The Time Dependent Reactor

Introduction

Up to now, we have concentrated on the physics analysis of the steady state critical reactor. Startup, shutdown, power maneuvers during normal operation, and the fuel burnup process are all equally important in the analysis and design of a power reactor -- and these processes are inherently time dependent. This section will look at some of these time dependent processes.

In general, there are three main areas of interest here:

Subject	Time Scale of Interest
Reactor Kinetics	seconds $\rightarrow$ minutes
Fission Product Poisoning	hours $\rightarrow$ days
Fuel Depletion	months $\rightarrow$ years

Reactor kinetics is the treatment of the time behavior of the neutron distribution over short periods of time. The variations are usually caused by changes in control rod positioning, soluble boron concentration, or coolant and fuel temperatures. Changes in these parameters add reactivity (positive or negative) to the core. This, in effect, perturbs the critical system so that the multiplication factor, k_{eff}, is no longer unity. Depending on the perturbation, k_{eff} can be slightly greater than or less than unity, and the neutron density will increase or decrease correspondingly.

The physics of the reactor in these situations is described by the time dependent neutron balance equation. In all our work up to now, we have set the time derivative term in the multigroup balance equations to zero. In kinetics work, however, the $\partial \phi_g(\vec{r}, t) / \partial t$ term becomes important.

When both space and time dependence is considered, one speaks of space-time reactor kinetics. However, in many applications (especially during normal operations), only small reactivity changes are made and no significant change in the spatial distribution is observed. In this situation, only the time dependence of the magnitude of the neutron density is important. This phenomenon is modeled using point kinetics, where the reactor is treated as a point in space. The essential assumption here is that the spatial flux distribution does not change significantly in time. In this course we will concentrate almost exclusively on point kinetics. The formal development of the space-time kinetics formulation is usually treated in graduate level courses.

Fission product poisoning is another important time dependent phenomenon. Fission products accumulate in a reactor from production via the fission reaction. These intermediate mass nuclides cause parasitic absorption in the core. This is especially important in thermal reactors, since most absorption cross sections are relatively high at thermal energies -- however, fission product poisoning is important in all systems. A few fission product nuclides play an especially important role in thermal systems because of their extremely large thermal absorption cross sections and relatively large yields. For example, Xe-135 has a 2200 m/s microscopic absorption cross section of about 2.65×10^6 barns, and for Sm-149, σ_a(2200 m/s) = 41000 barns. For reference, a typical (average) fission product has a 2200 m/s cross section of about 40-50 barns.

With cross sections this large, it is understandable that Xe-135 and Sm-149 play a significant role in everyday reactor operations for thermal systems. In steady state operation, the production rates of these nuclides exactly match their loss rates (from neutron absorption and radioactive

decay, if appropriate). However, any change from steady state (such as a simple reduction in power), perturbs this equilibrium condition and the resulting dynamics of the system, while trying to reach a new equilibrium, is extremely important.

The time scale of interest here is dominated by the flux level and absorption cross sections, $\Sigma_{ai}\phi$, the production rate from fission, $\gamma_i\Sigma_f\phi$, and from the decay of other fission products, $\lambda_j N_j$, and the decay constant, λ_i, of the isotope of interest, where i refers to this specific fission product nuclide. For the case of Xe-135 and Sm-149, the dynamics are important in the hours to days timeframe. We will see that this characteristic time is long enough so that the effects of the buildup and loss of fission products are not a safety consideration; but they do factor into the day-to-day operational procedures of every operating power plant (especially in large PWRs).

A third time dependent effect that requires consideration is the fuel burnup process. This phenomenon takes place over relatively long periods of time (months to years). Fresh fuel inserted into a reactor is usually free of fission product poisons and the higher actinides. However, once power operation begins, neutron fission, which produces the fission products, and neutron capture, which produces nuclides higher up the chain, alter the distribution of nuclides in the system. For example, in a thermal system fueled with low enriched uranium, there is a significant reduction in the original fissile element, U235, and a substantial buildup of fissile plutonium, Pu239.

This transmutation of the heavy elements and the continuous buildup of fission product nuclides certainly affect the instantaneous neutron balance within the system. To maintain criticality over the design cycle length, considerable excess fuel must be loaded initially. The excess reactivity is balanced by neutron poisons (typically soluble boron and burnable absorbers in a PWR). As the fuel depletes and the fission product poisons accumulate, the amount of the controlled poisons is reduced. By definition, the end-of-cycle is reached when the excess reactivity of the fuel is zero with no control in the core.

The interaction of these effects and the design of the system to produce power over a specified cycle length are associated with the phrase "incore fuel management". In today's environment, with a relatively large base of installed nuclear power around the world, much of the activity of the core physics engineer is focused on the subject of fuel management and reactor operations. For example, with nearly a hundred operational reactors in the United States that have 12-24 month cycles, there is a continuous effort required for the design and analysis of subsequent cycles.

The three main areas outlined above, as well as a brief discussion of some control considerations, are treated in more detail in subsequent parts of this set of Lecture Notes (and the associated files that address specific subjects). It will be clear by the time we finish that the dynamics of the core plays as important a role in the overall design as does the steady state considerations discussed in previous lessons. We will also see that such things as temperature effects and the concept of reactivity coefficients enter into the overall design of a power reactor. Many of these concepts will bring together much of the material we have studied thus far. In fact, this section will complete our beginning study of reactor physics and, hopefully, it will tie up a lot of loose ends. With the completion of this material, the student should have a good understanding of the key physics concepts required for the design and safe operation of a nuclear reactor core.

Reactor Kinetics

We will first elaborate on the subject of reactor kinetics. The starting point of our discussions will be the time-dependent diffusion equation that was developed in Ref. 1. In words, this equation states that the

$$\begin{matrix} \text{rate of change} \\ \text{of neutron density} \end{matrix} = \begin{matrix} \text{production rate of} \\ \text{neutrons per unit volume} \end{matrix} - \begin{matrix} \text{loss rate of} \\ \text{neutrons per unit volume} \end{matrix}$$

For energy group g, this can be written mathematically as

$$\frac{\partial}{\partial t} n_g = \frac{1}{v_g} \frac{\partial}{\partial t} \phi_g = \left[Q_g + \chi_g \sum_{g'} \nu\Sigma_{fg'} \phi_{g'} + \sum_{g' \neq g} \Sigma_{g' \to g} \phi_{g'} \right]$$
$$- \left[-\vec{\nabla} \cdot D_g \vec{\nabla}\phi_g + \Sigma_{ag}\phi_g + \sum_{g' \neq g} \Sigma_{g \to g'}\phi_g \right] \tag{1}$$

and, for the 1-group or 1-speed approximation, this simplifies to

$$\frac{\partial}{\partial t} n = \frac{1}{v} \frac{\partial}{\partial t} \phi = \left[Q + \nu\Sigma_f \phi \right] - \left[-\vec{\nabla} \cdot D\vec{\nabla}\phi + \Sigma_a \phi \right] \tag{2}$$

In developing these expressions we assumed that ν represented the sum of both the prompt and delayed yield from fission. ***These equations, in their present form, are substantially correct, but not precisely correct.*** To clarify this statement, we need to consider the time at which the neutrons are produced.

