Plotting $\phi(E)$ and $\phi(u)$

a. Using the data given for T = 300 C, carefully plot $\phi(E)$ vs. E and $\phi(u)$ vs. E for this situation, and clearly describe to the reader the behavior observed in each plot. Do the profiles make sense? Explain…

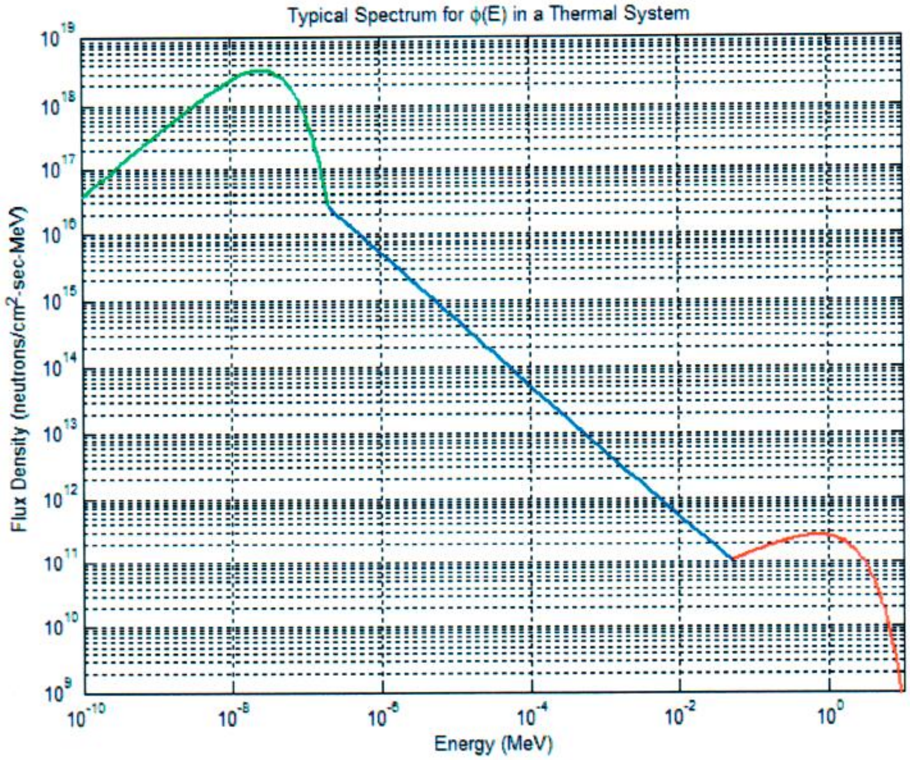
b. How does this flux spectrum compare to the one given in the Lectures Notes for T = 20 C -- that is, "Is it a harder or softer spectrum (be sure to explain what is meant by this terminology)?

Discussion for Parts a and b of problem (see figures below):

