Subcritical Multiplication

Introduction

Clearly it is important to monitor the neutron level in a reactor system at all times -- even during shutdown conditions. When a system is subcritical, neutrons introduced into the system still undergo all the same type of interactions -- capture, fission, scattering, and leakage from the system -- that occur during normal power operations. When subcritical, however, the neutron lost terms (absorption and leakage) dominate neutron production from fission, and the multiplication factor, k, is less than unity, where k is defined as

$$k = \frac{\text{neutron production rate from fission}}{\text{neutron loss rate}} \tag{1}$$

For monitoring shutdown operations, most reactors have a sufficiently large neutron source (either inherent within the fuel or externally controlled by the reactor operators) to insure that the neutron flux in the system is high enough for proper operation of the in-core instrumentation. A detector placed within the system will measure a count rate that is proportional to the total neutron level in the vicinity of the detector, which can consist of both the original source neutrons and/or the neutrons produced via fission with the original source particles and their progeny, depending upon the source-detector configuration and the reactivity level of the system.

To see the relationship between the steady state neutron level and the source strength, consider the following arguments (see the Appendix for two alternate derivations). First, let's assume that a steady state source of neutrons producing q neutrons per generation is available. At time t = 0, with no neutron population present, we turn on the source. Thus, initially, q neutrons are introduced into the multiplying system which can be characterized by the multiplication factor, k. Using the definition of k in eqn. (1), the total neutron population after one generation, n_1, after two generations, n_2, etc. can be written as the sum of the fission neutrons produced in the current generation and the original source neutrons added in that generation, or

$$n_o = q$$

$$n_1 = kn_o + q = (k+1)q$$

$$n_2 = kn_1 + q = (k^2 + k)q + q = (k^2 + k + 1)q$$

$$n_3 = kn_2 + q = (k^2 + k + 1)q + q = (k^3 + k^2 + k + 1)q$$

and, at steady state (after many generations), we have

$$n_\infty = (1 + k + k^2 + k^3 + \quad \cdots \quad)q = \left(\sum_{p=0}^{\infty} k^p\right)q \tag{2}$$

However, the infinite series in eqn. (2), for k < 1, is the binomial series, which reduces to

$$n_\infty = \frac{1}{1-k}q \tag{3}$$

For ease of discussion in the above development, we let q be the number of source neutrons added per generation. If the resultant expression is divided by the neutron generation time, Λ, (with units of seconds per generation), then the interpretation becomes more straightforward in terms of a standard neutron source with units of neutrons per second,

$$\frac{n_\infty}{\Lambda} = \frac{1}{1-k}\frac{q}{\Lambda}$$

or

$$N = \frac{1}{1-k} S = M S \tag{4}$$

where S is the original input neutron source strength (neutrons/sec), N is the total steady state neutron source present in the system (neutrons/sec), and M is the subcritical multiplication factor, which is given explicitly by

$$M = \frac{1}{1-k} \tag{5}$$

These last two expressions are extremely important when monitoring subcritical operations!

The two limiting conditions, $k \to 0$ and $k \to 1$, imply that no fuel is present and that the system is critical, respectively:

Non-Multiplying System:	$k \to 0$ and $M \to 1$ which indicates that the total neutron source is due only to the original source neutrons (i.e. no fuel is present).
Critical System:	$k \to 1$ and $M \to \infty$ which says that the total neutron source is dominated by the fission neutron source.

Usual subcritical operation lies somewhere between these limits, but it needs to be emphasized that the subcritical multiplication factor, M, and the total neutron source, N, clearly increase dramatically as k increases towards its upper limit of unity.

Because of the behavior of M as k approaches unity, one often considers 1/M instead of M directly. In this case, $1/M \to 0$ as the system nears critical. This relationship is very useful in observing "approach to critical" situations -- since simply plotting 1/M as a function of the parameter of interest (i.e. fuel elements loaded, control rod position, etc.) gives a very clear indication of where criticality will occur.

