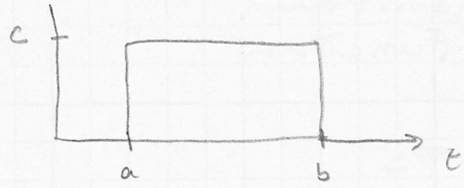


$f(t)$ is a pulse function as shown

Two methods:

① From basic defn

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt = c\int_a^b e^{-st}\, dt$$

$$= c\left[-\frac{1}{s} e^{-st}\right]_a^b$$

or $$\boxed{F(s) = \frac{c}{s}\left(e^{-as} - e^{-bs}\right)}$$ ans

② Let's write $f(t)$ with delayed step functions (Heaviside functions)

where $u(t)$ = unit step and $u(t) \Longleftrightarrow \frac{1}{s}$

and from the time shift property

$$f(t - t_0) \Longleftrightarrow e^{-st_0} F(s)$$

Then $$f(t) = c\, u(t-a) - c\, u(t-b)$$

$$\therefore \boxed{F(s) = \frac{c}{s} e^{-as} - \frac{c}{s} e^{-bs}}$$ ans

ok - same as above

A simple SISO system has a transfer function given by

$$G(s) = \frac{3s}{s^2+9}$$

ⓐ Analytically determine the system output for all time if

$$u(t) = \begin{cases} 0 & 0 \le t < 1 \\ 2 & 1 \le t \le 4 \\ 0 & t > 4 \end{cases}$$

Method I use the convolution integral

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau \qquad \text{where } h(t) = \mathcal{L}^{-1}\{G(s)\}$$

for $G(s) = \frac{3s}{s^2+9}$ $h(t) = 3\cos 3t$

Region 1 $t < 1$ $\boxed{y_1(t) = 0}$ since $u(t) = 0$

Region 2 $1 \le t \le 4$

$$y_2(t) = \int_1^t 3\cos 3(t-\tau)\,(2)\,d\tau$$

let $x = t-\tau$, $dx = -d\tau$

$$= 6\int_{t-1}^{0} \cos 3x\,(-dx)$$

switch limits

$$= 6\int_0^{t-1} \cos 3x\,dx$$

$$= \frac{6}{3}\sin 3x\Big|_0^{t-1}$$

$$\therefore \boxed{y_2(t) = 2\sin 3(t-1)}$$

Region 3 $t > 4$

$$y_3(t) = \int_1^4 6\cos 3(t-\tau)\,d\tau + \underbrace{\int_4^t 0\,d\tau}_{\to 0}$$

let $x = t-\tau$, $dx = -d\tau$

$$= 6\int_{t-4}^{t-1} \cos 3x\,dx$$

$$= 2\sin 3x\Big|_{t-4}^{t-1}$$

$$\therefore \boxed{y_3(t) = 2\left[\sin 3(t-1) - \sin 3(t-4)\right]}$$

Note With our defn of a causal time function we can simply write

$$y(t) = 2\left[\sin 3(t-1) - \sin 3(t-4)\right]$$

since any time function, $f(\tau)$, is zero for $\tau < 0$.

$\boxed{\text{Method II}}$ use the Time shift property of Laplace Transforms

$$y(t) = \mathcal{L}^{-1}\left\{ G(s)\,U(s) \right\}$$

But from Prob # 1, we know that

$$U(s) = \frac{2}{s}\left(e^{-s} - e^{-4s}\right)$$

$$\therefore \quad y(t) = \mathcal{L}^{-1}\left\{ \left(\frac{3\cancel{s}}{s^2+9}\right)\left(\frac{2}{\cancel{s}}\right)\left(e^{-s} - e^{-4s}\right) \right\}$$

But we note that from a table of L.T.

$$\mathcal{L}^{-1}\left\{ \frac{2(3)}{s^2+3^2} \right\} = 2 \sin 3t$$

and from the time shift property

$$f(t-t_0) \iff e^{-st_0} F(s)$$ } assuming causal time functions

$$\therefore \quad \boxed{y(t) = 2 \sin 3(t-1) - 2 \sin 3(t-4)}$$

Note that with the defn of causal time functions, this one statement of $y(t)$ covers all time for $t > 0$.