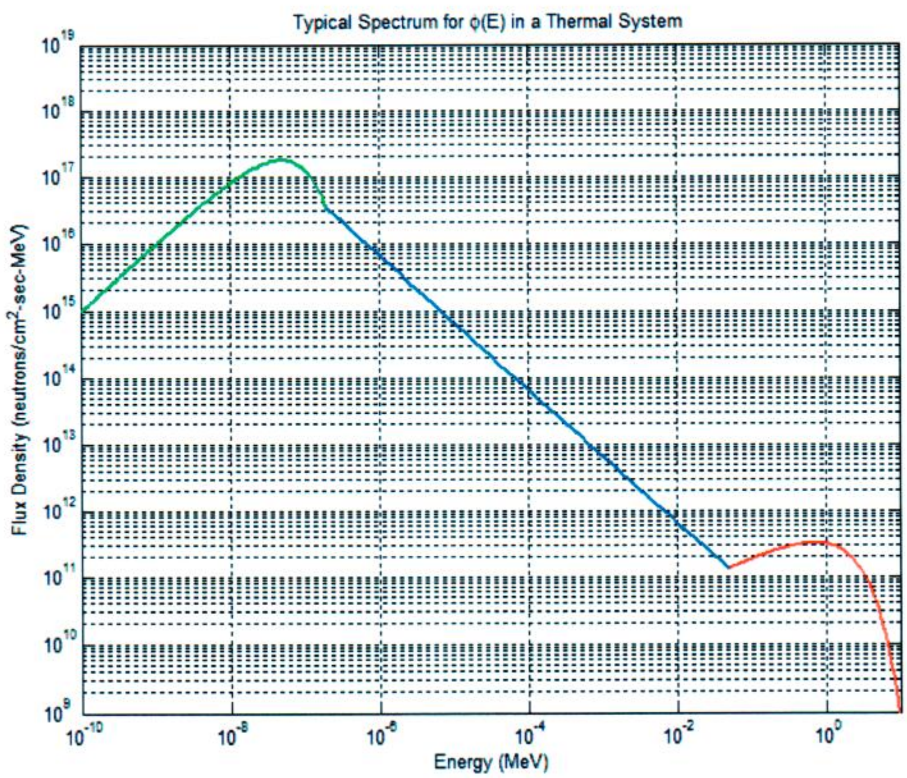
As described in the Lecture Notes, the $\phi(E)$ vs E "distribution clearly shows the three distinct spectrum functions that make up the composite $\phi(E)$ spectrum, with the fission spectrum at high energies, a clearly observable 1/E behavior at intermediate energies, and a Maxwellian distribution at low energy." However, for the spectrum at T = 300 C, the thermal component has a smaller contribution relative to the composite spectrum given in the Lecture Notes for T = 20 C because the temperature is larger and the thermal Maxwellian broadens as the temperature increases. This is simply due the increased kinetic energy of the "dilute" neutron gas which is at thermal equilibrium with its environment. As the neutron energy increases (on the average), the spectrum shifts to the right (i.e. higher energy) and this is referred to as hardening the spectrum. Thus, in comparing the two $\phi(E)$ plots, the one for T = 300 C has a larger epi-thermal and fast component, and a much lower thermal contribution -- and this is exactly as expected because of the increased (harder) average energy of the thermal neutrons.

Concerning the flux per unit lethargy plots [i.e. $\phi(u)$ vs. E], since $\phi(u) = E\phi(E)$, we expect the high energy region to be increased relative to the change in $\phi(E)$ at lower energies and, of course, since $\phi(E)$ is proportional to 1/E in the intermediate energy ranges, the $\phi(u)$ profile is constant at these energies. Thus, the composite $\phi(u)$ profile is quite different from the $\phi(E)$ curve, but the observed differences are easily explainable. Finally concerning the $\phi(u)$ profiles at two different temperatures, the above discussion about the shift in the Maxwellian spectrum still applies, and the decrease in the "thermal" component and the increase in the intermediate and fast contributions are quite apparent -- even more so than in the $\phi(E)$ plots.

Note: In all cases, the total energy-integrated flux is 10^{12} neutrons/cm^2-s.


Typical Spectrum for φ(E) in a Thermal System
Flux Density (neutrons/cm²-sec-MeV)
Energy (MeV)
10^19
10^18
10^17
10^16
10^15
10^14
10^13
10^12
10^11
10^10
10^9
10^-10
10^-8
10^-6
10^-4
10^-2
10^0