Measurement Considerations

In practice, measuring the parameters in eqn. (4) is not particularly easy. Usually, the value of k for the system is not known, the input source strength, S, is not readily available, and the total neutron source strength, N, is not directly measurable. Instead, what is available from the detector is a count rate, C, in counts per second that is proportional to the total neutron source level, or

$$C_i = \alpha_i N_i = \alpha_i \left(\frac{1}{1-k_i} \right) S_i = \alpha_i M_i S_i \tag{6}$$

where α_i is the proportionality constant and the subscript i refers to the i^{th} configuration (i.e. the count rate in a particular configuration is a function of the given configuration).

However, if the input source strength remains constant (independent of i) for a series of measurements, then a ratio of measurements in two specific configurations removes the dependence on S,

$$\frac{C_i}{C_o} = \frac{\alpha_i M_i S}{\alpha_o M_o S} = \frac{\alpha_i M_i}{\alpha_o M_o} = \frac{\alpha_i (1-k_o)}{\alpha_o (1-k_i)} \tag{7}$$

where C_o is the initial count rate in the base system.

Now, since the true value of the absolute subcritical multiplication factor, M, is often not available (need value of k for some base configuration), a common definition for the relative subcritical multiplication factor in the i^{th} configuration, M_{ri}, is

$$M_{ri} = \frac{\alpha_i M_i}{\alpha_o M_o} = \frac{C_i}{C_o} \tag{8}$$

which can be determined simply by the ratio of the count rate for configuration i to the initial count rate. Finally, taking the inverse of eqn. (8) gives

$$\frac{1}{M_{ri}} = \frac{C_o}{C_i} = \frac{\alpha_o (1-k_i)}{\alpha_i (1-k_o)} = \beta_i (1-k_i) \tag{9}$$

where β_i is just another (unknown) proportionality constant as implied in the above equation. The important feature here is that the inverse relative subcritical multiplication factor, $1/M_r$, is approximately a linear function of the neutron multiplication factor, k. In particular, a plot of $1/M_r$ using two known values can be easily extrapolated to the $1/M_r = 0$ point -- which gives a rough prediction of where the system will be critical.

Note #1: The use of eqn. (9) to extrapolate linearly to the $1/M_r = 0$ point implies that β_i (and α_i) is insensitive to the configuration change from the $(i\text{-}1)^{th}$ to the i^{th} arrangement. Although this is often a relatively poor assumption in the early stages of an approach to critical sequence, it usually becomes a reasonably good approximation as one slowly approaches the critical configuration. Thus, in practice, the $1/M_r$ plot is indeed a very useful tool for predicting where criticality will occur.

Note #2: As a final footnote to the above development, one should be aware that many of the above equations are often expressed in terms of reactivity, ρ, where

$$\rho = \frac{k-1}{k} \qquad \text{or} \qquad k = \frac{1}{1-\rho} \tag{10}$$

In particular, substituting these expressions into eqn. (7) gives

$$\frac{C_i}{C_o} = \frac{\alpha_i \left(1 - \frac{1}{1-\rho_o}\right)}{\alpha_o \left(1 - \frac{1}{1-\rho_i}\right)} = \frac{\alpha_i}{\alpha_o}\frac{\rho_o}{\rho_i}\frac{1-\rho_i}{1-\rho_o} \approx \frac{\alpha_i}{\alpha_o}\frac{\rho_o}{\rho_i} \tag{11}$$

where the last approximation assumes that $\rho << 1$. Furthermore, if the proportionality constant between the total neutron source, N_i, and detector count rate, C_i, is relatively insensitive to the configuration change, then eqn. (11) simplifies further to the following very useful expression,

$$\frac{C_i}{C_o} \approx \frac{\rho_o}{\rho_i} \tag{12}$$

This simple relationship says that, if the count rate is doubled, then the reactivity level is reduced by about one-half (and vice versa) -- and this is a very useful rule-of-thumb for monitoring and control of subcritical systems.

