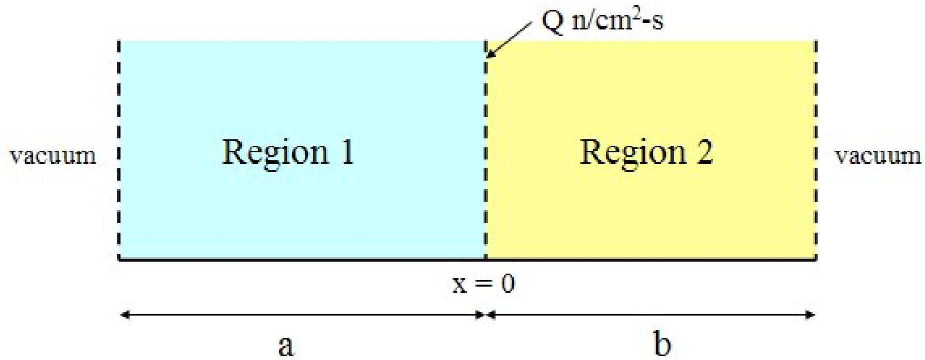
Two-Region Slab with a Planar Source at the Interface

Consider the simple two-region geometry sketched below. The origin of the 1-D slab geometry is at the interface between Region 1 and Region 2. An infinite isotropic planar source, Q n/cm^2-sec, is located at this point (with an infinite depth into the page). Water, graphite, or beryllium can occupy either region, where the appropriate material data can be obtained from Table 5.2 in Lamarsh (Ref. 1).



For 1-group theory, the diffusion equation for each moderating region is given by (see Ref. 1 or Ref. 2)

$$\nabla^2\phi - \frac{1}{L^2}\phi = \frac{-Q}{D} \tag{1}$$

However, the source is zero except at x = 0, therefore

$$\nabla^2\phi - \frac{1}{L^2}\phi = 0 \quad \text{for} \quad x \neq 0 \tag{2}$$

and, for 1-D Cartesian geometry, this becomes

$$\frac{d^2}{dx^2}\phi - \frac{1}{L^2}\phi = 0 \tag{3}$$

with general solutions

$$\phi_1(x) = A_1 \sinh\frac{x}{L_1} + A_2 \cosh\frac{x}{L_1} \qquad \text{for} \quad x < 0 \tag{4}$$

$$\phi_2(x) = A_3 \sinh\frac{x}{L_2} + A_4 \cosh\frac{x}{L_2} \qquad \text{for} \quad x > 0 \tag{5}$$

where ϕ_1 and ϕ_2 are the flux profiles in regions 1 and 2 respectively, and L_1 and L_2 are the diffusion lengths (material properties) in each region.

As apparent, this two-region system, when modeled using 1-group diffusion theory, requires four boundary conditions (BCs) to obtain a unique solution for the flux solution for any combination of materials in Regions 1 and 2 [notice the four unknown coefficients in eqns. (4) and (5)]. In particular, in a typical two-region problem, the flux and current must be continuous at the interface between the two material regions, and standard vacuum BCs are applied at both exterior boundaries (i.e. flux vanishes at the extrapolated boundary). For the usual problem, these four conditions allow computation of the four unknown coefficients. For this problem, however, the singular source at x = 0 requires that a special source condition be applied instead of the usual continuity of current condition -- that is, the current may be discontinuous at x = 0 because of the discontinuous source at this point. For 1-D Cartesian geometry, this source condition can be written at x = 0 as

$$\lim_{\Delta x \to 0}\left\{\begin{matrix}\text{leakage per unit area from}\\ \text{left side of a thin box}\end{matrix} + \begin{matrix}\text{leakage per unit area from}\\ \text{right side of a thin box}\end{matrix}\right\} = \begin{matrix}\text{source within the thin box of width}\\ \text{and unit cross sectional area}\end{matrix} \quad (6)$$

or, in mathematical terms, this translates to the condition

$$\lim_{x\to 0}\left[\vec{J}(x<0)\cdot(-\hat{i}) + \vec{J}(x>0)\cdot(\hat{i})\right] = \lim_{x\to 0}\left[-J(x<0)+J(x>0)\right] = Q \quad (7)$$