OK, same as before

(b) This system can be simulated in Matlab using LSIM, with the transfer function form converted to state space form.

→ see Tdelay_in1.m } solutions are identical

(c) We should also be able to solve this problem using an ODE solver.

First convert the transfer function form into an ODE

$$Y(s) = G(s)\,U(s)$$

$$Y(s) = \frac{3s}{s^2+9}\,U(s)$$

$$\left(s^2+9\right) Y(s) = 3s\,U(s)$$

or $\boxed{y'' + 9y = 3u'}$ ← This is the ODE

To convert this to state form, we first define the states

$$x_1 = y$$

$$x_2 = y' - 3u \quad \} \text{ by inspection of the given ODE}$$

then
$$\frac{dx_1}{dt} = y' = x_2 + 3u$$

$$\frac{dx_2}{dt} = y'' - 3u' = -9y = -9x_1$$

$$\therefore \boxed{\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u}$$

state representation of original ODE

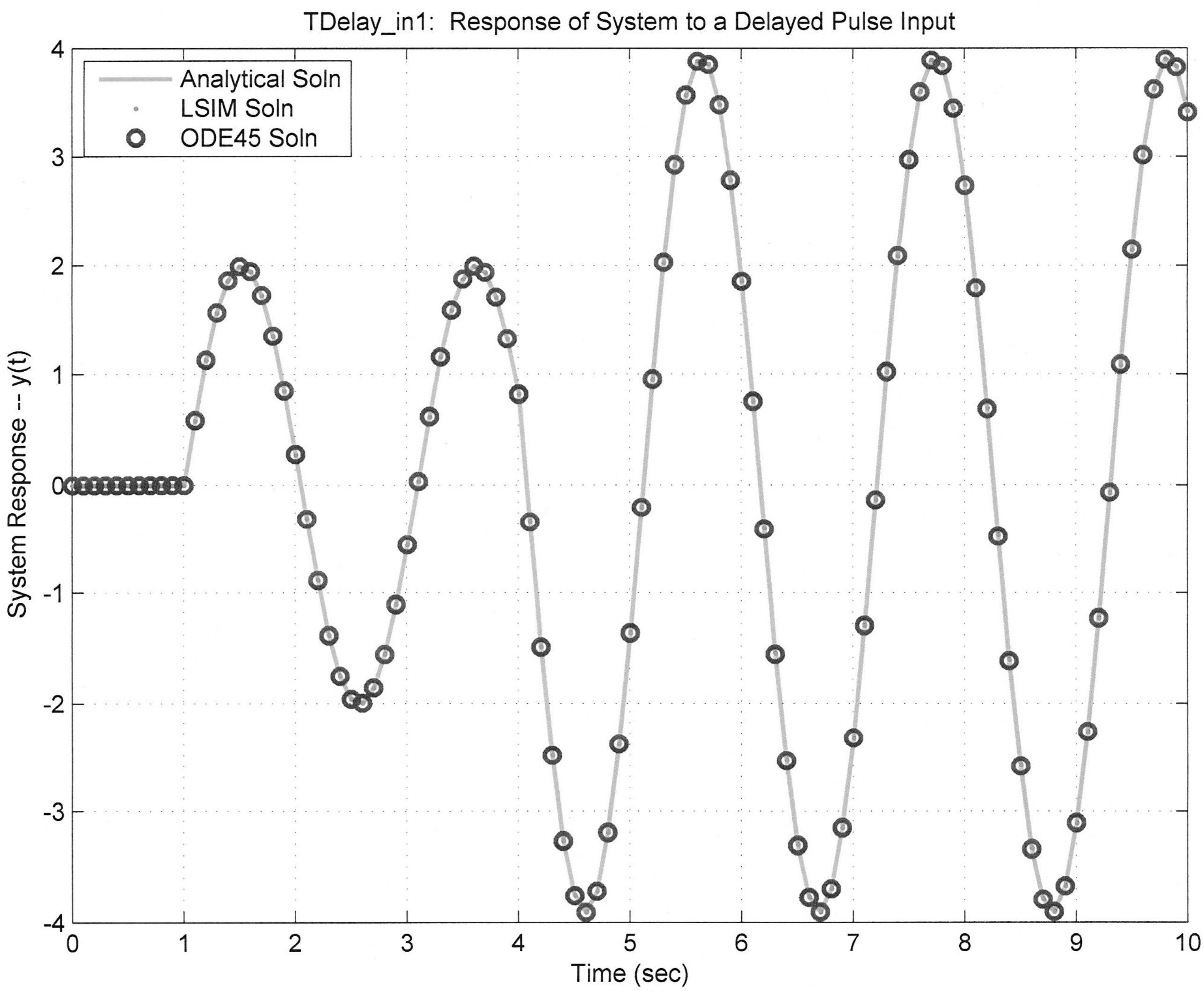
Now, because of the discrete nature of $u(t)$ for this case, we can simply make multiple calls to ODE45 over each piecewise continuous interval and use the ICs for the current interval as the endpoint condition from the previous interval.

