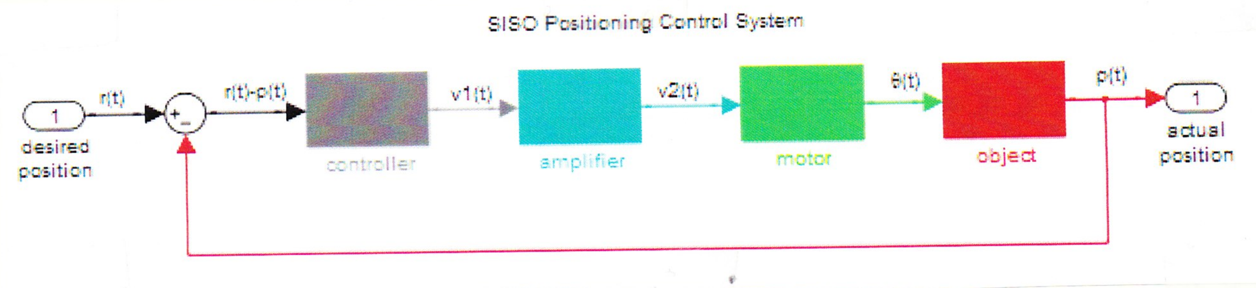
SISO Positioning Control System



$$v_1 = k_1(r-p) \qquad v_2 = k_2 v_1$$

$$\frac{d\theta}{dt} = 0.4 v_2 \qquad \frac{d^2p}{dt} + \frac{dp}{dt} + 4p = \theta$$

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let's start with a simple substitution

$$\frac{d\theta}{dt} = 0.4 v_2 = 0.4 k_2 v_1 = 0.4 k_2 k_1 (r-p)$$

$$\frac{d^2p}{dt} + \frac{dp}{dt} + 4p = \theta$$

Now define the state vector and input vector as

$$\underline{x} = \begin{bmatrix} \theta \\ p \\ \frac{dp}{dt} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad u = r$$

Now we have three state eqns

$$\frac{dx_1}{dt} = 0.4 k_2 k_1 (u - x_2)$$

$$\frac{dx_2}{dt} = x_3$$

$$\frac{dx_3}{dt} = x_1 - 4x_2 - x_3$$

original eqns

$$\frac{d\theta}{dt} = 0.4 k_2 k_1 (r-p)$$

$$\frac{dp}{dt} = x_3 = \frac{dp}{dt}$$

$$\frac{d}{dt}\left(\frac{dp}{dt}\right) = \theta - 4p - \frac{dp}{dt}$$

and these can be written in matrix form

with $u = r$ and $y = p = x_2$

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,u$$

$$y = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,u$$

where $\underline{\underline{A}} = \begin{bmatrix} 0 & -0.4 k_2 k_1 & 0 \\ 0 & 0 & 1 \\ 1 & -4 & -1 \end{bmatrix}$

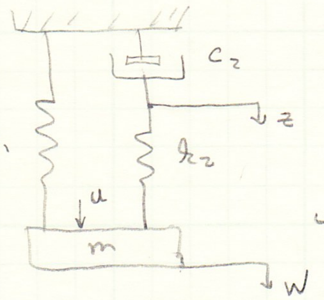
$$\underline{\underline{B}} = \begin{bmatrix} 0.4 k_2 k_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\underline{C}} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$\underline{\underline{D}} = \begin{bmatrix} 0 \end{bmatrix}$$

→ Put the following eqns. for the mechanical system shown below into standard state form

(a) System # 1



$$m\ddot{w} + k_1 w + k_2 (w - z) = u$$

$$k_2 (w - z) = c_2 \dot{z}$$

u is in units of kN

with $m = 1000\,kg$ $K_1 = 60\,kN/m$ $k_2 = 40\,kN/m$ $C_2 = 4\,\frac{kN\text{-}s}{m}$

$$\ddot{w} + 60w + 40(w - z) = u$$

$$40(w - z) = 4\dot{z}$$

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let $x_1 = w$

$x_2 = dw/dt = dx_1/dt$

$x_3 = z$

	units	
$\therefore \quad \dfrac{dx_1}{dt} = x_2$	$\frac{m}{s}$	ok
$\dfrac{dx_2}{dt} = \dfrac{d^2w}{dt^2} = -\left(\dfrac{k_1 + k_2}{m}\right) w + \dfrac{k_2}{m} z + \dfrac{u}{m}$	$\frac{m}{s^2}$	ok
$= -\left(\dfrac{k_1 + k_2}{m}\right) x_1 + \dfrac{k_2}{m} x_3 + \dfrac{u}{m}$		
$= -100x_1 + 40x_3 + u$		
$\dfrac{dx_3}{dt} = \dfrac{dz}{dt} = \dfrac{k_2}{c_2}(w - z) = \dfrac{k_2}{c_2}(x_1 - x_3)$	$\frac{m}{s}$	ok

$$= 10x_1 - 10x_3$$

$$\therefore \quad \frac{d}{dt}\underline{x} = \begin{bmatrix} 0 & 1 & 0 \\ -100 & 0 & 40 \\ 10 & 0 & -10 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \underline{x} + [0]\, u$$