Thus, the required four boundary conditions needed to get a unique solution to this system are as follows:

1. The vacuum condition on the left says that $\phi_1(-x_1) = 0$ where $x_1 = a + d_1$ (d_1 is the extrapolation distance in region 1), or

$$A_1 \sinh\frac{-x_1}{L_1} + A_2 \cosh\frac{-x_1}{L_1} = 0 \quad (8)$$

2. The vacuum condition on the right says that $\phi_2(x_2) = 0$ where $x_2 = b + d_2$ (d_2 is the extrapolation distance in region 2), or

$$A_3 \sinh\frac{x_2}{L_2} + A_4 \cosh\frac{x_2}{L_2} = 0 \quad (9)$$

3. The requirement for continuity of flux at the center gives

$$\phi_1(0) = \phi_2(0) \qquad \text{or} \qquad \phi_1(0) - \phi_2(0) = 0 \quad (10)$$

which reduces to

$$A_2 - A_4 = 0 \quad (11)$$

4. And, finally, the source condition given in eqn. (7) gives

$$-J_1(0) + J_2(0) = Q \quad (12)$$

But the currents are given by

$$\vec{J}_1(x) = -D_1 \frac{d\phi_1}{dx}\hat{i} = \frac{-D_1}{L_1}\left(A_1 \cosh\frac{x}{L_1} + A_2 \sinh\frac{x}{L_1}\right)\hat{i} \quad (13)$$

$$\vec{J}_2(x) = -D_2 \frac{d\phi_2}{dx}\hat{i} = \frac{-D_2}{L_2}\left(A_3 \cosh\frac{x}{L_2} + A_4 \sinh\frac{x}{L_2}\right)\hat{i} \quad (14)$$

And, when evaluated at x = 0, eqn. (12) gives

$$\frac{D_1}{L_1}A_1 - \frac{D_2}{L_2}A_3 = Q \tag{15}$$

Putting these four constraint equations [eqns. (8), (9), (11), and (15)] into matrix form gives

$$\begin{bmatrix} \sinh\frac{-x_1}{L_1} & \cosh\frac{-x_1}{L_1} & 0 & 0 \\ 0 & 0 & \sinh\frac{x_2}{L_2} & \cosh\frac{x_2}{L_2} \\ 0 & 1 & 0 & -1 \\ \frac{D_1}{L_1} & 0 & \frac{-D_2}{L_2} & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Q \end{bmatrix} \tag{16}$$

Solution of this system of four equations gives the desired flux profile for a given material and geometry arrangement [i.e., gives the coefficients for the $\phi_1(x)$ and $\phi_2(x)$ flux profiles in eqns. (4) and (5)].