OK, this should work ...

see tdelay_m1.m } all solutions are identical

Note
Another option is to build in the discrete functional behavior of $u(t)$ within the ODE function file explicitly and then just make one call to the ODE solver

I used the first option since it was consistent with the use of the lsim routine


TDelay_in1: Response of System to a Delayed Pulse Input
Analytical Soln
LSIM Soln
ODE45 Soln
System Response -- y(t)
Time (sec)
4
3
2
1
0
-1
-2
-3
-4
0
1
2
3
4
5
6
7
8
9
10

```
%
%   TDELAY_IN1.M  Dynamic System with a Time Delay in the Input  (Case 1)
%
%   Program written by J. R. White, UMass-Lowell (April 2014)
%

      clear all,   close all,   nfig = 0;
%
%   define discrete time intervals
      dt1 = 1;    dt2 = 3;    dt3 = 6;
      nt1 = 11;   t1 = linspace(0,dt1,nt1)';
      nt2 = 31;   t2 = linspace(0,dt2,nt2)';
      nt3 = 61;   t3 = linspace(0,dt3,nt3)';
      t = [t1; t2+dt1; t3+dt1+dt2];    nt = length(t);
%
%   analytical solution from hand development (see notes)
      ya = zeros(size(t));
      for i = 1:nt
        if t(i) >= 1 && t(i) <= 4;   ya(i) = 2*sin(3*(t(i)-1));  end
        if t(i) > 4;   ya(i) = 2*(sin(3*(t(i)-1)) - sin(3*(t(i)-4)));  end
      end
%
%   system definition: transfer function representation ==> G(s) = 3s/(s^2 + 9)
      n1 = [3 0];    d1 = [1 0 9];    sys1 = tf(n1,d1);
      sys2 = ss(sys1);  % convert to state form (so we can include ICs)

%   define input vectors and use LSIM to simulate system
      yn1 = zeros(size(t1));
      xn2o = [0 0];          u2 = 2*ones(size(t2));
      [yn2,tt,xn2] = lsim(sys2,u2,t2,xn2o);
      xn3o = xn2(nt2,:);  u3 = zeros(size(t3));
      yn3 = lsim(sys2,u3,t3,xn3o);
      yn_lsim = [yn1; yn2; yn3];
%
%   define inputs and use ODE45 to simulate system
      clear yn1 yn2 yn3 xn2
      yn1 = zeros(size(t1));
      xn2o = [0 0];          U = 2;   ftx = @(t,x) [x(2) + 3*U;-9*x(1)];
      [tt,xn2] = ode45(ftx,t2,xn2o);  yn2 = xn2(:,1);
      xn3o = xn2(nt2,:);  U = 0;   ftx = @(t,x) [x(2) + 3*U;-9*x(1)];
      [tt,xn3] = ode45(ftx,t3,xn3o);  yn3 = xn3(:,1);
      yn_ode45 = [yn1; yn2; yn3];
%
%   plot various solutions for y(t)
      nfig = nfig+1;   figure(nfig)
      plot(t,ya,'g-',t,yn_lsim,'r.',t,yn_ode45,'bo','LineWidth',2),grid
      title('TDelay\_in1:  Response of System to a Delayed Pulse Input')
      xlabel('Time (sec)'),ylabel('System Response -- y(t)')
      legend('Analytical Soln','LSIM Soln','ODE45 Soln','Location','NorthWest')
%   end of program
```