Recall that delayed neutrons are produced from the decay of certain nuclides (called precursors) that are produced in the fission process (see Refs. 1 and 2 for further details). The delayed neutron precursors are usually grouped into six separate groups with six effective decay constants, λ_i, and yields, β_i. The total delayed neutron fraction is given by $\beta = \sum\beta_i$, where

$$\beta_i = \frac{\nu_i}{\nu_T} = \frac{\text{delayed neutrons from precursor group i per fission}}{\text{total neutrons emitted per fission}}$$
$$= \frac{\text{fraction of total neutrons emitted that}}{\text{result from the decay of precursor group i}}$$

With these definitions, one has

β = fraction of total neutrons that are delayed

$1 - \beta$ = fraction of total neutrons that are prompt

Thus, the obvious choice for the fission source for the 1-speed case becomes

$$S_{fis}^{total} = S_{fis}^{prompt} + S_{fis}^{delayed} = (1-\beta)\nu\Sigma_f\phi + \beta\nu\Sigma_f\phi = \nu\Sigma_f\phi$$

which agrees with that given in eqn. (2). However, there is a subtle problem here which relates to the timing of the neutron production terms. The prompt term, $(1 - \beta)\nu\Sigma_f\phi$, accounts for the instantaneous release of prompt neutrons at the time of fission. The delayed term, $\beta\nu\Sigma_f\phi$ (or $\beta_i\nu\Sigma_f\phi$ if we consider an individual precursor group), is not really the instantaneous delayed neutron production term but, instead, it is the precursor production rate. The delayed neutrons,

in turn, result from the decay of the precursors (which have characteristic decay constants, λ_i). Since each precursor decay produces one delayed neutron, we have

$$S_{fis}^{delayed} = \sum_{i=1}^{6} \lambda_i C_i \tag{3}$$

where C_i = concentration of delayed neutron precursors in group i

$\lambda_i C_i$ = decay rate of precursor group i (which is also the production rate of delayed neutrons from precursor group i)

$\sum_i \lambda_i C_i$ = total delayed neutron production rate from all precursor groups, and

$\beta_i \nu \Sigma_f \phi$ = production rate of precursor group i (for the one-speed approximation)

Thus, from this discussion, clearly it is the fission source terms in eqns. (1) and (2) that are incorrect. For the multigroup case, the fission source can be written as (assuming β to be essentially independent of energy),

$$S_g^{fis} = \chi_g \sum_{g'} (1-\beta) \nu \Sigma_{fg'} \phi_{g'} + \sum_i \chi_{gi} \lambda_i C_i \tag{4}$$

where we have taken into account the observation that the prompt and delayed spectra are different (recall that the delayed neutron spectra peak at a substantially lower energy relative to the prompt fission neutrons -- see Ref. 2).

For the one energy group case, this simply becomes

$$S^{fis} = (1-\beta) \nu \Sigma_f \phi + \sum_i \lambda_i C_i \tag{5}$$

In summary, we shall write the complete (and correct) space-time kinetics equations as follows:

Neutron Balance (1-speed approximation)

$$\frac{1}{v} \frac{\partial}{\partial t} \phi = \left[Q + (1-\beta) \nu \Sigma_f \phi + \sum_i \lambda_i C_i \right] - \left[-\vec{\nabla} \cdot D \vec{\nabla} \phi + \Sigma_a \phi \right] \tag{6}$$

Precursor Balance

$$\frac{\partial C_i}{\partial t} = \beta_i \nu \Sigma_f \phi - \lambda_i C_i \qquad \text{for } i = 1, 2, \cdots 6 \tag{7}$$

where, in general, all the variables here are functions of both space and time. These coupled equations (seven of them) represent a set of partial differential equations in space and time and, in general, they are rather difficult to solve. There are Space-Time Kinetics codes available for the solution of these equations -- but the details of these codes are beyond the scope of this course. However, eqns. (6) and (7) can be easily reduced to the Point Kinetics approximation -- which represents a spatially integrated or point model of the system. This simplification is much easier to deal with, but it will still give us a lot of insight into the basic phenomena of interest in reactor kinetics studies.

Point Kinetics

The actual development and solution of the point kinetics model are given in three separate sets of Lecture Notes (Refs. 3-5). The first file (Ref. 3) develops the general point kinetics model using the one-speed diffusion equation (by formal integration over the spatial variable) and the usual Lifetime and Generation Time Formulations of Point Kinetics (from definition of the prompt neutron lifetime, l_p, and the prompt neutron generation time, Λ_p). The second set of notes (Ref. 4) addresses the normalization of the Generation Time Formulation so that the total reactor power (in watts) and the total neutron source strength (in neutrons/second) appear directly within the equations. Finally, Ref. 5 discusses the solution of the Generation Time Formulation for the case of a step change in reactivity. The details in these references certainly represent the bulk of our treatment of this subject, so it is essential that the student study these lessons with some care -- there is a lot of good stuff here and mastery of the material in Refs. 3-5 will give you a solid foundation in the area of reactor kinetics…

Please make sure you are comfortable with the material in Refs. 3-5 before continuing!!!

Note: Concerning the treatment of this subject in Section 7.2 of Lamarsh (Ref. 6), we should note that all the same key points are illustrated, such as the new terminology introduced (reactivity equation, reactor period, prompt jump/drop, etc.) and the expected behavior following a step change in reactivity. However, Lamarsh unnecessarily assumes an infinite reactor model (to remove the spatial dependence) and some of the resulting notation and discussion is somewhat awkward and confusing. Since some of the notation from Lamarsh (especially the use of k_∞ within many of the equations) is not in common use, I recommend that you focus on the material in Refs. 3-5 (rather than Ref. 6) to get a good understanding of the subject of Point Kinetics…

Note: Concerning small reactivities, this subject is not treated explicitly in Ref. 5. However, this is a useful approximation that is often valid, so it is appropriate to address this briefly here. In particular, for $\rho \approx 0$ (either positive or negative), the magnitude of the most positive root of the reactivity equation is small compared to the magnitude of all the λ_i values (i.e. $|\omega| << |\lambda_i|$). With this result, eqn. (6) from Ref. 5 becomes

$$\rho = \Lambda\omega + \sum_i \frac{\beta_i\omega}{\omega+\lambda_i} \approx \Lambda\omega + \omega\sum_i \frac{\beta_i}{\lambda_i} = \omega\left(\Lambda + \sum_i \frac{\beta_i}{\lambda_i}\right)$$

and, since the reactor period, τ, is just the inverse of the most positive root, ω_1, we have

$$\tau = \frac{1}{\rho}\left(\Lambda + \sum_i \frac{\beta_i}{\lambda_i}\right) \qquad \text{(for small } \rho) \qquad\qquad (8)$$

Also, in all practical cases, the generation time, Λ, is small compared to the 2nd term inside the brackets. Thus, we can estimate the reactor period as

$$\tau = \frac{1}{\rho}\sum_i \frac{\beta_i}{\lambda_i} = \frac{t_d\beta}{\rho} \qquad \text{(for small } \rho) \qquad\qquad (9)$$

where t_d is the mean lifetime of delayed neutrons -- that is, $t_d = \left(\sum \beta_i/\lambda_i\right)/\beta$. Therefore, if the reactivity is indeed small, then the reactor period is simply inversely proportional to the

reactivity inserted. This inverse relationship is shown nicely in the left portion of Fig. 4 in Ref. 5 (i.e. for $|\rho| < 0.10\ \beta$).