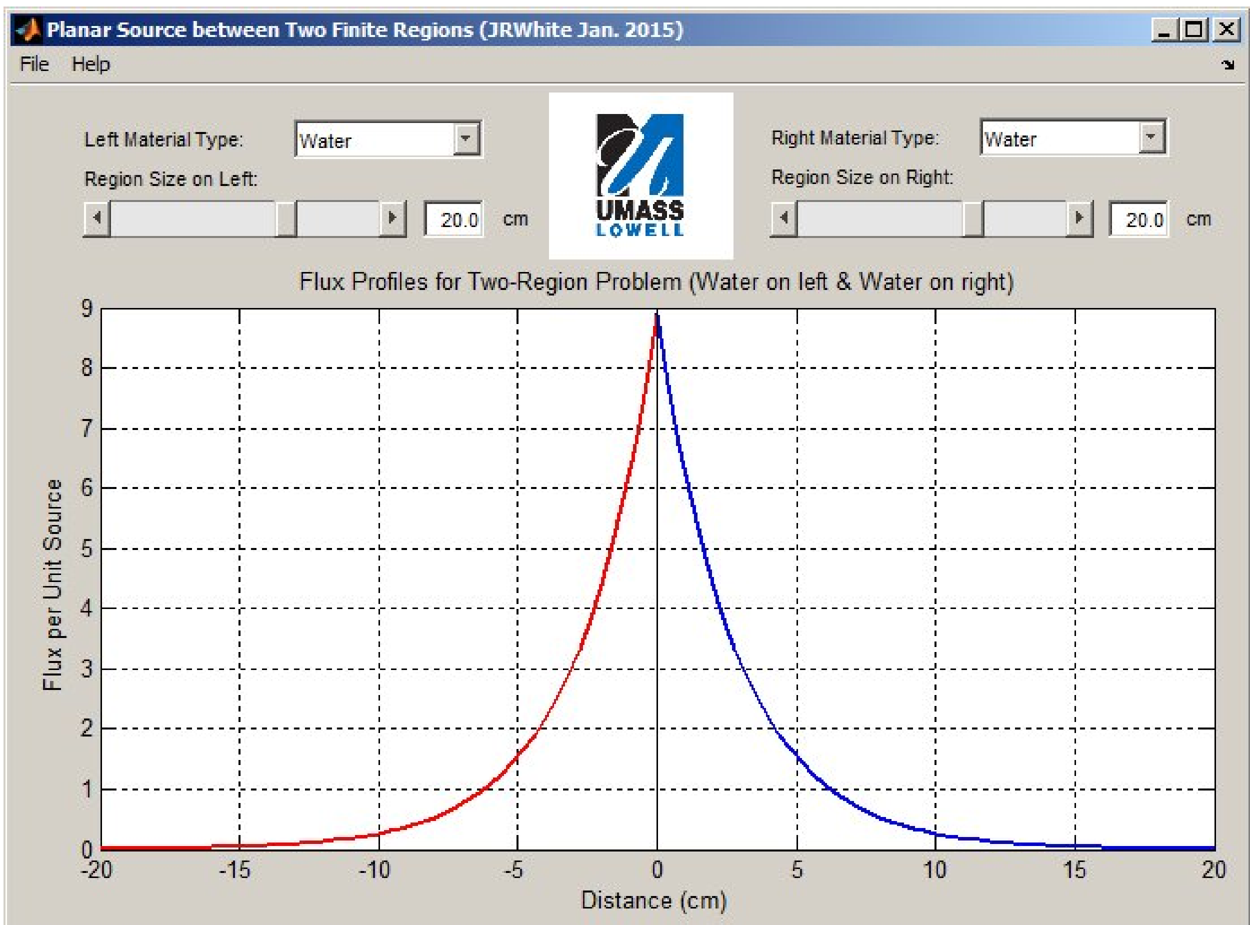


Fig. 1 User interface for the two_regions GUI.

A Matlab graphical user interface called **two_regions_gui** was designed to setup and solve eqn. (16) and then to evaluate and plot eqns. (4) and (5) for a variety of moderator materials (water, graphite, or beryllium) for user-specified values for the region dimensions (a and b). The GUI is pretty simple, as shown in Fig.1, but it is quite useful for understanding this particular asymmetric diffusing medium problem. Each half plane essentially operates as a finite region with a planar source at x = 0 but, because of the possibility of different materials or different region sizes, the flux profiles are generally not symmetric. Of course, when a = b and the material selections are identical in both regions, then a symmetric solution is indeed obtained (as seen in Fig. 1). The user should explore the various possibilities here and try to gain a good understanding of the overall behavior for this relatively simple two-region system.

References

1. J. R. Lamarsh and A. J. Baratta, ***Introduction to Nuclear Engineering***, 3rd Edition, Prentice Hall (2001).
2. J. R. White, "The Multigroup Neutron Balance Equation," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.