System Dynamics (22.554 and 24.509)
Exam #2 Spring 2014

Problem #1. (20 points)

Consider the thick-walled refrigerated compartment shown in the rough sketch given below. In a simple model of this system, we have the following variable definitions:

$T_c(t)$ = compartment temperature

$T_w(t)$ = wall temperature

$T_a(t)$ = ambient temperature

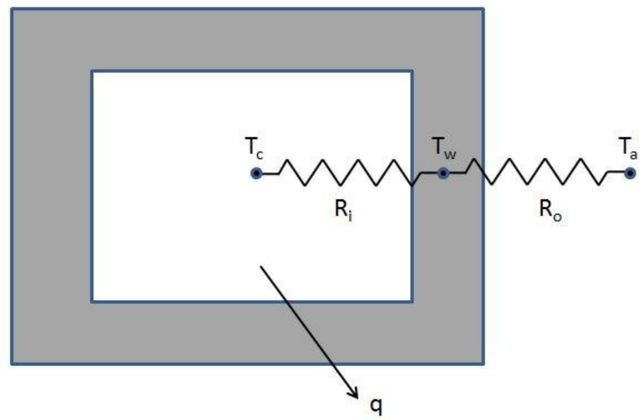
$q(t)$ = energy removal rate from compartment

m_c = mass of air in refrigerated compartment

m_w = mass of wall material

c_c, c_w = material specific heats

R_i, R_o = effective thermal resistances



a. Assuming that $T_a(t)$ and $q(t)$ are independent inputs to this system, develop the energy balance equations for this system (i.e. for the wall and refrigerated compartment). Also, with $T_c(t)$ as the only model output response, put these balance equations into state-space form -- that is, explicitly define the A, B, C, and D matrices for this LTI system.

b. Now modify the system in Part a such that the compartment temperature is controlled by extracting energy from the compartment interior based on the difference between some desired reference temperature, $T_{ref}(t)$, and the current compartment temperature. In mathematical form, this gives

$$q(t) = \begin{cases} k\left(T_c(t) - T_{ref}(t)\right) & T_c > T_{ref} \\ 0 & T_c \le T_{ref} \end{cases}$$

where k is a proportionality constant.

If the outputs of this new system are $T_c(t)$ and $q(t)$, put this new model into state-space form and briefly describe how you would simulate this system using a standard ODE solver.

c. Modeling the compartment as simply air is not very realistic -- we often must model the "stuff" being cooled that is put into the cold environment. For example, assume that the cold chamber is used in a large supermarket to keep the meat chilled before it is put out in the display cases for the consumer. In particular, how would you modify the system in Part b if 450 kg of freshly cut steaks were placed on a series of thin aluminum racks within the otherwise empty cooling chamber?

Note that a new set of state equations is NOT required here, but a good description of what is necessary to model this new system is expected. Be as explicit as possible so that the reader can easily understand and visualize the resultant model for the situation described here...

Problem #2. (15 points)

The eigenvalues of a 3rd order closed loop system are as follows (where time is measured in seconds):

$$\lambda_1 = -3.0 \qquad \lambda_{2,3} = -0.42 \pm j0.428$$

For this system, answer the following questions:

a. Explain why the complex conjugate pair of eigenvalues is referred to as "dominant", and estimate the rise time, settling time, and maximum overshoot that would be associated with the step response of this system. Show your work.

b. Estimate the number of oscillations (i.e. number of oscillatory cycles) observed before the system settles to within ±1% of the final value. Again, explain your logic…

c. Would the expected step response of the system change significantly if $\lambda_1 = -0.1$ sec^{-1} instead of the value given above? Would the response time be faster or slower than in the original system? Explain…

Problem #3. (20 points)

We showed that one way to evaluate the matrix exponential is via the expression

$$e^{\underline{\underline{A}}t} = L^{-1}\left\{\left(s\underline{\underline{I}} - \underline{\underline{A}}\right)^{-1}\right\}$$

a. Derive this result by formally taking the Laplace transform of $e^{\underline{\underline{A}}t}$ using the basic definition of a Laplace transform. Be formal in your development.