Subcritical Multiplication

Our treatment of point kinetics has focused on variations from critical, where the external source is assumed to be negligible relative to the fission source (recall that the first step in Ref. 5 was to set the external source to zero). Clearly, however, in a subcritical system, the external source is extremely important, and this term definitely cannot be ignored. Here we will not formally treat the full subcritical kinetics problem (simply due to a lack of time). However, a separate set of Lecture Notes (Ref. 7) has been prepared to address the subcritical steady state problem. The primary goal here was to develop an expression for the so-called subcritical multiplication factor and to relate this to the value of k_{eff} or ρ. In particular, Ref. 7 shows that the total neutron source, N, is simply related to the subcritical multiplication factor, M, by the expression

$$N = S_{fis} + S_{ext} = M\ S_{ext} \qquad \text{with} \qquad M = \frac{1}{1-k} \tag{10}$$

Reference 7 also discusses the importance of this result and how a simple extrapolated plot of 1/M can lead to an estimate of when criticality will be achieved. These are important practical concepts that are used in a variety of "approach to critical" situations. Thus, you should definitely study Ref. 7 to glean as much insight as possible from this development and the associated applications…

> ***If appropriate, be sure you are comfortable with the material in Ref. 7 before continuing!!!
> (Note that, in some semesters, this subject is not treated in detail due to lack of time.)***

Reactivity Coefficients

Reference 5 gives a brief comparison of the transient behavior of a reactor due to a step change in reactivity for cases involving no feedback and negative feedback. In those comparisons, a generic power feedback coefficient, $\alpha_P = \partial\rho/\partial P$, was applied to treat, in a collective fashion, a number of feedback effects, such as to account for the reactivity effect due to changes in the fuel temperature, moderator temperature, and/or coolant voiding. In practice, however, the individual coefficients for each separate effect are needed since the time constant associated with each reactivity feedback can be significantly different -- which can indeed be important when considering certain transient situations. For example, in a power excursion, the fuel temperature is the first to respond to an increased fission power, then the coolant temperature, and finally the temperature of the structural components, and the time delay associated with the various heat transport mechanisms can be important when addressing the time dependence of the reactivity feedback within a formal solution of the point kinetics formulation.

Each reactivity coefficient is defined in a similar fashion. For a temperature effect, for example, we simply write the temperature coefficient of reactivity as

$$\alpha_T = \frac{\partial\rho}{\partial T}$$

where the temperature might be associated with the fuel, coolant, or structural materials. Since $\rho = (k-1)/k$, this can be written as

$$\alpha_T = \frac{\partial \rho}{\partial T} = \frac{\partial}{\partial T}\left(1 - \frac{1}{k}\right) = \frac{1}{k^2}\frac{\partial k}{\partial T} \approx \frac{1}{k}\frac{\partial k}{\partial T} \tag{11}$$

where the last approximation ($k^2 \approx k$) is valid for a near-critical or critical reference state (i.e. $k_{ref} \approx 1.000$). And, from the basic definition of a reactivity coefficient, α_T, we see that

$$\rho_f(t) = \Delta\rho(t) \approx \frac{\partial \rho}{\partial T}\Delta T(t) = \alpha_T\left\{T(t) - T_{ref}\right\} \tag{12}$$

Thus, once the reactivity coefficients are known, they can be used to approximate the inherent feedback reactivity within the system as implied by eqn. (12).

In practice, the temperature coefficients (with units of $\Delta k/k$ per unit temperature) are not really very easy to quantify. Often these are computed using sophisticated computer codes that attempt to model the reactor configuration in as much detail as possible. Usually two discrete temperatures are chosen and the representative cross sections and atom densities are determined for each temperature. The neutron balance equation is then solved using these data sets to obtain two values of k_{eff}. For example, given the T-k combinations,

T_1 = reference temperature $\rightarrow$ k_1 = reference k_{eff}

T_2 = perturbed temperature $\rightarrow$ k_2 = perturbed k_{eff}

the average temperature coefficient over the given temperature range is

$$\bar{\alpha}_T = \frac{\int_{T_1}^{T_2}\alpha_T(T)\,dT}{\int_{T_1}^{T_2}dT} = \frac{\int_{T_1}^{T_2}\frac{1}{k}\frac{\partial k}{\partial T}\,dT}{T_2 - T_1} = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2}\frac{dk}{k} = \frac{\ln\left(k_2/k_1\right)}{T_2 - T_1} \tag{13}$$

As implied here, the temperature coefficient is a function of temperature, so eqn. (13) may be evaluated over several T-k pairs to develop the rough behavior of $\bar{\alpha}_T$ vs. temperature. Alternatively, one can plot k vs. T for a set of discrete T-k pairs and, via a curve fit or a finite difference approximation, form $\bar{\alpha}_T$ vs. T from the basic definition given in eqn. (11).

For rough qualitative estimates or to simply help physically explain some observed behavior, one can use the 6-factor formula to break α_T into its various subcomponents. Recall that the 6-factor formula is given by

$$k = k_\infty P_T P_F \tag{14}$$

and, taking the natural logarithm of both sides, gives

$$\ln k = \ln k_\infty + \ln P_T + \ln P_F$$

Now, taking the partial derivative with respect to temperature (holding all other variables constant), gives

$$\alpha_T = \frac{1}{k}\frac{\partial k}{\partial T} = \frac{1}{k_\infty}\frac{\partial k_\infty}{\partial T} + \frac{1}{P_T}\frac{\partial P_T}{\partial T} + \frac{1}{P_F}\frac{\partial P_F}{\partial T}$$

or

$$\alpha_T = \alpha_T\big|_{k_\infty} + \alpha_T\big|_{P_T} + \alpha_T\big|_{P_F} \tag{15}$$

Note also that, via the same type of argument, the k_∞ component can be broken into the four individual sub-factors associated with the elements of the 4-factor formula, or

$$\alpha_T\big|_{k_\infty} = \alpha_T\big|_{\eta_T} + \alpha_T\big|_f + \alpha_T\big|_p + \alpha_T\big|_\varepsilon \tag{16}$$

Thus, the fuel temperature coefficient, for example, involves summing the individual effects associated with how a change in T_f affects the neutron reproduction factor, the fuel utilization, the resonance escape probability, etc…

Lamarsh (Ref. 6) does a nice job discussing the importance of these individual terms to the overall temperature coefficient of reactivity -- with a focus on establishing/justifying the sign (positive, negative, or essentially zero) of the separate components. In particular, when treating the fuel temperature coefficient -- which is often referred to as the prompt temperature coefficient (or Doppler coefficient) since the fuel temperature feedback usually has the fastest response time -- Lamarsh argues that the dominate contribution here is due to the change in the resonance escape probability. To see this, we note that in thermal systems using low enriched uranium, there is a significant amount of U238 present and a single particularly large resonance at about 6.67 eV plays a dominant role in the overall inherent safety of these systems.

