Given $\underline{\underline{A}} = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

→ Calc det $\underline{\underline{A}}$

$$\det \underline{\underline{A}} = \sum_j a_{ij} c_{ij} \quad \leftarrow \text{expanding along row 1}$$

where $c_{ij} = (-1)^{i+j} M_{ij}$, M_{ij} = minor of ij element

$$= (-1)(+1) M_{13}$$

$$= -\left[0 + 2 \right] = \boxed{-2} \quad \underline{\text{ans}}$$

→ Calc $\underline{\underline{A}}^{-1}$

$$\underline{\underline{A}}^{-1} = \frac{\underline{\underline{C}}^T}{\det \underline{\underline{A}}} \quad \text{where } \underline{\underline{C}} = \text{cofactor matrix}$$

$$\underline{\underline{C}} = \begin{bmatrix} 1 & -(6) & 2 \\ -(0) & -2 & -(0) \\ 1 & -(2) & 0 \end{bmatrix} = \begin{bmatrix} 1 & -6 & 2 \\ 0 & -2 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

$$\therefore \underline{\underline{A}}^{-1} = -\frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ -6 & -2 & -2 \\ 2 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ 3 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}} \quad \underline{\text{ans}}$$

check $\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}$ ok

not required

→ Estimate new determinant if a_{11} becomes .5

$$\frac{d}{da_{ij}} (\det \underline{\underline{A}}) = \frac{d}{da_{ij}} \sum_j a_{ij} c_{ij} \quad \text{for row } i$$

$$= c_{ij}$$

$$\therefore \Delta \det A = \frac{d}{da_{ij}} (\det \underline{\underline{A}})\, \Delta a_{ij} = c_{ij}\, \Delta a_{ij}$$

$\therefore$ for present case $c_{11} = 1$ and $\Delta a_{ij} = .5 - 0 = .5$

$$\therefore \Delta \det A \simeq .5$$

and $\det \underline{\underline{A}}' = \det \underline{\underline{A}} + \Delta \det A = \boxed{-1.5}$ ans

→ note that this is an exact calc as verified by direct substitution

Given $\underline{\underline{A}} = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

→ calculate the eigenvalues and eigenvectors of $\underline{\underline{A}}$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{bmatrix} -\lambda & 0 & -1 \\ 2 & 1-\lambda & 2 \\ -2 & 0 & 1-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) &= -\lambda(1-\lambda)(1-\lambda) - 1(2)(1-\lambda) \\ &= (1-\lambda)\left[-\lambda + \lambda^2 - 2\right] \\ &= (1-\lambda)\left[(\lambda-2)(\lambda+1)\right] \\ &= 0 \end{aligned}$$

$\therefore \boxed{\lambda_1 = -1 \qquad \lambda_2 = 1 \qquad \lambda_3 = 2}$ ← eigenvalues

eigenvectors

for $\lambda_1 = -1$ $\quad (\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\underline{x}_1 = 0$

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_1 = 0 \qquad \therefore \boxed{\underline{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}} \text{ ans}$$

$x_1 - x_3 = 0 \qquad x_1 = x_3$

$2(x_1 + x_2 + x_3) = 0 \qquad x_2 = -(x_1 + x_3) = -2x_1$

for $\lambda_2 = 1$ $\quad (\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})\underline{x}_2 = 0$

$$\begin{bmatrix} -1 & 0 & -1 \\ 2 & 0 & 2 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_2 = 0 \qquad \therefore \boxed{\underline{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \text{ ans}$$

$-x_1 - x_3 = 0 \qquad x_1 = -x_3$

$-2x_1 = 0 \qquad x_1 = 0 \qquad x_2 = \text{arbitrary}$

for $\lambda_3 = 2$ $\quad (\underline{\underline{A}} - \lambda_3 \underline{\underline{I}})\underline{x}_3 = 0$

$$\begin{bmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_3 = 0 \qquad \therefore \boxed{\underline{x}_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}} \text{ ans}$$

$-2x_1 - x_3 = 0 \qquad x_3 = -2x_1$

$2x_1 - x_2 + 2x_3 = 0 \qquad x_2 = 2(x_1 + x_3) = -2x_1$

Given $\underline{\underline{A}} = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