Example Application within the UMLRR

As an illustration of the use of the above development, we briefly review the initial critical loading analysis that was performed for the conversion of the UMass-Lowell research reactor (UMLRR) from high-enriched uranium (HEU) fuel to low-enriched uranium (LEU) fuel. The actual fuel conversion took place in August 2000 using an approach that was based roughly on a previous computational study of the expected core loading sequence. The details of the pre-analysis and the results of the actual startup testing are reported in more detail in Refs. 1-3.

After a fairly comprehensive design process, a 21-element core with 19 full fuel elements and 2 partial elements was chosen as the best candidate for the initial startup of the LEU-fueled UMLRR.[1] As part of the design studies, both 2-D and 3-D VENTURE[4] models were developed and these were used to predict a variety of neutronics characteristics within the new core. A top view of the material layout for the initial reference core is shown in Fig. 1 along with a legend to help identify the various components within the system. This figure depicts the final goal of the initial loading procedure -- this was the target configuration for the new LEU core.

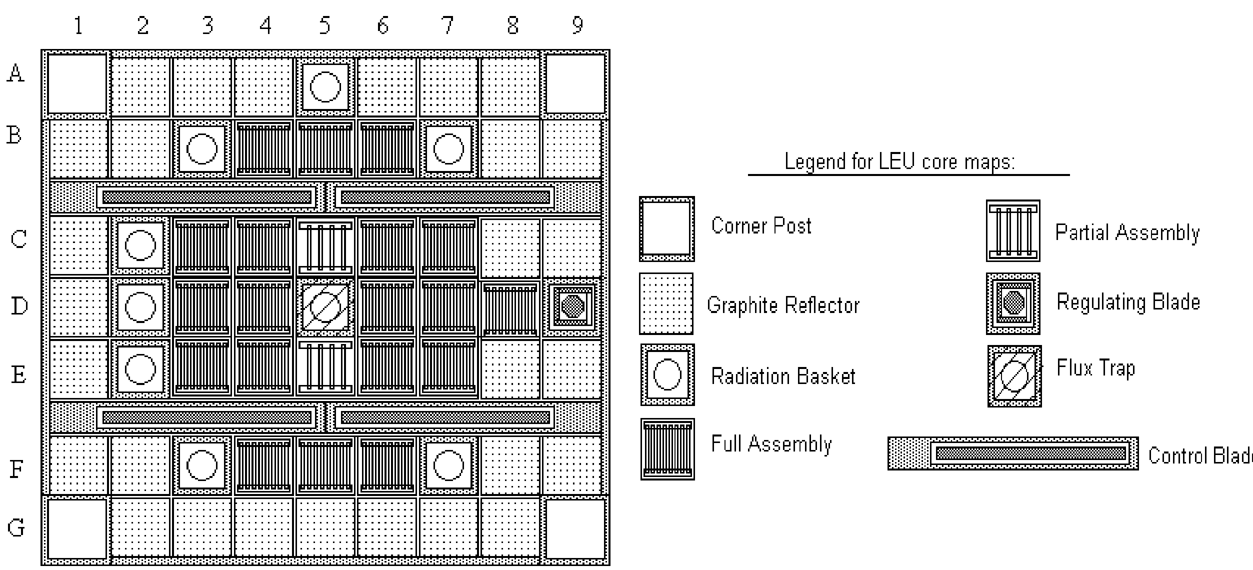

Fig. 1 Target LEU core design for the UMLRR.

A set of formal in-house procedures is used by the operations staff when performing a critical loading for a new configuration. The basic idea is to carefully monitor the subcritical multiplication associated with each new configuration as one goes from only a few elements towards a configuration that leads to a critical system. As the number of fuel elements in the core increases, the subcritical multiplication increases, eventually approaching infinity as a critical configuration is reached -- and, as $M \rightarrow \infty$, $1/M \rightarrow 0$ following the development from above.