T = 20 C


Typical Spectrum for φ(E) in a Thermal System
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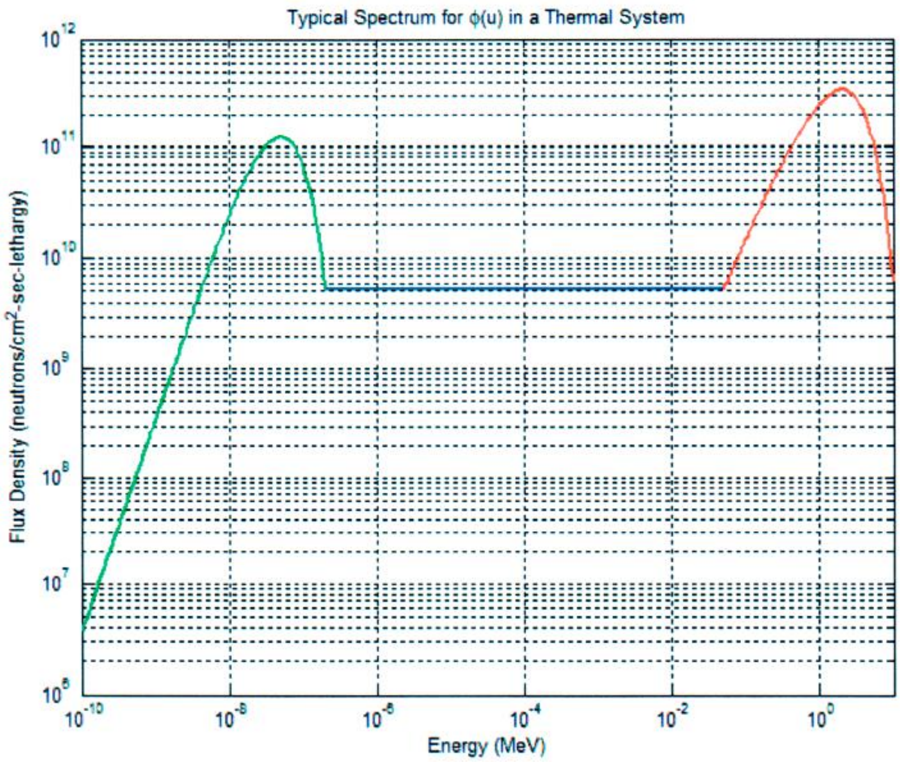
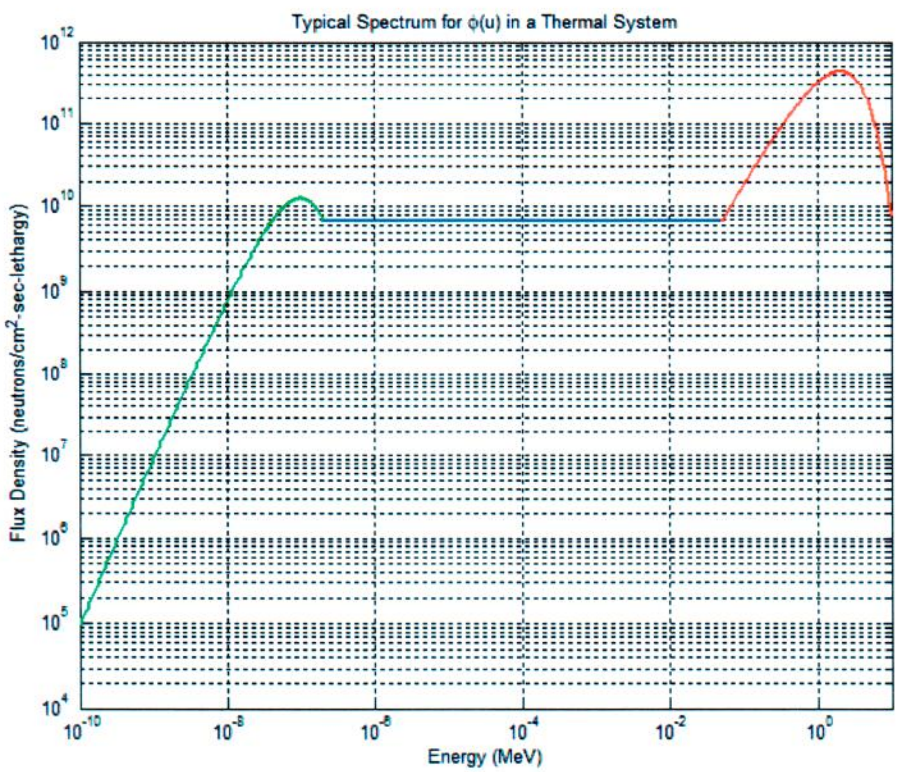
10^15
10^14
10^13
10^12
10^11
10^10
10^9
10^-10
10^-8
10^-6
10^-4
10^-2
10^0

T = 300 C



Typical Spectrum for φ(u) in a Thermal System
Flux Density (neutrons/cm2-sec-lethargy)
Energy (MeV)

T = 20 C



Typical Spectrum for φ(u) in a Thermal System
Flux Density (neutrons/cm2-sec-lethargy)
Energy (MeV)

T = 300 C

Neutron Current
+
Leakage

from Lamarsh

$\boxed{5-20}$ Given $\phi_T(x,y,z) = A\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\cos\left(\frac{\pi z}{a}\right)$

($a \rightarrow$ length of side of cube)

→ What is thermal neutron current?