Consider the following SISO system transfer function

$$G(s) = \frac{5e^{-2s}}{s+3}$$

ⓐ If $u(t) = \sin t$, determine the response of this system using analytical means

$$Y(s) = G(s)\,U(s)$$

$$\sin t \iff \frac{1}{s^2+1}$$

$$= \left(\frac{5e^{-2s}}{s+3}\right)\left(\frac{1}{s^2+1}\right)$$

Let's write this as a product of 2 terms

$$Y(s) = F(s)\,e^{-2s}$$

$\therefore$ we know that $y(t) = f(t-2)$ } from the time shift property (assuming causal time functions)

Now let's determine $f(t)$

$$F(s) = \left(\frac{5}{s+3}\right)\left(\frac{1}{s^2+1}\right)$$

$$= \frac{A}{s+3} + \frac{Bs+C}{s^2+(1)^2}$$

$$A = \left.\frac{5}{s^2+1}\right|_{s=-3} = \frac{1}{2}$$

Now clearing fractions

$$5 = \tfrac{1}{2}(s^2+1) + (Bs+C)(s+3)$$

$$= \tfrac{1}{2}(s^2+1) + Bs^2 + 3Bs + Cs + 3C$$

$$= \left(\tfrac{1}{2}+B\right)s^2 + (3B+C)s + 3C + \tfrac{1}{2}$$

$\therefore$ $B + \frac{1}{2} = 0$ $\qquad$ $3B + C = 0$ $\qquad$ $3C + \frac{1}{2} = 5$

$\boxed{B = -\frac{1}{2}}$ $\qquad$ $C = -3B$ $\qquad$ $3\left(\frac{3}{2}\right) + \frac{1}{2} = 5$

$\boxed{C = \frac{3}{2}}$ $\qquad$ $\frac{10}{2} = 5$ ✓

$$\therefore F(s) = \frac{1}{2}\,\frac{1}{s+3} - \frac{1}{2}\,\frac{s}{s^2+(1)^2} + \frac{3}{2}\,\frac{1}{s^2+(1)^2}$$

and

$$f(t) = \frac{1}{2}e^{-3t} - \frac{1}{2}\cos t + \frac{3}{2}\sin t$$

(cont.)

Now including the time delay, we have

$$\boxed{y(t) = \frac{1}{2}\left[e^{-3(t-2)} - \cos(t-2) + 3\sin(t-2)\right]}$$

ans

(b) Using the same input function, determine the system ~~response~~ using Matlab's LSIM function and several low order Pade' approximations to the e^{-2s} factor. See help for PADE and CONV.

for this case $u(t) = \sin t$

$$G(s) = \frac{5}{s+3} e^{-2s}$$

$$= \frac{5}{s+3} P(s, \tau, n)$$

where $P(s,\tau,n)$ is the n^{th} order Pade' approximation to $e^{-\tau s}$

The matlab command is

[num2, den2] = PADE(τ, n) here τ = 2 sec

If we set num1 = 5 den1 = [1 3]
then the total transfer function can be described as

num = conv (num1, num2)
den = conv (den1, den2)

We can then use LSIM with the transfer function format to simulate the desired system

→ let's try 3 approximations n = 1, 2, 3

→ see tdelay_tf1 for the simulation

(c) Develop the differential eqn that describe this system. Use Matlab's ODE solver to simulate/solve this system for $u(t) = \sin t$.