The numerical simulation of the initial critical loading experiment for the new LEU core involved the computation of the neutron multiplication factor, k_i, for several different loading configurations using a 2-D VENTURE model. With this information, a standard 1/M plot could be generated; simulating what might be expected during the actual approach to critical for the new LEU core. During actual loading, the detector count rate, C_i, for each configuration was available and this also allowed the development of the 1/M plot based on actual reactor measurements.

The resultant 1/M plots for the VENTURE simulation and the actual reactor measurements are given in Fig. 2. The data for the simulated plot were generated with 2-D VENTURE k_{eff} calculations for 14 different assembly configurations. The actual reactor loading approached a critical configuration with 16 discrete core configurations. Although the core layouts for the simulation and the actual loading were not identical, the trend towards a critical configuration was expected to be quite similar, especially as one approaches the critical state.

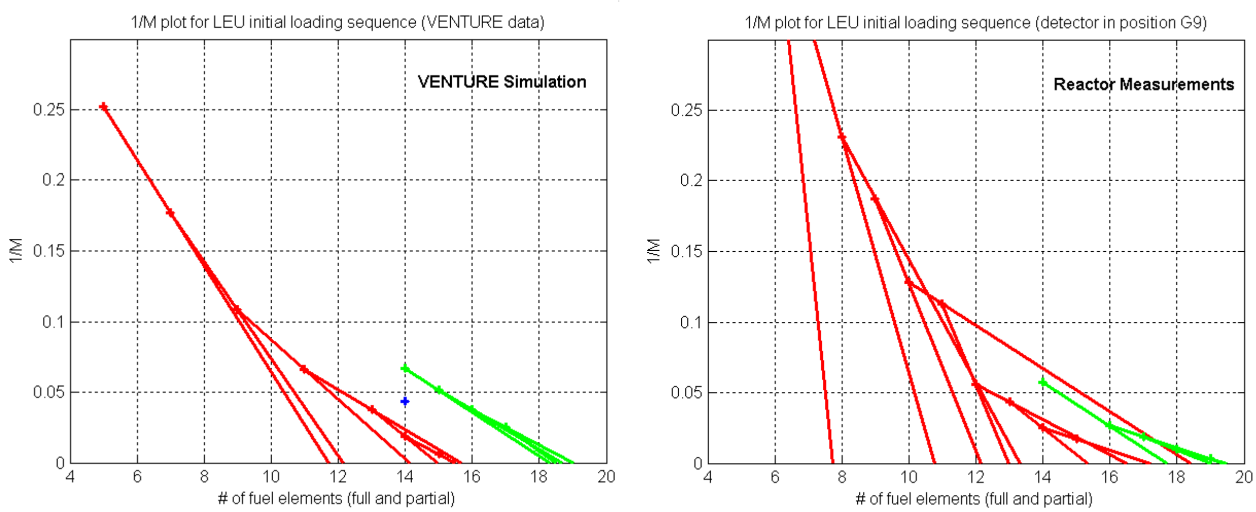

Fig. 2 1/M plots for the initial loading sequence (simulation vs. measurement).

The process started by loading the core periphery with an arrangement of graphite reflectors and radiation basket assemblies that was consistent with the proposed final core configuration as shown in Fig. 1. This arrangement left 22 centrally located grid positions available for the placement of full and partial fuel assemblies and the central flux trap irradiation facility. Initially, only full fuel elements were loaded into the core, starting in the central core region. New assemblies were then added in a systematic manner, trying to maintain as much symmetry as possible, until the core was nearly critical. At this point, the full fuel assembly in D5 was removed and replaced with the flux trap assembly. In addition, the full fuel elements in C5 and E5 were exchanged with two partial assemblies. Both these moves decreased the multiplication factor, k, and increased the inverse subcritical multiplication, 1/M, in the system. This discontinuity is apparent in Fig. 2 with the start of a new set of 1/M lines with 14 assemblies loaded (12 full and 2 partial fuel elements). After this configuration change, the normal systematic loading of full fuel assemblies was continued until a critical core was reached. The final step, of course, was to load all remaining assemblies to achieve the loading pattern given in Fig. 1 to provide some excess reactivity for normal operation of the reactor over an extended period of time.