$$\vec{J} = -D\vec{\nabla}\phi$$

where in Cartesian geometry

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\therefore \vec{J}(x,y,z) = DA\left(\frac{\pi}{a}\right)\Big[\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\cos\left(\frac{\pi z}{a}\right)\hat{i} + \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)\cos\left(\frac{\pi z}{a}\right)\hat{j} + \cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right)\sin\left(\frac{\pi z}{a}\right)\hat{k}\Big]$$

ans

→ What is the leakage from each side of the reactor?

$$\text{leakage} = \int_A \vec{J}\cdot\hat{n}\,dA$$

for rt. side (for example) $\hat{n}\,dA = dy\,dz\,\hat{i}$ } where $\hat{i}$ is the outward pointing normal vector and $dy\,dz$ represents a differential area

$$\therefore \begin{matrix}\text{leakage}\\ \text{rt. face}\end{matrix} = DA\left(\frac{\pi}{a}\right)\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}\int_{-a/2}^{a/2}\cos\frac{\pi y}{a}\,dy\int_{-a/2}^{a/2}\cos\frac{\pi z}{a}\,dz$$

and

$$\int_{-a/2}^{a/2}\cos\frac{\pi y}{a}\,dy = \frac{a}{\pi}\sin\frac{\pi y}{a}\Big|_{-a/2}^{a/2} = \frac{a}{\pi}\big(1-(-1)\big) = \frac{2a}{\pi}$$

$$\therefore \begin{matrix}\text{leakage}\\ \text{rt. face}\end{matrix} = DA\left(\frac{\pi}{a}\right)\left(\frac{2a}{\pi}\right)^2 = \boxed{\frac{4DA\,a}{\pi}}$$

ans

→ What is total leakage?

from symmetry total leakage = 6 * leakage from one face

$$\therefore \begin{matrix}\text{total thermal}\\ \text{leakage}\end{matrix} = \boxed{\frac{24DA\,a}{\pi}}$$

ans

For neutron balance

$$\text{Source} = \int_0^H Q(x)\,dx = \int_0^H A\, e^{-\alpha x}\,dx$$

$$= -\frac{A}{\alpha} e^{-\alpha x}\Big|_0^H = -\frac{A}{\alpha}\left(e^{-\alpha H} - 1\right)$$

$$= \boxed{\frac{A}{\alpha}\left(1 - e^{-\alpha H}\right)} \quad \underline{\text{total source}}$$

per unit area in yz plane

$$\text{absorbed} = \int_0^H \Sigma_a\, \phi(x)\,dx$$

$$= \Sigma_a \int_0^H \left(C_1 \cosh\frac{x}{L} + C_2 \sinh\frac{x}{L} + C_3 e^{-\alpha x}\right) dx$$

$$= \Sigma_a \left[C_1 L \sinh\frac{x}{L} + C_2 L \cosh\frac{x}{L} - \frac{C_3}{\alpha} e^{-\alpha x}\right]\Bigg|_0^H$$

$$= \Sigma_a \left[C_1 L \sinh\frac{H}{L} + C_2 L \cosh\frac{H}{L} - \frac{C_3}{\alpha} e^{-\alpha H} - C_2 L + \frac{C_3}{\alpha}\right] \quad \text{Total absorption}$$

leakage at $x = H$ $= \int \vec{J}\cdot\hat{n}\,dA = J(H)$ here $\hat{n} = \hat{i}$

$\int dA = \underline{1}$ per unit area

leakage at $x = 0$ $= \int \vec{J}\cdot\hat{n}\,dA = -J(0)$ here $\hat{n} = -\hat{i}$

where

$$J(x) = -D\frac{d}{dx}\phi(x) = -D\frac{d}{dx}\left[C_1 \cosh\frac{x}{L} + C_2 \sinh\frac{x}{L} + C_3 e^{-\alpha x}\right]$$

$$= -D\left[\frac{C_1}{L}\sinh\frac{x}{L} + \frac{C_2}{L}\cosh\frac{x}{L} - C_3\,\alpha\, e^{-\alpha x}\right]$$

rt face $\therefore$ leakage at $x = H$ $= -D\left[\frac{C_1}{L}\sinh\frac{H}{L} + \frac{C_2}{L}\cosh\frac{H}{L} - C_3\,\alpha\, e^{-\alpha H}\right]$

left face leakage at $x = 0$ $= +D\left[\frac{C_2}{L} - C_3\,\alpha\right]$

note $L^2 = \frac{D}{\Sigma_a}$

$\therefore D = L^2 \Sigma_a$

Given $\phi(r) = A \frac{\sin Br}{r}$

with $B = \frac{\pi}{R}$ and $R = R_0 + d$

(a) What is the neutron current?