→ Calc $e^{\underline{\underline{A}}t}$

$$e^{\underline{\underline{A}}t} = \sum_{i=1}^{N} P(\lambda_i) \prod_{j \neq i}^{N} \frac{(\underline{\underline{A}} - \lambda_j \underline{\underline{I}})}{(\lambda_i - \lambda_j)}$$ for distinct eigenvalues

since $\lambda_1 = -1 \quad \lambda_2 = 1 \quad \lambda_3 = 2$

we have

for $i=1$ $\Rightarrow$ $$e^{-t}\left\{ \underbrace{\frac{\begin{bmatrix} -1 & 0 & -1 \\ 2 & 0 & 2 \\ -2 & 0 & 0 \end{bmatrix}}{-1-(1)}}_{j=2} \underbrace{\frac{\begin{bmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ -2 & 0 & -1 \end{bmatrix}}{-1-(2)}}_{j=3} \right\} = \frac{1}{6} e^{-t} \begin{bmatrix} 4 & 0 & 2 \\ -8 & 0 & -4 \\ 4 & 0 & 2 \end{bmatrix}$$

for $i=2$ $\Rightarrow$ $$e^{t}\left\{ \underbrace{\frac{\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \\ -2 & 0 & 2 \end{bmatrix}}{1-(-1)}}_{j=1} \underbrace{\frac{\begin{bmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ -2 & 0 & -1 \end{bmatrix}}{1-(2)}}_{j=3} \right\} = -\frac{1}{2} e^{t} \begin{bmatrix} 0 & 0 & 0 \\ -4 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for $i=3$ $\Rightarrow$ $$e^{2t}\left\{ \underbrace{\frac{\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \\ -2 & 0 & 2 \end{bmatrix}}{2-(-1)}}_{j=1} \underbrace{\frac{\begin{bmatrix} -1 & 0 & -1 \\ 2 & 0 & 2 \\ -2 & 0 & 0 \end{bmatrix}}{2-(1)}}_{j=2} \right\} = \frac{1}{3} e^{2t} \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

use this just so I can factor out a $\frac{1}{3}$

$$\therefore e^{\underline{\underline{A}}t} = \frac{1}{3} e^{-t} \begin{bmatrix} 2 & 0 & 1 \\ -4 & 0 & -2 \\ 2 & 0 & 1 \end{bmatrix} + \frac{1}{3} e^{t} \begin{bmatrix} 0 & 0 & 0 \\ 6 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{3} e^{2t} \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

or $$\boxed{e^{\underline{\underline{A}}t} = \frac{1}{3} \begin{bmatrix} 2e^{-t} + e^{2t} & 0 & e^{-t} - e^{2t} \\ -4e^{-t} + 6e^{t} - 2e^{2t} & 3e^{t} & -2e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & 0 & e^{-t} + 2e^{2t} \end{bmatrix}}$$ and

→ check with MATLAB for $t=1$

$$e^{\underline{\underline{A}}t} = \begin{bmatrix} 2.7083 & 0 & -2.3404 \\ .0200 & 2.7183 & 4.6808 \\ -4.6808 & 0 & 5.0427 \end{bmatrix}$$ ok

$$\boxed{\begin{aligned} e^{-1} &= .3679 \\ e^{1} &= 2.7183 \\ e^{2} &= 7.3891 \end{aligned}}$$

→ also for $t=0$ $e^{\underline{\underline{A}}t} \to I$ (ok)

Matrix Exponential (cont.)

$\rightarrow$ Compute $\frac{d}{dt}\{e^{\underline{\underline{A}}t}\}$

$$\frac{d}{dt}\{e^{\underline{\underline{A}}t}\} = \underline{\underline{A}}\,e^{\underline{\underline{A}}t}$$

$$\therefore \frac{d}{dt}\{e^{\underline{\underline{A}}t}\} = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}\left(\frac{1}{3}\right)\begin{bmatrix} 2e^{-t}+e^{2t} & 0 & e^{-t}-e^{2t} \\ -4e^{-t}+6e^{t}-2e^{2t} & 3e^{t} & -2e^{-t}+2e^{2t} \\ 2e^{-t}-2e^{2t} & 0 & e^{-t}+2e^{2t} \end{bmatrix}$$

or

$$\boxed{\frac{d}{dt}\{e^{\underline{\underline{A}}t}\} = \frac{1}{3}\begin{bmatrix} -2e^{-t}+2e^{2t} & 0 & -e^{-t}-2e^{2t} \\ 4e^{-t}+6e^{t}-4e^{2t} & 3e^{t} & 2e^{-t}+4e^{2t} \\ -2e^{-t}-4e^{2t} & 0 & -e^{-t}+4e^{2t} \end{bmatrix}}$$ ans

$\rightarrow$ note that if we differentiate each term of $e^{\underline{\underline{A}}t}$, we get the same result. OK