In particular, as shown in the sketch (from Ref. 6) of the U238 capture cross section in the vicinity of the 6.67 eV resonance, the peak cross section tends to decrease and the wings of the resonance tend to broaden as the temperature is increased. This broadening is due to the increased relative motion of the U238 nuclei as the temperature and average kinetic energy increase (this is often referred to as Doppler broadening). Note, however, that although the shape of the resonance changes, the integral under the $\sigma_c(E)$ curve remains constant.
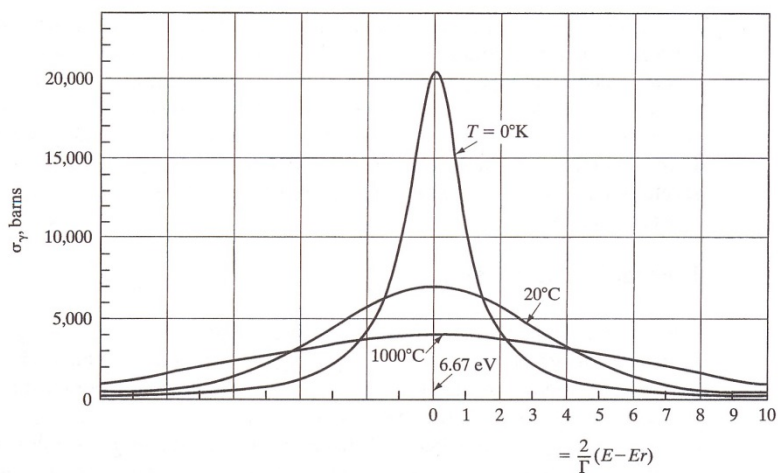
Thus, the absorption rate associated with this single resonance can be written as

$$F_a = \int N\sigma_c(E)\phi(E)dE = \phi_{ave}\int \Sigma_c(E)dE = constant \times \phi_{ave}$$

where

$$\phi_{ave} = \frac{\int \Sigma_c(E)\phi(E)dE}{\int \Sigma_c(E)dE}$$

Now, the key observation here is that, as the resonance peak decreases, we see less of a dip in the local flux, $\phi(E)$, within the resonance (i.e. less resonance self shielding), so the effective average flux defined above increases -- which, in turn, increases the overall absorption rate in the

resonance. Thus, for an increase in the fuel temperature, we see a broadening of the resonance, which increases the average flux and overall absorption rate, ultimately resulting in a decreased resonance escape probability and decreased reactivity, or

$$\alpha_{T_f} = \frac{1}{k}\frac{\partial k}{\partial T_f} \approx \frac{1}{p}\frac{\partial p}{\partial T_f} < 0$$

Thus, the prompt temperature coefficient or Doppler coefficient is negative in low enriched uranium-fueled thermal systems and this is indeed a practical and essential inherent safety feature in all systems of this type.

For fast reactors, things are a little more complicated because there are both fission and capture resonances in the fuel material (U235, U238, Pu239, etc.) and important capture resonances in the structure and coolant (Fe and Na, for example). The broadening of the fission resonances represents a positive reactivity addition and, as discussed above, the capture resonances represent a negative effect. Although the capture (loss) component tends to dominate, all the resonances need to be treated carefully and the positive and negative contributions need to be added appropriately to give the composite reactivity coefficient.

Although of lesser immediate consequence than α_{prompt}, the coolant/moderator temperature coefficient is also very important and, along with the Doppler coefficient, it tends to drive the ultimate behavior of the system over slightly longer periods of time. In particular, in water-cooled and water-moderated systems, a change in the moderator temperature, which either increases or decreases the water density, affects the multiplication factor in several ways -- it can change the value of the thermal utilization, $f = \Sigma_{aF}/(\Sigma_{aF} + \Sigma_{aM})$, by changing the relative absorption rates of the fuel and non-fuel (moderator) materials, it can change the resonance escape probability, $p = \Sigma_{1\to2}/(\Sigma_{a1} + \Sigma_{1\to2})$, by changing the relative distribution between the fast absorption rate and downscatter rate, and it can change the overall non-leakage probability, $P_F P_T \approx 1/(1 + B^2 M_T^2)$, since neutrons diffuse more readily through less dense materials. Thus, we can write the moderator temperature coefficient in terms of these components as

$$\alpha_{T_m} = \alpha_T\big|_f + \alpha_T\big|_p + \alpha_T\big|_{P_F} + \alpha_T\big|_{P_T}$$

For the fuel utilization component in water-moderated systems, an increase in moderator temperature leads to a decrease in density which tends to reduce the number of absorptions in the moderator. Thus, f tends to increase with an increase in moderator temperature, and $\alpha_T\big|_f$ is positive.

However, for the resonance escape and non-leakage probabilities, just the opposite is true. For example, an increase in moderator temperature decreases the moderator density, which decreases $\Sigma_{1\to2}$, with a subsequent decrease in the resonance escape probability, p. Similarly, this same decrease in density increases the neutron leakage and decreases the P_F and P_T non-leakage probabilities. Thus, the remaining three components, $\alpha_T\big|_p + \alpha_T\big|_{P_F} + \alpha_T\big|_{P_T}$, of the overall temperature coefficient, α_{Tm}, are all negative.

In deciding the ultimate sign of α_{Tm}, one must balance the positive fuel utilization component with the other three negative terms. Here, the question becomes whether the "moderator absorbs more that it moderates" or "moderates more than it absorbs". For the usual case, the second

option is true, since the main purpose of the moderator is to slow down neutrons while minimizing parasitic absorption -- thus, here $\alpha_{Tm} < 0$. However, in a PWR with a high soluble boron loading (which often occurs at the beginning of a new fuel cycle), this situation may be reversed (i.e. the moderator and soluble poison material may absorb more than it moderates) and α_{Tm} can become slightly positive. Clearly this situation needs to be avoided under full power operational conditions -- thus there is usually an upper limit on the soluble boron loading to assure that α_{Tm} remains negative under most conditions [note that a small positive temperature coefficient is allowed under very restricted conditions during reactor startup at the beginning of cycle (BOC)].

Other reactivity coefficients are also important -- see the discussion in Lamarsh, for example, on the void coefficient for both water-cooled and sodium-cooled systems -- and, collectively, these parameters are extremely important to the routine operation and inherent safety of all nuclear systems. Clearly, special care and effort must be taken to assure that all the important reactivity coefficients are properly bounded (as required) and that the systems are always operated within strict guidelines for the safe operation of the facility.

Control Considerations

Reactivity control via burnable poisons, soluble boron, and/or discrete control rods or blades is necessary to allow full operator control of the fission chain reaction at all times -- to facilitate normal operations and power maneuvers, to compensate for changes in the fissile and fission product inventories over time, and to shut down the system for scheduled and unscheduled maintenance of both the primary and secondary systems within the overall reactor facility.