Given $G(s) = \dfrac{5e^{-2s}}{s+3}$

but $G(s) = \dfrac{Y(s)}{U(s)}$

$\therefore \quad (s+3)\,Y(s) = 5e^{-2s}\,U(s)$

Taking the inverse Transform of each term gives

$$\frac{dy}{dt} + 3y = 5\,u(t-2)$$

where this is $u(t)$ delayed by 2 units

This also assumes zero initial conditions

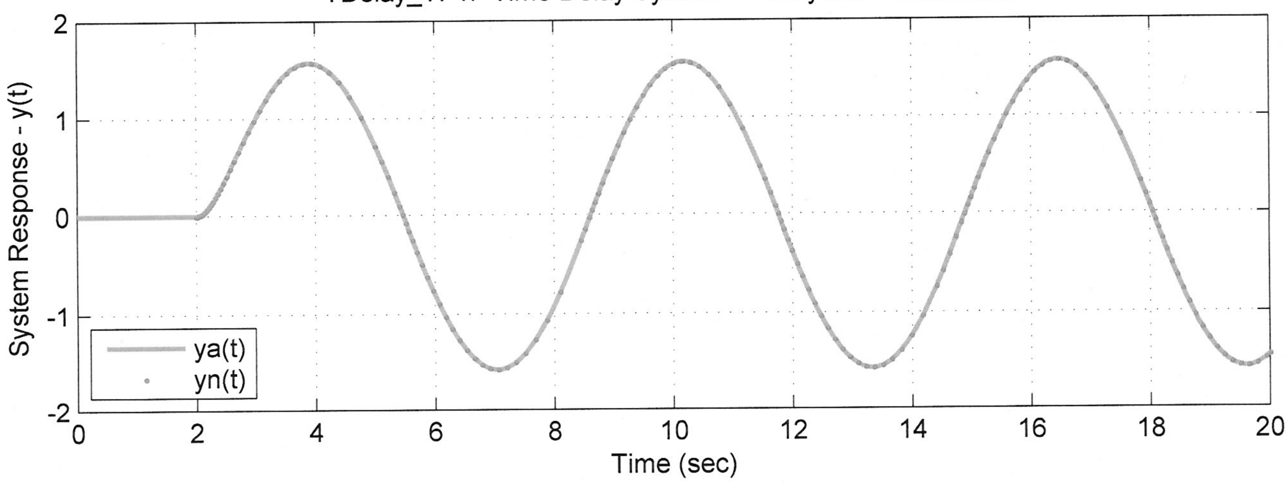
$\therefore \quad y(0) = 0$ — as implied with the original Transfer function description

→ this can be easily modeled in ODE's

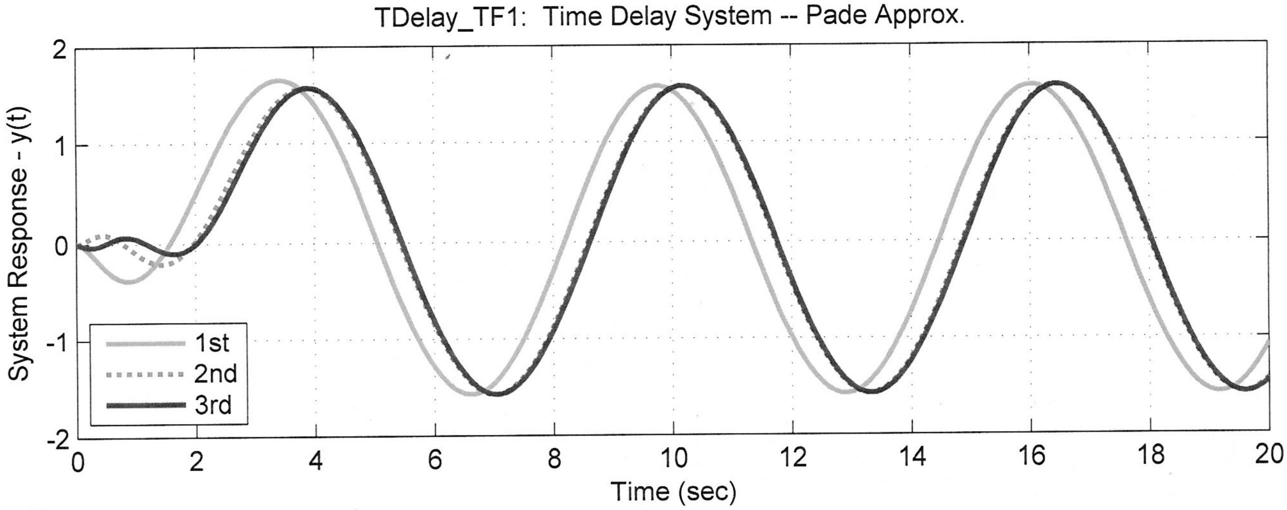
→ see tdelay_tf1.m — note that to start at 2 sec to account for the time delay...