$\rightarrow$ Compute $\int e^{\underline{\underline{A}}t}\,dt$

$$\int e^{\underline{\underline{A}}t}\,dt = \underline{\underline{A}}^{-1}e^{\underline{\underline{A}}t}$$

and from before $\underline{\underline{A}}^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ -6 & -2 & -2 \\ 2 & 0 & 0 \end{bmatrix}$

$$\therefore \int e^{\underline{A}t}\,dt = -\frac{1}{6}\begin{bmatrix} 1 & 0 & 1 \\ -6 & -2 & -2 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2e^{-t}+e^{2t} & 0 & e^{-t}-e^{2t} \\ -4e^{-t}+6e^{t}-2e^{2t} & 3e^{t} & -2e^{-t}+2e^{2t} \\ 2e^{-t}-2e^{2t} & 0 & e^{-t}+2e^{2t} \end{bmatrix}$$

or

$$\boxed{\int e^{\underline{A}t}\,dt = -\frac{1}{6}\begin{bmatrix} 4e^{-t}-e^{2t} & 0 & 2e^{-t}+e^{2t} \\ -8e^{-t}-12e^{t}+2e^{2t} & -6e^{t} & -4e^{-t}-2e^{2t} \\ 4e^{-t}+2e^{2t} & 0 & 2e^{-t}-2e^{2t} \end{bmatrix}}$$ ans

$\rightarrow$ note that if we integrate each term of $e^{\underline{\underline{A}}t}$, we get the same result. OK

Functions of a Square Matrix

Given $A = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

→ Compute A^2

<u>Traditional Approach</u>

$$\begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 0 & -1 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}}$$ <u>ans</u>

<u>Sylvester's Thm</u> (from calc of e^{At})

$$A^2 = (-1)^2 \frac{1}{3}\begin{bmatrix} 2 & 0 & 1 \\ -4 & 0 & -2 \\ 2 & 0 & 1 \end{bmatrix} + (1)^2 \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 6 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ (2)^2 \frac{1}{3}\begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 6 & 0 & -3 \\ -6 & 3 & 6 \\ -6 & 0 & 9 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 0 & -1 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}}$$ OK

→ Compute $\sqrt{A}$

Using Sylvester's Thm again

$$\sqrt{A} = \frac{\sqrt{-1}}{3}\begin{bmatrix} 2 & 0 & 1 \\ -4 & 0 & -2 \\ 2 & 0 & 1 \end{bmatrix} + \frac{\sqrt{1}}{3}\begin{bmatrix} 0 & 0 & 0 \\ 6 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\sqrt{2}}{3}\begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

i

$$= \frac{1}{3}\begin{bmatrix} \sqrt{2} + \sqrt{-4} & 0 & -\sqrt{2} + \sqrt{-1} \\ \sqrt{36} - \sqrt{8} - \sqrt{-16} & 3 & \sqrt{8} - \sqrt{-4} \\ -\sqrt{8} + \sqrt{-4} & 0 & \sqrt{8} + \sqrt{-1} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 0.4714 + 0.6667i & 0 & -0.4714 + 0.3333i \\ 1.0572 - 1.3333i & 1 & 0.9428 - 0.6667i \\ -0.9428 + 0.6667i & 0 & 0.9428 + 0.3333i \end{bmatrix}}$$ <u>ans</u>

<u>OK</u> checked with MATLAB

Matrix Diagonalization

Given $A = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$ with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$

and eigenvectors $\underline{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ $\underline{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\underline{x}_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

from Prob #1

→ Show that $\underline{\underline{D}} = \underline{\underline{M}}^{-1} \underline{\underline{A}} \, \underline{\underline{M}}$

where $\underline{\underline{M}}$ = modal matrix and $\underline{\underline{D}}$ = diagonal matrix of eigenvalues

---

from above $\underline{\underline{M}} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix}$

need to find $\underline{\underline{M}}^{-1}$ where $\underline{\underline{M}}^{-1} = \dfrac{\underline{\underline{C}}^T}{\det \underline{\underline{M}}}$

$$\underline{\underline{C}} = \begin{bmatrix} -2 & -(6) & -1 \\ -(0) & -3 & -(0) \\ -1 & -(0) & 1 \end{bmatrix} \quad \text{or} \quad \underline{\underline{C}}^T = \begin{bmatrix} -2 & 0 & -1 \\ -6 & -3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$\det \underline{\underline{M}} = 1(-2) + 1(-1) = \boxed{-3}$

$$\therefore \boxed{\underline{\underline{M}}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 0 & 1 \\ 6 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix}}$$

check

$$\underline{\underline{M}}^{-1} \underline{\underline{M}} = \frac{1}{3} \begin{bmatrix} 2 & 0 & 1 \\ 6 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