$$\vec{J} = -D \vec{\nabla} \phi$$

but in spherical geometry, the gradient is given as

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \psi}\hat{a}_\psi$$

see Appendix III in Lamarsh (wikipedia on web)

For this case, there is no θ and ψ dependence, thus we have

$$\vec{J}(r) = -D \frac{\partial \phi}{\partial r}\hat{a}_r$$

and $\frac{d}{dr}\left[\frac{\sin Br}{r}\right] = \frac{1}{r} B\cos Br - \frac{1}{r^2}\sin Br$

$$\therefore \boxed{\vec{J}(r) = DA\left[\frac{\sin Br}{r^2} - \frac{B\cos Br}{r}\right]\hat{a}_r}$$

ans

(b) Compute the leakage at $r = R_0$ using the surface integral formulation

$$\text{leakage} = \int_A \vec{J}\cdot\hat{n}\, dA$$

surface location

for 1-D spherical geometry $\hat{n} = \hat{a}_r$ $\quad dA = R_0^2 \sin\theta\, d\theta\, d\psi$

and, since $\vec{J}$ is independent from θ and ψ, we have

$$\int dA = R_0^2 \underbrace{\int_0^\pi \sin\theta\, d\theta}_{2} \underbrace{\int_0^{2\pi} d\psi}_{2\pi} = 4\pi R_0^2$$

and, with $\hat{a}_r \cdot \hat{a}_r = 1$, we have

$$\boxed{\text{leakage} = 4\pi R_0^2 \left\{DA\left[\frac{\sin BR_0}{R_0^2} - \frac{B\cos BR_0}{R_0}\right]\right\}}$$

ans

c. Develop an expression for $\vec{\nabla}\cdot\vec{J}$ (divergence of $\vec{J}$)

$$\vec{\nabla}\cdot\vec{J} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J_r) + \underbrace{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, J_\theta)}_{\to 0} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(J_\phi)}_{\to 0}$$

see App III Lamarsh (wikipedia)

$$= \frac{1}{r^2}\frac{d}{dr}\left\{ r^2 DA\left[\frac{\sin Br}{r^2} - \frac{B\cos Br}{r}\right]\right\}$$

$$= \frac{DA}{r^2}\frac{d}{dr}\left\{\sin Br - Br\cos Br\right\}$$

$$= \frac{DA}{r^2}\left[\cancel{B\cos Br} - B\left(-Br\sin Br + \cancel{\cos Br}\right)\right]$$

$$\boxed{\vec{\nabla}\cdot\vec{J} = DAB^2\frac{\sin Br}{r}}$$

d. Compute leakage using the volume integral formulation

$$\text{leakage} = \int_V \vec{\nabla}\cdot\vec{J}\, d\vec{r}$$

here $d\vec{r} = r^2\sin\theta\, dr\, d\theta\, d\phi = 4\pi r^2 dr$ — see Part b for no θ and ϕ dependence

$$\therefore \text{leakage} = \int_0^{R_0} DAB^2\frac{\sin Br}{r}\, 4\pi r^2\, dr$$

$$= 4\pi DAB^2 \underbrace{\int_0^{R_0} r\sin Br\, dr}_{\left.\frac{1}{B^2}\sin Br - \frac{1}{B} r\cos Br\right|_0^{R_0}}$$

table lookup

$$\boxed{\text{leakage} = 4\pi DA\left[\sin BR_0 - BR_0\cos BR_0\right]}$$

e. Compare results for Parts b and d

clearly these are identical — That is

$$\int_A \vec{J}\cdot\hat{n}\, dA = \int_V \vec{\nabla}\cdot\vec{J}\, d\vec{r}$$

Divergence Thm

this calc validates the Divergence Thm for one specific case ...

a. Is upscatter possible within this group structure? Explain…

Yes -- upscatter can occur when there are multiple thermal groups (i.e. multiple low energy groups below about 1 eV). This occurs when a neutron gains energy when scattering from a target atom. In this case, scattering from group 4 to group 3 is possible. This, however, is the only upscatter of importance here.

b. Give explicit expressions for the inscatter rates to group 2 and to group 3.