→ Put the following 3rd order systems into standard state space form: (cont)

$\boxed{\text{system \# 2}}$

(b) balance eqn $$\frac{mc_2}{k_2}\frac{d^3z}{dt^3} + m\frac{d^2z}{dt^2} + \left(\frac{k_1+k_2}{k_2}\right)c_2\frac{dz}{dt} + k_1 z = u$$

response $$y = \frac{c_2}{k_2}\frac{dz}{dt} + z$$

→ let $m = 1000\,kg$ $k_1 = 60\,KN/m$ $k_2 = 40\,KN/m$ $c_2 = 4\,KN\text{-}s/m$

then $$\boxed{\frac{d^3z}{dt^3} + 10\frac{d^2z}{dt^2} + 100\frac{dz}{dt} + 600z = 10u}$$

$$\boxed{y = \frac{1}{10}\frac{dz}{dt} + z}$$

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let $x_1 = z$ $x_2 = \frac{dz}{dt} = \frac{dx_1}{dt}$ $x_3 = \frac{d^2z}{dt} = \frac{dx_2}{dt}$

$\therefore$ $\frac{dx_1}{dt} = x_2$ and $y = \frac{1}{10}x_2 + x_1$

$\frac{dx_2}{dt} = x_3$

$\frac{dx_3}{dt} = -600x_1 - 100x_2 - 10x_3 + 10u$

or
$$\boxed{\begin{aligned}\frac{d}{dt}\underline{x} &= \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -600 & -100 & -10\end{bmatrix}\underline{x} + \begin{bmatrix}0\\0\\10\end{bmatrix}u \\ y &= \begin{bmatrix}1 & \frac{1}{10} & 0\end{bmatrix}\underline{x} + [0]\,u\end{aligned}}$$

ans

System # 3

(c) balance $\frac{m c_2}{k_2} \frac{d^3 w}{dt^3} + m \frac{d^2 w}{dt^2} + \left(\frac{k_1 + k_2}{k_2}\right) c_2 \frac{dw}{dt} + k_1 w = \frac{c_2}{k_2}\frac{du}{dt} + u$

response is simply $y(t) = w(t)$

with data $m = 1000$ kg $\quad k_1 = 60$ kN/m $\quad k_2 = 40$ kN/m $\quad c_2 = 4$ kN-s/m

we have $\frac{m c_2}{k_2} = \left(\frac{1}{10}\right)\left(10^3\right)$ kg-s $\qquad \frac{k_1 + k_2}{k_2} c_2 = 10 \frac{\text{kN-s}}{\text{m}}$

and $\frac{c_2}{k_2} = \frac{1}{10}$ s

then $$\boxed{\underbrace{\frac{d^3 w}{dt^3} + 10}_{a_1} \frac{d^2 w}{dt^2} + \underbrace{100}_{a_2} \frac{dw}{dt} + \underbrace{600}_{a_3} w = \underbrace{\frac{du}{dt}}_{b_2} + \underbrace{10}_{b_3} u}$$

$b_0 = b_1 = 0$

from standard form $x_n = \frac{d}{dt} x_{n-1} - \beta_{n-1} u$ for $j > 1$

with $x_1 = y - \beta_0 u$ $\quad \therefore \quad x_1 = w - \beta_0 u$

$x_2 = \frac{dw}{dt} - \beta_0 \frac{du}{dt} - \beta_1 u$

and $\beta_n = b_n - a_1 \beta_{n-1} \cdots - a_n \beta_n$ $\qquad x_3 = \frac{d^2 w}{dt^2} - \beta_0 \frac{d^2 u}{dt^2} - \beta_1 \frac{du}{dt} - \beta_2 u$