ok

Now compute $\underline{\underline{M}}^{-1} \underline{\underline{A}} \, \underline{\underline{M}}$

$$\underline{\underline{A}} \, \underline{\underline{M}} = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & -4 \\ -1 & 0 & -4 \end{bmatrix}$$

$$\underline{\underline{M}}^{-1} \underline{\underline{A}} \, \underline{\underline{M}} = \frac{1}{3} \begin{bmatrix} 2 & 0 & 1 \\ 6 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & -4 \\ -1 & 0 & -4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\therefore \boxed{\underline{\underline{D}} = \underline{\underline{M}}^{-1} \underline{\underline{A}} \, \underline{\underline{M}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

ok

```
%
%   MATRIX_DEMO_CS3.M  Demo for performing several simple matrix operations
%                          in Matlab (Case 3)
%
%   This file does a number of simple matrix operations in Matlab.  These same
%   operations were done by hand for validation purposes.
%
%   File prepared by J. R. White, UMass-Lowell (Jan. 2014)
%

      clear all,   close all,   nfig = 0;
%
%   define matrix and perform desired operations (use Case 3 data)
      format compact
      A = [0 0 -1; 2 1 2; -2 0 1];
      disp('Given matrix:'),                       A
      disp('determinant'),                         det(A)
      disp('inverse'),                             inv(A)
      disp('eigenvalues and eigenvectors'),        [M,D] = eig(A)
      disp('matrix exponential = exp(At) for t=1'),expm(A)
      disp('derivative of exp(At) for t=1'),       A*expm(A)
      disp('integral of exp(At) for t=1'),         inv(A)*expm(A)
      disp('A*A'),                                 A*A
      disp('sqrt(A)'),                             sqrtm(A)
      disp('validate D = inv(M)*A*M'),             inv(M)*A*M
%
%   for fun, let's also generate exp(At) using the symbolic toolbox (not required)
      syms t
      S = expm(A*t)
      pretty(S)
%
%  end of simulation
```

```
>> matrix_demo_cs3
iven matrix:
A =
     0     0    -1
     2     1     2
    -2     0     1
determinant
ans =
    -2
inverse
ans =
   -0.5000         0   -0.5000
    3.0000    1.0000    1.0000
   -1.0000         0         0
eigenvalues and eigenvectors
M =
         0   -0.4082    0.3333
    1.0000    0.8165   -0.6667
         0   -0.4082   -0.6667
D =
     1     0     0
     0    -1     0
     0     0     2
matrix exponential = exp(At) for t=1
ans =
    2.7083   -0.0000   -2.3404
    0.0200    2.7183    4.6808
   -4.6808         0    5.0487
derivative of exp(At) for t=1
ans =
    4.6808         0   -5.0487
   -3.9250    2.7183   10.0973
  -10.0973    0.0000    9.7294
integral of exp(At) for t=1
ans =
    0.9863    0.0000   -1.3541
    3.4641    2.7183    2.7083
   -2.7083    0.0000    2.3404
A*A
ans =
     2     0    -1
    -2     1     2
    -2     0     3
sqrt(A)
ans =
   0.4714 + 0.6667i  -0.0000             -0.4714 + 0.3333i
   1.0572 - 1.3333i   1.0000 - 0.0000i    0.9428 - 0.6667i
  -0.9428 + 0.6667i   0.0000              0.9428 + 0.3333i
validate D = inv(M)*A*M
 ns =
    1.0000         0         0
```

```
         0   -1.0000   -0.0000
         0         0    2.0000

S =
[                2/(3*exp(t)) + exp(2*t)/3,      0,     1/(3*exp(t)) - exp(2*t)/3]
[ 2*exp(t) - (2*exp(2*t))/3 - 4/(3*exp(t)), exp(t), (2*exp(2*t))/3 - 2/(3*exp(t))]
[            2/(3*exp(t)) - (2*exp(2*t))/3,      0, 1/(3*exp(t)) + (2*exp(2*t))/3]

  +-                                                                   -+
  |          2        exp(2 t)                       1        exp(2 t)  |
  |      -------- + --------,              0,    -------- - --------    |
  |      3 exp(t)       3                        3 exp(t)       3       |
  |                                                                     |
  |            2 exp(2 t)       4                2 exp(2 t)       2     |
  |  2 exp(t) - ---------- - --------, exp(t), ---------- - --------    |
  |                 3         3 exp(t)               3         3 exp(t) |
  |                                                                     |
  |        2        2 exp(2 t)                    1        2 exp(2 t)   |
  |    -------- - ----------,              0,  -------- + ----------    |
  |    3 exp(t)         3                      3 exp(t)         3       |
  +-                                                                   -+
>>
```