The control rods in typical systems have a variety of complicated physical designs -- such as the cluster-type control rod "fingers" in PWRs, the cruciform geometry containing several individual control rods or pins in BWRs, and the large blade-like structures used in many research reactors (see accompanying figures and sketches). In thermal reactors, in particular, these discrete control configurations are often referred to as black absorbers (similar to a black hole) -- that is, like light in the vicinity of a black hole, once a neutron enters a black absorber, it does not return.

Diffusion theory does not do a good job in these situations, so transport theory computations are often applied to compute the rod worths directly or to compute "effective" homogenized cross sections for use in diffusion theory calculations. Either way, relatively detailed computer modeling is generally needed to get reasonably accurate estimates of control rod worths.

However, there are several situations where the neutron poison material can be treated as a homogenous absorber -- which then makes the problem amenable to simple diffusion theory analyses. For example, for fast reactors, the poison rods are not as "black" because the absorption cross sections are much lower at higher neutron energies. Thus, the mean free paths, even in the vicinity of the control material, are relatively large so that the flux depression in and near the rods is considerably reduced. In this case, it may be a reasonable approximation to homogenize the poison material over a larger region, where diffusion theory is adequate. Thus, as a first approximation (for simple hand calculations), control rod worths in fast reactors can be treated as homogeneous absorbers. And, of course, since the fission product poisons in all

reactors and the soluble boron distributions in PWRs are already physically well distributed throughout the core, these situations are quite accurately represented as homogeneous poisons. Thus, there are indeed several applications where a simple approximate homogeneous poison treatment can shed light on the approximate control/poison worths within the system.
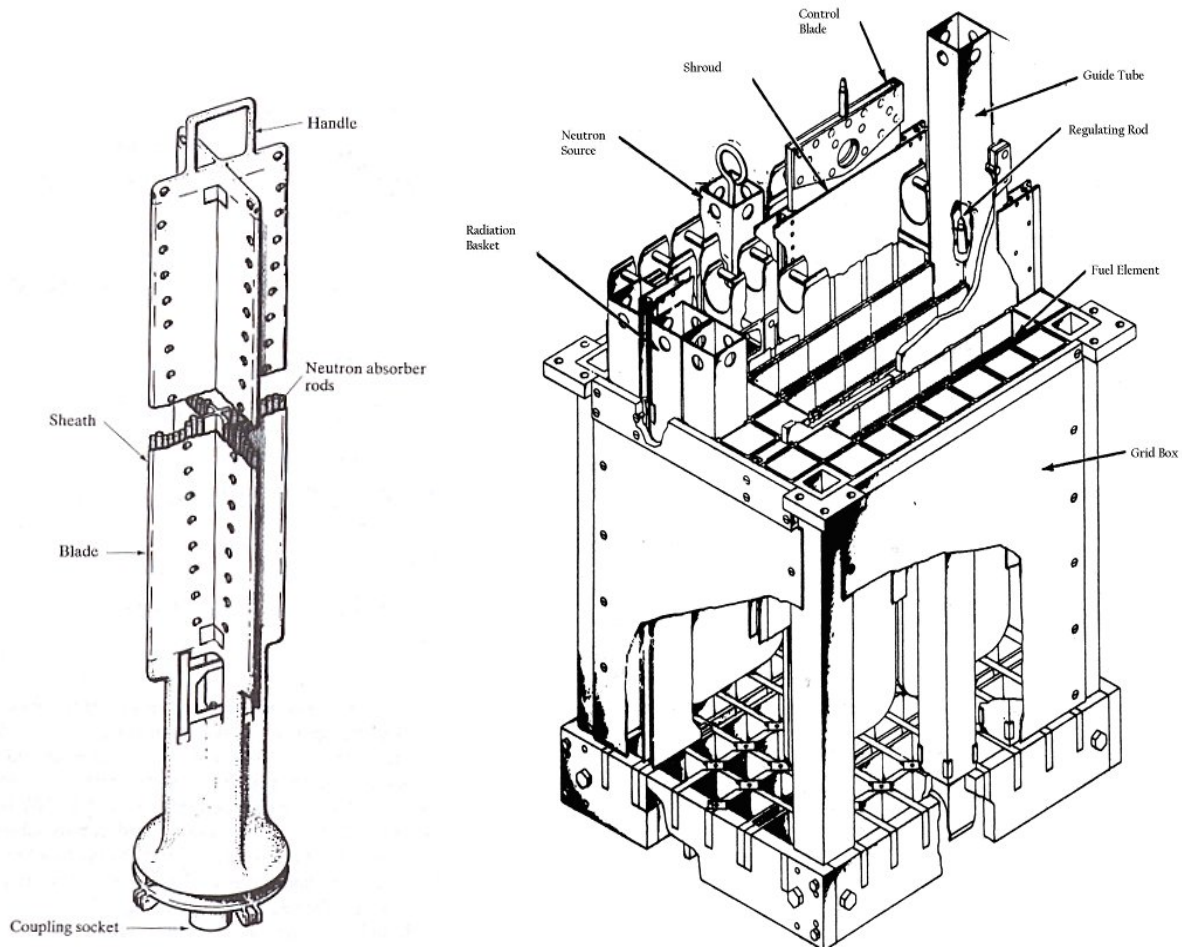
To see this, let's first consider a fast system represented via 1-group diffusion theory. In this case, the multiplication factor can be written as

$$k = k_\infty P_L = \eta f P_L \qquad \textbf{(1-group k}_\textbf{eff}\textbf{)} \qquad (17)$$

and a change in reactivity is given by

$$\rho_w = \rho - \rho_o = \frac{k-1}{k} - \frac{k_o - 1}{k_o} = \frac{k - k_o}{k k_o} \approx \frac{k - k_o}{k} \qquad (18)$$

where ρ_w is the reactivity worth of the poison or control material, ρ is the reactivity level with control, ρ_o is the reference poison-free reactivity state, and the last approximation assumes that the reference (poison-free state) is nearly critical with $k_o \approx 1$.

However, since η and P_L do not change significantly with the introduction of a homogeneous poison (i.e. the flux shape does not change significantly), an estimate of the reactivity worth of the control/poison can be given by

$$\rho_w = \frac{\eta f P_L - \eta f_o P_L}{\eta f P_L} = \frac{f - f_o}{f} \tag{19}$$

Note also that the reactivity worth is often written as a positive value for convenience and, if control is being inserted we know that the worth is negative, and if it is being removed, then the reactivity change is positive. If this convention is followed, then eqn. (19) becomes

$$\rho_w = \frac{f_o - f}{f} \tag{20}$$

since the un-poisoned fuel utilization, f_o, will be greater than the poisoned value, f.

