matrix'), rank(M)
%
%  3. Find the state feedback gain matrix for a set of given closed loop poles.
   clp = [-2+2*sqrt(3)*j  -2-2*sqrt(3)*j  -6];
   Ks = place(A,B,clp);
   disp('Desired closed loop poles for state feedback controller');   clp
   disp('State feedback gains needed to give desired poles (Matlab Generated)'); Ks
   disp('Calculated eigenvalues of system with state feedback'); eig(A-B*Ks)
%
%  4. Find the response of the closed loop system due to a unit step change in rd
   to = 0;  tff = 5;
   t = linspace(to,tff,201);
   sysclm = ss(A-B*Ks,B,C,D);
   ym = step(sysclm,t);         % Matlab generated response
%
%  5. Find the response of the system designed by hand
   disp('Plant in State Space Form (Generated by hand)')
   AA = [0 1 0;0 0 1;-2 -5 -4],  BB = [0;0;8],  CC = [1 0 0],  DD = 0
   disp('Feedback Gain Matrix (Generated by Hand)'), KKs = [47/4  35/8  3/4]
   disp('Eigenvalues of hand generated system with state feedback'); eig(AA-BB*KKs)
   syscl h = ss(AA-BB*KKs,BB,CC,DD);
   yh = step(syscl h,t);         % hand generated response
%
%  compare the results of the two systems
```

```
    nfig = nfig+1;      figure(nfig)
    plot(t,ym,'r-',t,yh,'g--','LineWidth',2),grid,
    title('SF\_2:  Closed Loop Response due to Step Change in Setpoint')
    xlabel('Time '),ylabel('Output Response')
    legend('Matlab Generated','Hand Generated')
%
%   6. Calc setpoint gain and redo simulations/comparisons
    Nrm = C*inv(A-B*Ks)*B;  Nrm = -1/Nrm;      % setpoint gain using Matlab data
    Nrh = CC*inv(AA-BB*KKs)*BB;  Nrh = -1/Nrh; % setpoint gain using hand calcs
    disp('Setpoint gain (Matlab Generated)');  Nrm
    disp('Setpoint gain (Generated by Hand)'); Nrh
    sysclmsg = ss(A-B*Ks,B*Nrm,C,D);
    ymsg = step(sysclmsg,t);         % Matlab generated response with setpoint gain
    sysclhsg = ss(AA-BB*KKs,BB*Nrh,CC,DD);
    yhsg = step(sysclhsg,t);         % hand generated response with setpoint gain
%
%   compare the results of the two systems
    nfig = nfig+1;      figure(nfig)
    plot(t,ymsg,'r-',t,yhsg,'g--','LineWidth',2),grid,
    title('SF\_2:  Closed Loop Response due to Step Change in Setpoint (with Nr)')
    xlabel('Time '),ylabel('Output Response')
    legend('Matlab Generated','Hand Generated')
%
%   end of simulation
```