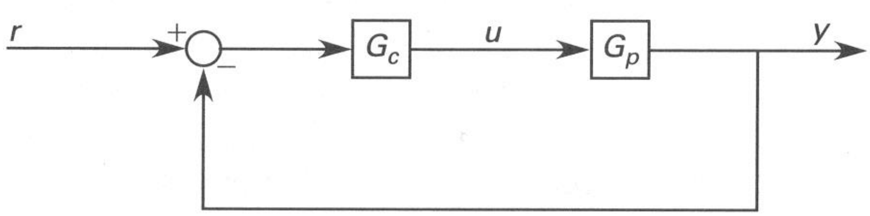
b. Based on this result, determine an explicit closed form expression for $e^{\underline{\underline{A}}t}$ given that

$$\underline{\underline{A}} = \begin{bmatrix} -2 & -5/2 \\ 2 & 0 \end{bmatrix}$$

Problem #4. (25 points)

Consider the plant and control system shown in the diagram, where G_c is the controller transfer function and G_p is the plant transfer function and

$$G_c(s) = \frac{K}{s+2} \qquad \text{and} \qquad G_p(s) = \frac{1}{10s+1}$$



Open Loop System:

a. Determine the response of just the plant (denoted by G_p) due to a unit step input in u(t) -- this is the step response of the open loop plant (that is, determine y(t) due to a step change in u(t) assuming that the feedback loop is not present).

b. Sketch the response computed in Part a and determine how long the open loop response takes to get to 90% of the final value.

Closed Loop System:

c. Determine the value of K in the controller transfer function so that the impulse response (i.e. natural response) of the closed loop system is critically damped (i.e. gives repeated roots). For this value of K, what is the closed loop transfer function? Show your work.

d. For the correct value of K from Part c, the closed loop transfer function can be approximated by

$$G(s) \approx \frac{0.9}{(s+1)^2}$$

Note: This is only an approximation to the result from Part c, but it simplifies subsequent calculations.

Using this approximate closed loop transfer function, determine the response of the overall system to a unit step change in the setpoint, r(t). Does your result seem reasonable?

e. Estimate the steady state error for this particular design. Also, estimate the time needed to reach 90 % of the final value for the closed loop step response of the system. Compare this result to that for the open loop step response and explain the purpose of the controller in this example.

Problem #5. (20 points)

Consider the system defined by

$$\frac{d}{dt}\underline{x} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u} \qquad \text{and} \qquad \underline{y} = \underline{\underline{C}}\,\underline{x}$$

where

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -6 & 0 \end{bmatrix} \qquad \underline{\underline{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \underline{\underline{C}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The goal here is to design a full-order state observer. The error dynamics associated with the state observer is given by

$$\frac{d}{dt}\underline{e} = \left(\underline{\underline{A}} - \underline{\underline{L}}\,\underline{\underline{C}}\right)\underline{e}$$

where $\underline{e} = \underline{x} - \hat{\underline{x}}$ is the error vector.

The observer gain matrix, $\underline{\underline{L}}$, can be chosen to place the observer poles at any desired location if the observability matrix, $\underline{\underline{H}}$, has rank n (n is the order of the system), where $\underline{\underline{H}}$ is given by

$$\underline{\underline{H}} = \begin{bmatrix} \underline{\underline{C}} \\ \underline{\underline{C}}\,\underline{\underline{A}} \\ \underline{\underline{C}}\,\underline{\underline{A}}^2 \\ \mathrm{M} \\ \underline{\underline{C}}\,\underline{\underline{A}}^{n-1} \end{bmatrix}$$

a. Within this context, we know that the pole placement method can only be applied if the system is completely state observable. Based on your understanding of this term/concept, do you think the above system is indeed state observable? Explain your logic. This should be a rationalization based on the physical interactions in the system, not on a formal mathematical proof. Be explicit…

b. Now actually compute the observability matrix for the above system and show that its rank is n = 3 -- be sure to show your work. Is this consistent with your rationalization from Part a?

c. Assuming state observability (this is the result you should have demonstrated above), determine the observer gain matrix that gives observer poles at

$$\mu_1 = -10, \quad \mu_2 = -10, \quad \text{and} \quad \mu_3 = -15$$

Again, show your work and be careful with the arithmetic!!!