Now, for a homogeneous system, appropriate expressions for the fuel utilization can be inserted into eqn. (20), for example, to give

$$\rho_w = \frac{\dfrac{\Sigma_{aF}}{\Sigma_{aF} + \Sigma_{aM}} - \dfrac{\Sigma_{aF}}{\Sigma_{aF} + \Sigma_{aM} + \Sigma_{aP}}}{\dfrac{\Sigma_{aF}}{\Sigma_{aF} + \Sigma_{aM} + \Sigma_{aP}}} = \frac{\Sigma_{aF}^2 + \Sigma_{aF}\Sigma_{aM} + \Sigma_{aF}\Sigma_{aP} - \Sigma_{aF}^2 - \Sigma_{aF}\Sigma_{aM}}{\Sigma_{aF}\left(\Sigma_{aF} + \Sigma_{aM}\right)}$$

or $\qquad \rho_w = \dfrac{\Sigma_{aP}}{\Sigma_{aF} + \Sigma_{aM}} \tag{21}$

Equation (21) gives the reactivity worth of a homogeneous poison using 1-group theory.

For thermal systems, the above general development can be adapted to approximate the worth of homogeneous poisons (soluble boron and fission product poisons), but not for control rods. In this case, we argue that the well-distributed poison material mostly affects the thermal utilization within the six-factor formula, which leads to essentially the same result as above (with care taken to use the average thermal cross sections), or

$$\rho_w = \frac{\overline{\Sigma}_{aP}}{\overline{\Sigma}_{aF} + \overline{\Sigma}_{aM}} \tag{22}$$

In thermal systems, several alternate forms, in addition to eqn. (22), can also be used. For example, recalling that $f_o = \overline{\Sigma}_{aF}/\left(\overline{\Sigma}_{aF} + \overline{\Sigma}_{aM}\right)$, eqn. (22) can be written as

$$\rho_w = \frac{\overline{\Sigma}_{aP}}{\overline{\Sigma}_{aF}} f_o \tag{23}$$

Also, if we make the assumption that the un-poisoned system is critical, then

$$k_o = \eta_T f_o p \varepsilon P_T P_F = \frac{\nu \overline{\Sigma}_f}{\overline{\Sigma}_{aF}} f_o p \varepsilon P_T P_F = 1$$

and $\quad f_o = \dfrac{\overline{\Sigma}_{aF}}{\nu\overline{\Sigma}_f p\varepsilon P_T P_F}$

Thus, eqn. (23) can be written as

$$\rho_w = \frac{\overline{\Sigma}_{aP}/\overline{\Sigma}_f}{\nu p\varepsilon P_T P_F} \tag{24}$$

This latter form, in particular, is often used to estimate the worth of fission product poisons in thermal systems (we will use this form in a later lesson to quantify the reactivity effect of some common fission product chains…).

One final expression for ρ_w can also be developed by starting with eqn. (22) and systematically eliminating any reference to the absorption rate in the fuel. In particular, we have

$$\rho_w = \frac{\overline{\Sigma}_{aP}}{\overline{\Sigma}_{aF} + \overline{\Sigma}_{aM}} = \frac{\overline{\Sigma}_{aP}/\overline{\Sigma}_{aM}}{\overline{\Sigma}_{aF}/\overline{\Sigma}_{aM} + 1}$$

But,

$$\frac{\overline{\Sigma}_{aF}}{\overline{\Sigma}_{aM}} = \frac{\overline{\Sigma}_{aF}/\left(\overline{\Sigma}_{aF} + \overline{\Sigma}_{aM}\right)}{\overline{\Sigma}_{aM}/\left(\overline{\Sigma}_{aF} + \overline{\Sigma}_{aM}\right)} = \frac{f_o}{1-f_o}$$

Thus,

$$\rho_w = \frac{\overline{\Sigma}_{aP}}{\overline{\Sigma}_{aM}}\left(\frac{1}{\dfrac{f_o}{1-f_o} + 1}\right) = \frac{\overline{\Sigma}_{aP}}{\overline{\Sigma}_{aM}}\frac{1-f_o}{f_o + 1 - f_o} = \left(1-f_o\right)\frac{\overline{\Sigma}_{aP}}{\overline{\Sigma}_{aM}} \tag{25}$$

As a specific application, for example, eqn. (25) can be used to determine an approximate value for the soluble boron worth in a PWR system. In particular, boric acid (H_3BO_3) is soluble in water and this homogeneous poison can be used to help override the initial excess reactivity of the fuel and to compensate for fuel depletion and fission product buildup -- and, since it is distributed evenly throughout the coolant, it influences the reactivity without significantly affecting the flux and power distributions.

Once dissolved in the water, the concentration of the boron within the system is usually given in parts per million, where *1 ppm implies 1 gram of boron per 10^6 grams of water*. If we let C be the soluble boron concentration in ppm, then the ratio of the boron atom density to the moderator (water) atom density is given by

$$\frac{N_B}{N_M} = \frac{m_B \dfrac{N_A \text{ atoms of B}}{10.8\,\text{g of B}}}{m_W \dfrac{N_A \text{ molecules of } H_2O}{18\,\text{g of } H_2O}} = \frac{18}{10.8}\frac{m_B}{m_W} = \frac{18\times10^{-6}}{10.8}C \tag{26}$$

where m_B is the mass of boron and m_W is the mass of water (and $m_B/m_W = 10^{-6}\,C$).

If we also include estimates of the microscopic absorption cross sections (from Lamarsh), we have

$$\frac{\overline{\Sigma}_{aB}}{\overline{\Sigma}_{aM}} = \frac{N_B}{N_M}\frac{\overline{\sigma}_{aB}}{\overline{\sigma}_{aM}} = \frac{18\times10^{-6}}{10.8}C\left(\frac{759}{0.66}\right) = 1.92\times10^{-3}C \tag{27}$$

Finally, substituting eqn. (27) into eqn. (25), gives

$$\rho_w = 1.92\times10^{-3}\left(1-f_o\right)C \tag{28}$$

as the approximate worth of C ppm of soluble boron distributed evenly throughout a PWR core.

As a numerical example, a typical PWR with 3 – 5 w/o enriched fuel will usually have an un-poisoned thermal utilization in the range of 0.90 to 0.95 (this is usually fairly large by design). Putting these values into eqn. (28) gives

For $f_o = 0.90$: $\rho_w = 1.92\times10^{-3}(0.10)(1) = 0.0192$ %Δk/k per ppm ≈ 0.02 %Δk/k per ppm

For $f_o = 0.95$: $\rho_w = 1.92\times10^{-3}(0.05)(1) = 0.0096$ %Δk/k per ppm ≈ 0.01 %Δk/k per ppm

And, since a common "unit of reactivity" is the pcm (where 1 pcm stands for **percent milli ρ** or, precisely 1 pcm = 10^{-5} Δk/k), this simple calculation indicates that the soluble boron worth in PWRs is usually about 10 – 20 pcm/ppm (a rough "rule-of-thumb").