inscatter to group 2: $\Sigma_{1\to2}\phi_1$ (downscatter from group 1)

inscatter to group 3: $\Sigma_{1\to3}\phi_1 + \Sigma_{2\to3}\phi_2 + \Sigma_{4\to3}\phi_4$ (includes downscatter from groups 1 and 2 and upscatter from group 4)

c. What is the removal rate from group 2? What about from group 3?

removal = absorption + outscatter

removal from group 2: $\Sigma_{a2}\phi_2 + \Sigma_{2\to3}\phi_2 + \Sigma_{2\to4}\phi_2 = (\Sigma_{a2} + \Sigma_{2\to3} + \Sigma_{2\to4})\phi_2 = \Sigma_{R2}\phi_2$

removal from group 3: $\Sigma_{a3}\phi_3 + \Sigma_{3\to4}\phi_3 = (\Sigma_{a3} + \Sigma_{3\to4})\phi_3 = \Sigma_{R3}\phi_3$

d. What is a reasonable distribution of numerical values for the multigroup fission spectrum? State any assumptions.

$\chi_1 = 1.0$ and $\chi_2 = \chi_3 = \chi_4 = 0.0$ (all neutrons are born above 1 keV)

e. Give explicit expressions for the fission rate in group 2 and the fission source in group 2. What is the difference in these two quantities?

fission rate in group 2: $\Sigma_{f2}\phi_2$

fission source in group 2: $\chi_2(\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2 + \nu\Sigma_{f3}\phi_3 + \nu\Sigma_{f4}\phi_4) = 0$ (since $\chi_2 = 0.0$)

The fission rate in a group gives the number of fissions per second in the group (per unit volume or integrated over space, as desired). However, the fission source gives the number of fission neutrons born in the group of interest from fission in all groups within the given volume.

f. Write an expression for the overall 1-group average absorption cross section.

$$\overline{\Sigma}_a = \frac{\int_0^\infty \Sigma_a(E)\phi(E)dE}{\int_0^\infty \phi(E)dE} = \frac{\sum_{g=1}^{4}\Sigma_{ag}\phi_g}{\sum_{g=1}^{4}\phi_g} = \frac{\Sigma_{a1}\phi_1 + \Sigma_{a2}\phi_2 + \Sigma_{a3}\phi_3 + \Sigma_{a4}\phi_4}{\phi_1 + \phi_2 + \phi_3 + \phi_4}$$

g. In an infinite homogeneous system, the multiplication factor simply becomes a property of the materials within the system. Within this context, write an expression for k_∞ for a 4-group, 1-region problem. Explain any assumptions/simplifications that may be needed.

$$k_\infty = \frac{\sum_{g=1}^{4}\left\langle \nu\Sigma_{fg}\phi_g \right\rangle}{\sum_{g=1}^{4}\left\langle \Sigma_{ag}\phi_g \right\rangle} = \frac{\left\langle \nu\Sigma_{f1}\phi_1 \right\rangle + \left\langle \nu\Sigma_{f2}\phi_2 \right\rangle + \left\langle \nu\Sigma_{f3}\phi_3 \right\rangle + \left\langle \nu\Sigma_{f4}\phi_4 \right\rangle}{\left\langle \Sigma_{a1}\phi_1 \right\rangle + \left\langle \Sigma_{a2}\phi_2 \right\rangle + \left\langle \Sigma_{a3}\phi_3 \right\rangle + \left\langle \Sigma_{a4}\phi_4 \right\rangle} = \frac{\overline{\nu\Sigma_f}}{\overline{\Sigma}_a}$$

where the 1-group average cross sections are defined as above.

h. Write an expression for the power density within the system.

$$P(\vec{r}) = \kappa\sum_{g=1}^{4}\Sigma_{fg}\phi_g = \kappa\left[\Sigma_{f1}\phi_1 + \Sigma_{f2}\phi_2 + \Sigma_{f3}\phi_3 + \Sigma_{f4}\phi_4\right]$$

where κ is the recoverable energy per fission with units of joules/fission (Lamarsh uses E_R here). This expression simply says that

$$P(\vec{r}) \Rightarrow \left(\frac{\text{J}}{\text{fission}}\right)\left(\frac{\text{fissions / s}}{\text{cm}^3}\right) = \frac{\text{J / s}}{\text{cm}^3} = \frac{\text{W}}{\text{cm}^3} = \text{power density}$$