$\therefore \beta_0 = b_0 = 0$

$\beta_1 = b_1 - a_1 \beta_0 = 0$

$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0 = 1$

$\beta_3 = b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0$

$= 10 - 10(1)$

$= 0$

$\therefore$ $x_1 = w$ — here w is the dependent variable

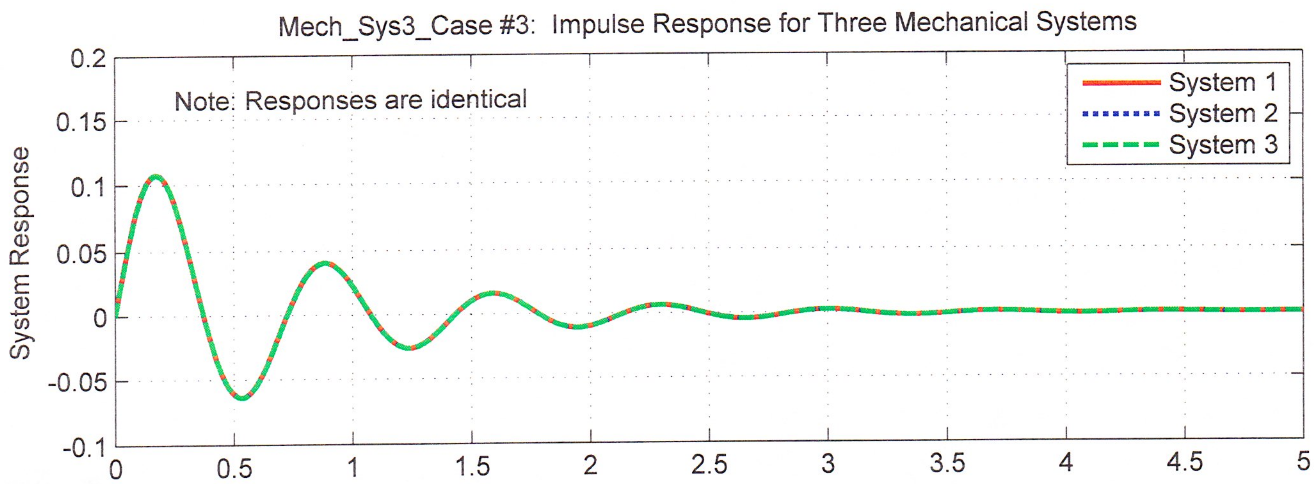
$x_2 = \frac{dw}{dt} = \frac{dx_1}{dt}$

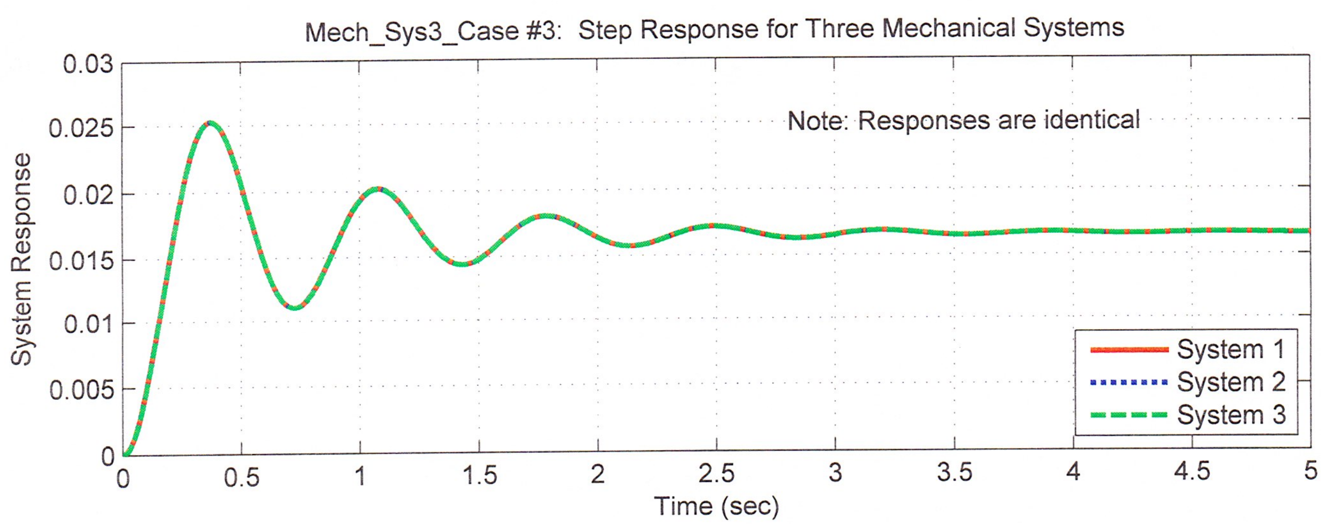
$x_3 = \frac{d^2 w}{dt^2} - u = \frac{dx_2}{dt} - u$

$$\therefore \boxed{\begin{aligned} \frac{d}{dt}\underline{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -600 & -100 & -10 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \underline{x} + [0]\, u \end{aligned}}$$