To complete our brief discussion of control considerations in both fast and thermal systems, we need to say a little about the worth versus position of a partially inserted discrete control rod or blade. Even though diffusion theory does not allow an accurate quantitative treatment of the control rod worth (as noted above), it does permit a reasonably good qualitative picture of the worth distribution versus insertion depth in typical systems. In particular, using **Perturbation Theory Methods** (see brief overview in the Appendix), it can be shown that the worth of a material inserted to an axial depth z within the reactor is proportional to the product of the forward and adjoint fluxes integrated over the perturbed domain. In particular, assuming 1-group theory and that movement of the control rod only perturbs the absorption cross section, we have

$$\rho_w(z) = \alpha\int_0^z \phi^*(z')\Delta\Sigma_a(z')\phi(z')\,dz' \tag{29}$$

where α is a proportionality constant and ϕ^* is known as the adjoint flux or importance function.

However, since the 1-group diffusion equation is self-adjoint, the adjoint and forward fluxes are identical, and eqn. (29) becomes

$$\rho_w(z) = \alpha\int_0^z \phi^2(z')\Delta\Sigma_a(z')\,dz' \tag{30}$$

Now, for a bare 1-D homogeneous critical reactor of total height H, the flux profile is given by

$$\phi(z) = A\sin Bz \quad \text{with} \quad B^2 = \left(\frac{\pi}{H}\right)^2$$

where z is measured from the top of the reactor (for simplicity, we have ignored the extrapolation distance in this simple development). Finally, if the rod absorption cross section is constant, then combining the flux profile for a homogeneous system with eqn. (30) gives

$$\rho_w(z) = C \int_0^z \sin^2 \frac{\pi z'}{H} dz' = C \left[\frac{z'}{2} - \frac{H}{4\pi} \sin \frac{2\pi z'}{H} \right]_0^z$$

$$= C \frac{H}{2} \left(\frac{z}{H} - \frac{1}{2\pi} \sin \frac{2\pi z}{H} \right)$$

where C is just a new proportionality constant.

To evaluate this constant, we let $\rho_w(z)|_{z=H} = \rho_w(H)$, which is the total rod worth. With this constraint we have

$$\rho_w(H) = C \frac{H}{2}(1-0) \qquad \text{or} \qquad C = \frac{2}{H} \rho_w(H)$$

and the so-called *ideal integral worth distribution* becomes

$$\rho_w(z) = \rho_w(H) \left(\frac{z}{H} - \frac{1}{2\pi} \sin \frac{2\pi z}{H} \right) \tag{31}$$

where $\rho_w(z)$ is the worth of a partially inserted rod to depth z [this is the relationship given in Chapter 7 of Lamarsh without much justification]. Finally, if one plots the relationship $\rho_w(z)/\rho_w(H)$, the ideal S-shaped normalized integral rod worth curve is obtained (as shown in the sketch below from Ref. 6).

Also of interest is the rate of change of $\rho_w(z)$ per unit distance. This differential worth can easily be obtained by differentiation of eqn. (31), or

$$\frac{d}{dz} \rho_w(z) = \frac{\rho_w(H)}{H} \left(1 - \cos \frac{2\pi z}{H} \right) \tag{32}$$

This function, when plotted, gives the familiar differential rod worth curve (as shown below in the sketch from Ref. 6).
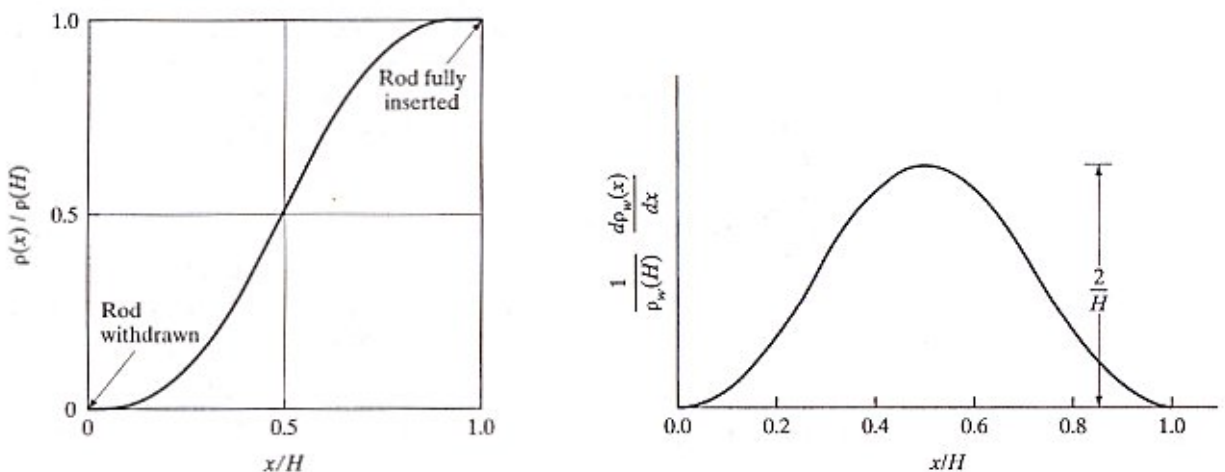
In practice, of course, the integral and differential worth curves for real reactor systems differ somewhat from the ideal curves shown above (note that these were developed using first-order perturbation theory for a bare homogeneous 1-group system -- a pretty idealized situation indeed). However, they do give a good qualitative view of what to expect for a real system, with low differential worth near the upper and lower boundaries (where the flux and importance functions are relatively low) and a peak differential worth near the core center (where we expect the highest flux and the largest neutron importance).

In contrast, Ref. 8 discusses the Stable Period Method that was utilized to measure the differential and integral blade worth curves for the UMLRR for its first 35+ years of operation (a new method based on Inverse Kinetics [Ref. 9] has been in use since 2013). In addition, Ref. 8 discusses the **blade_worth_gui** program, which is a graphical user interface developed in Matlab to display the differential and integral blade worth curves for all the control blades within the UMLRR. The code can be used to get a good understanding of typical differential and integral worth curves and it can also be used to convert raw measured data from a blade calibration experiment using the Stable Period Method within the UMLRR into useful graphical data.

In particular, it shows, using real measured data, that the differential worth curves within the UMLRR tend to follow a slightly bottom-skewed bell-shaped curve:

> The bell-shaped profile is due to the higher neutron flux at core center relative to the axial endpoints of the fuel, and the fact that neutrons in this central region contribute more to the system's criticality (i.e. they are more important) than neutrons near the periphery -- since a larger portion stay in the core rather than leak out of the fueled region.

> The slight downward skew is associated with the remaining control blades that are partially inserted into the upper portion of the core to offset any excess fuel reactivity that may be present (and this changes as a function of burnup). This partial insertion causes a slightly bottom-peaked flux distribution and differential blade worth profile.

Overall, however, the worth of a partially inserted control blade within the UMLRR does indeed behave qualitatively as expected from the simple theory described above -- but for quantitative evaluation, real measured data are always required…

Fission Product Poisoning

<div align="center">To be continued…</div>

References

1. J. R. White, "The Multigroup Neutron Balance Equation," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.

2. J. R. White, "Neutron Interactions with Matter (i.e. Neutron Cross Sections)," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.

3. J. R. White, "One-Speed Point Kinetics Equations," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.

4. J. R. White, "Normalization of the Generation Time Formulation of the One-Speed Point Kinetics Equations," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.