→ see attached comparison plot.

TDelay_TF1: Time Delay System -- Analytical & Numerical



TDelay_TF1: Time Delay System -- Pade Approx.



```
>> tdelay_tf1
Polynomials for Various Pade Approximations to original system
 First Order

n1 =
   -5     5

d1 =
    1     4     3

 Second Order

n2 =
    5   -15    15

d2 =
    1     6    12     9

 Third Order

n3 =
   -5    30   -75    75

d3 =
    1     9    33    60    45

>>
```

$$\frac{5}{s+3}e^{-2s} \simeq \frac{-5s+5}{s^2+4s+3}$$

$$\frac{5}{s+3}e^{-2s} \simeq \frac{-5s^2-15s+15}{s^3+6s^2+12s+9}$$

$$\frac{5}{s+3}e^{-2s} \simeq \frac{-5s^3+30s^2-75s+75}{s^4+9s^3+33s^2+60s+45}$$

```
%
%  TDELAY_TF1.M  Dynamic System with a Time Delay in the Transfer Function  (Case 1)
%
%  Program written by J. R. White, UMass-Lowell (April 2014)
%

   clear all,   close all,   nfig = 0;
%
%  analytical solution from hand development (see notes)
   nt = 201;  to = 0;  tfinal = 20;   t = linspace(to,tfinal,nt)';
   ya = zeros(size(t));
   for i = 1:nt
     if t(i) >= 2;
       ya(i) = 0.5*(exp(-3*(t(i)-2)) - cos(t(i)-2) + 3*sin(t(i)-2));
     end
   end
%
%  system definition (transfer function representation with Pade approx))
%  G(s) = 5/(s+3) * num(s)/den(s) from various Pade approximations
   n0 = 5;   d0 = [1 3];   T = 2;
   [n1a,d1a] = pade(T,1);   n1 = conv(n0,n1a);   d1 = conv(d0,d1a);
   [n2a,d2a] = pade(T,2);   n2 = conv(n0,n2a);   d2 = conv(d0,d2a);
   [n3a,d3a] = pade(T,3);   n3 = conv(n0,n3a);   d3 = conv(d0,d3a);
   disp('Polynomials for Various Pade Approximations to original system')
   disp('  First Order');   n1, d1
   disp('  Second Order');  n2, d2
   disp('  Third Order');   n3, d3
%
%  set input and simulate systems using LSIM
   u = sin(t);
   sys1 = tf(n1,d1);   yn1 = lsim(sys1,u,t);
   sys2 = tf(n2,d2);   yn2 = lsim(sys2,u,t);
   sys3 = tf(n3,d3);   yn3 = lsim(sys3,u,t);
%
%  numerical solution with ODE45 (Note: the time delay has been shifted to the input)
   yo = 0;   tol = 1e-4;
   options = odeset('RelTol',tol);
   fty = @(t,y)  -3*y + 5*sin(t-2);
   [tn,yn] = ode23(fty,[2,tfinal],yo,options);  % note the start at t = 2 secs
%
%  plot both numerical and analytical results for y(t)
   nfig = nfig+1;   figure(nfig)
   subplot(2,1,1),plot(t,ya,'g-',tn,yn,'r.','LineWidth',2),grid
   title('TDelay\_TF1:  Time Delay System -- Analytical & Numerical')
   ylabel('System Response - y(t)'),xlabel('Time (sec)')
   legend('ya(t)','yn(t)','Location','SouthWest')
%
   subplot(2,1,2),plot(t,yn1,'g-',t,yn2,'r:',t,yn3,'b-','LineWidth',2),grid
   title('TDelay\_TF1:  Time Delay System -- Pade Approx.')
   xlabel('Time (sec)'),ylabel('System Response - y(t)')
   legend('1st','2nd','3rd','Location','SouthWest')
%
```