→ See Matlab listing, diary file, and plots

These 3 systems are indeed identical !!! OK


Mech_Sys3_Case #3: Impulse Response for Three Mechanical Systems
Note: Responses are identical
System 1
System 2
System 3
System Response
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


Mech_Sys3_Case #3: Step Response for Three Mechanical Systems
Note: Responses are identical
System 1
System 2
System 3
System Response
0.03
0.025
0.02
0.015
0.01
0.005
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Time (sec)

```
>> mech_sys3_cs3
*** Various Forms of a 3rd Order Mechanical System (Case 3 Data) ***
State space representation and eigenvalues for System 1
A1 =
     0     1     0
  -100     0    40
    10     0   -10
B1 =
     0
     1
     0
C1 =
     1     0     0
D1 =
     0
e1 =
 -1.2898e+000 +8.8991e+000i
 -1.2898e+000 -8.8991e+000i
 -7.4204e+000
State space representation and eigenvalues for System 2
A2 =
     0     1     0
     0     0     1
  -600  -100   -10
B2 =
     0
     0
    10
C2 =
  1.0000e+000  1.0000e-001            0
D2 =
     0
e2 =
 -7.4204e+000
 -1.2898e+000 +8.8991e+000i
 -1.2898e+000 -8.8991e+000i
State space representation and eigenvalues for System 3
A3 =
     0     1     0
     0     0     1
  -600  -100   -10
B3 =
     0
     1
     0
C3 =
     1     0     0
D3 =
     0
e3 =
 -7.4204e+000
 -1.2898e+000 +8.8991e+000i
 -1.2898e+000 -8.8991e+000i
>>
```

```
%
%  MECH_SYS3_CS3.M  Various Forms of a 3rd Order Mechanical System (Case 3)
%
%  The purpose of this file is to show the equivalency of three different
%  state space representations of the same 3rd order mechanical system.  The
%  state equations are derived in the written development (see notes) -- here
%  we simply implement the three different forms and show that:
%     1. the eigenvalues of the state matrix are identical
%     2. the impulse responses of the three systems are identical
%     3. the step responses of the three systems are identical
%  where all the simulations assume zero initial conditions.
%
%  File prepared by J. R. White, UMass-Lowell (Feb. 2014)
%

%
    clear all,   close all,   nfig = 0;
%
%  define the A, B, C, and D matrices for each system (Case 3 Data)
    A1 = [0 1 0;-100 0 40;10 0 -10];    B1 = [0 1 0]';
    C1 = [1 0 0];                       D1 = 0;
%
    A2 = [0 1 0;0 0 1;-600 -100 -10];   B2 = [0 0 10]';
    C2 = [1 0.10 0];                    D2 = 0;
%
    A3 = [0 1 0;0 0 1;-600 -100 -10];   B3 = [0 1 0]';
    C3 = [1 0 0];                       D3 = 0;
%
%  compute eigenvalues
    format compact
    disp('*** Various Forms of a 3rd Order Mechanical System (Case 3 Data) ***')
    e1 = eig(A1);     e2 = eig(A2);     e3 = eig(A3);
    disp('State space representation and eigenvalues for System 1')
    A1, B1, C1, D1, e1
    disp('State space representation and eigenvalues for System 2')
    A2, B2, C2, D2, e2
    disp('State space representation and eigenvalues for System 3')
    A3, B3, C3, D3, e3
%
%  simulate different time domain responses (for zero initial conditions)
    t = linspace(0,5,501);
    sys1 = ss(A1,B1,C1,D1);   y1i = impulse(sys1,t);    y1s = step(sys1,t);
    sys2 = ss(A2,B2,C2,D2);   y2i = impulse(sys2,t);    y2s = step(sys2,t);
    sys3 = ss(A3,B3,C3,D3);   y3i = impulse(sys3,t);    y3s = step(sys3,t);
%
%  plot impulse responses
    nfig = nfig+1;    figure(nfig)
    subplot(2,1,1),plot(t,y1i,'r-',t,y2i,'b:',t,y3i,'g--','LineWidth',2),grid
    title('Mech\_Sys3\_Case #3:  Impulse Response for Three Mechanical Systems')
    ylabel('System Response')
    text(0.25,0.165,'Note: Responses are identical')
    legend('System 1','System 2','System 3','Location','NorthEast')
```

```
%
%  plot step responses
     subplot(2,1,2),plot(t,y1s,'r-',t,y2s,'b:',t,y3s,'g--','LineWidth',2),grid
     title('Mech\_Sys3\_Case #3:  Step Response for Three Mechanical Systems')
     xlabel('Time (sec)'),ylabel('System Response')
     text(2.8,.025,'Note: Responses are identical')
     legend('System 1','System 2','System 3','Location','SouthEast')
%
%  end of program
```