5. J. R. White, "Solution of the Point Kinetics Equations," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.

6. J. R. Lamarsh and A. J. Baratta, ***Introduction to Nuclear Engineering***, 3rd Edition, Prentice Hall (2001).

7. J. R. White, "Subcritical Multiplication," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.

8. J. R. White, "Blade Worth Calibration within the UMLRR," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***blade_worth_gui*** Matlab program.

9. T. P. Michaud, "Implementation of the Inverse Kinetics Method for Reactivity Calculations at the UMLRR," M.S. Thesis in Energy Engineering (Nuclear Option), UMass-Lowell, (Dec. 2011).

Appendix

A Brief Introduction to Perturbation Theory

The operator form of the diffusion equation is given as

$$(L - \lambda F)\phi = 0 \tag{A.1}$$

Now, if we multiply this by an arbitrary space and energy dependent weight function, w, and integrate over all space and energy, we have the usual expression for the eigenvalue λ, or

$$\langle w(L - \lambda F)\phi \rangle = 0 \qquad \text{or} \qquad \lambda = \frac{\langle wL\phi \rangle}{\langle wF\phi \rangle} \tag{A.2}$$

Note that in previous work we have written λ as $\lambda = \langle L\phi \rangle / \langle F\phi \rangle$ -- this just assumes that the weight function, w, is unity. Thus, the above expression is just a slight generalization of our previous representation.

Now, if a perturbation is made in the absorption cross section, for example, in some region in the core, then both the L operator and the flux distribution changes, and the first-order variation in λ can be written as

$$\Delta\lambda = \left\langle \frac{\partial\lambda}{\partial\Sigma_a}\Delta\Sigma_a \right\rangle + \left\langle \frac{\partial\lambda}{\partial\phi}\Delta\phi \right\rangle + \text{ higher-order terms } \approx \left\langle \frac{\partial\lambda}{\partial\Sigma_a}\Delta\Sigma_a \right\rangle + \left\langle \frac{\partial\lambda}{\partial\phi}\Delta\phi \right\rangle \tag{A.3}$$

where we have replaced the variation in L by the change in the absorption cross section. Now if we expand this approximate equation for $\Delta\lambda$ using the definition of λ from eqn. (A.2), we have

$$\Delta\lambda \approx \frac{\langle w\Delta\Sigma_a\phi \rangle}{\langle wF\phi \rangle} + \frac{\langle wL\Delta\phi \rangle}{\langle wF\phi \rangle} - \frac{\langle wL\phi \rangle\langle wF\Delta\phi \rangle}{\langle wF\phi \rangle^2} = \frac{\langle w\Delta\Sigma_a\phi \rangle}{\langle wF\phi \rangle} + \frac{\langle w(L - \lambda F)\Delta\phi \rangle}{\langle wF\phi \rangle} \tag{A.4}$$

where we have retained the integration over space and energy with the $< \ldots >$ notation. However, we note that the integral containing $\Delta\Sigma_a$ involves only a *local integration* and the integral containing the $\Delta\phi$ distribution is a *global integral* -- since $\Delta\Sigma_a$ is non-zero only in the location of the perturbation, yet this variation generally causes a flux perturbation everywhere in the system.

Now, to evaluate this expression for a given $\Delta\Sigma_a$, we need to know $\Delta\phi$. However, to avoid the recalculation of $\phi' = \phi + \Delta\phi$ for every variation in the cross sections, we can be a little clever in our choice of weight function, w (which has been arbitrary up to now). In particular, let's define H* as the *adjoint* of operator H, where H* is defined precisely by the equality

$$< vHu > = < uH * v > + \text{boundary terms} \tag{A.5}$$

where u and v are functions defined over the same phase space and, in most cases of practical interest, the boundary terms vanish with appropriate definition of the boundary conditions for the operator equations -- with $Hu = (L - \lambda F)\phi$ for our current application involving the neutron diffusion equation.

Now, with this general definition of an adjoint operator, we can write the numerator in the last term of the $\Delta\lambda$ expression in eqn. (A.4) as

$$< w\left(L-\lambda F\right)\Delta\phi> = <\Delta\phi\left(L-\lambda F\right)*w> + 0 \tag{A.6}$$

Finally, we now define a very specific weight function, $w = \phi*$ -- which is called the ***adjoint flux*** -- such that $\phi*$ is the solution to

$$\left(L-\lambda F\right)*\phi* = 0 \tag{A.7}$$

Clearly, if $w = \phi*$ satisfies this equation, then, via the equality in eqn. (A.6), the whole term containing $\Delta\phi$ vanishes identically, and the expression for $\Delta\lambda$ reduces to

$$\Delta\lambda = \frac{\langle w\Delta\Sigma_a\phi\rangle}{\langle wF\phi\rangle} = \frac{\langle \phi*\Delta\Sigma_a\phi\rangle}{\langle \phi*F\phi\rangle} \tag{A.8}$$

Thus, if the chosen weight function is the adjoint flux (or the ***importance function*** as it is often called), then the expression for $\Delta\lambda$ simplifies greatly, and we indeed see that the change in reactivity is simply proportional to the product of the forward and adjoint fluxes, as stated above.

To see this explicitly, we first write $\Delta\lambda$ in terms of the reactivity worth ρ_w

$$\Delta\lambda = \lambda - \lambda_o = \frac{1}{k} - \frac{1}{k_o} = \frac{k_o - k}{kk_o} = -\rho_w \tag{A.9}$$

and then, via the above development, we have

$$\rho_w = -\frac{\langle \phi*\Delta\Sigma_a\phi\rangle}{\langle \phi*F\phi\rangle} = -\frac{1}{\langle \phi*F\phi\rangle}\int \phi*(\vec{r})\Delta\Sigma_a(\vec{r})\phi(\vec{r})\,d\vec{r} = \alpha\int \phi*(\vec{r})\Delta\Sigma_a(\vec{r})\phi(\vec{r})\,d\vec{r} \tag{A.10}$$

and this is the form used in eqn. (29) in the main document (where the last two forms assume 1-group theory).

Note: The goal of this brief introduction to Perturbation Theory Methods was to justify the form of the expression used in eqn. (29) in the main document for the worth of a partially inserted control rod -- and we have just completed this task. However, the reader should be aware that there are many things that we have not addressed in any real detail -- such as, How do we find the needed adjoint operators, $H* = (L - \lambda F)*$ and, in particular, show that the 1-group diffusion theory operator is self-adjoint (i.e. that $H* = H$ for 1-group theory)? What are the "boundary terms" noted above and how do they go to zero? How do we solve the adjoint flux equation, $(L - \lambda F)*\phi* = 0$, and what does the adjoint flux look like? How do we evaluate the above perturbation integrals for the general multigroup problem? Etc., Etc..

Indeed, these are all very important questions that are really beyond the scope of this course (they are usually addressed as part of a graduate-level course in Reactor Physics). However, you are now at least aware of this important subject area and, if you can't wait for graduate school, you can always do a little self-study on your own -- Perturbation Theory is indeed a very fascinating specialization within the field of Reactor Theory that can be utilized in a wide range of interesting applications (and you are certainly encouraged to explore further, as desired)…