From Fig. 2, with just full fuel present, we see that VENTURE predicted the LEU core would be critical with 16 elements. However, actual measured data show that criticality would not be achieved until 17 full fuel elements are loaded. Thus, the VENTURE calculations are somewhat conservative, with a small over-prediction of the fuel reactivity. After replacing the full fuel elements in positions C5, D5, and E5 with the flux trap and two partial fuel elements (with half the U235 loading of a full element), criticality is not reached until 19 and 20 elements were loaded, respectively, for the VENTURE simulations and actual initial loading. Thus, again, the VENTURE data slightly over-predict the core reactivity. The actual initial critical configuration, with 18 full and 2 partial assemblies was designated as the M-1-1 core. As a last step, the final full element was placed in position F6 to give the proposed initial core configuration shown in Fig. 1 (referred to as the M-1-2 configuration).

The above simulated and actual experimental analyses were performed to guarantee a safe and predictable startup for the new LEU core that was installed within the UMLRR in the summer of 2000. The 1/M plots generated here are quite typical of any approach to critical sequence. In this example, the number of fuel elements loaded was the variable of interest, but similar behavior would be expected in other cases (for the determination of the critical height of a control blade, for example). Thus, the application given here is typical of any approach to critical sequence, and it shows a real application of the concepts and equations developed in this set of Lecture Notes on Subcritical Multiplication.

Summary/Conclusions

This unit on Subcritical Multiplication should give the reader a good understanding of this important subject. The focus here is on the steady state behavior of subcritical systems, with emphasis on the definition and application of the subcritical multiplication factor, M. The specific application to the UMass-Lowell research reactor (UMLRR) adds some specificity to the subject and it shows how a 1/M plot can be used to predict where criticality will occur in a particular system. The Appendix also gives two alternate derivations of the expression for M, and these should give additional insight and understanding of this important parameter -- especially for those interested in a more mathematical approach to the subject. Overall, the reader should be leaving this unit with a better general understanding of subcritical systems, and how knowledge of M and 1/M can be quite useful in a number of practical situations. This increased understanding of the topic was the primary purpose of this unit on **Subcritical Multiplication** -- hopefully we were successful in achieving this goal…

References

1. J. R. White, et. al., “Calculational Support for the Startup of the LEU-Fueled UMass-Lowell Research Reactor,” Advances in Reactor Physics and Mathematics and Computation, Pittsburgh, PA (May 2000).
2. “Report on the HEU to LEU Conversion of the University of Massachusetts Lowell Research Reactor,” submitted to the US Nuclear Regulatory Commission in fulfillment of Amendment No. 12 to License No. R-125 (April 2001).
3. J. R. White and L. Bobek, “Startup Test Results and Model Evaluation for the HEU to LEU Conversion of the UMass-Lowell Research,” 24th International Meeting on Reduced Enrichment for Research and Test Reactors (RERTR 2002), San Carlos de Bariloche, Argentina (Nov. 2002).
4. “VENTURE-PC - A Reactor Analysis Code System,” Radiation Safety Information Computational Center, CCC-654 (1997).

Appendix

Alternate Derivations of the Formula for the Subcritical Multiplication Factor

In this Appendix we provide two alternate derivations of the formula for the subcritical multiplication factor, M. The starting point is two different formulations of the point kinetics equations (the lifetime and generation time formulations, as detailed in Ref. A.1) that describe the dynamics of critical, as well as subcritical nuclear systems. In both cases, the time-dependent equations are reduced to the situation of interest -- that is, for a steady state subcritical system -- and the resultant formula for M is identical to the one developed using the more intuitive approach in the main body of these Lecture Notes [see eqn. (5)]. The goal here is to offer additional insight into the subject of subcritical systems and to provide a brief look at a more mathematical approach to the study of this class of nuclear systems.

The Lifetime Formulation

From Ref. A.1, the defining equations for the Lifetime Formulation of point kinetics are:

$$\frac{dT}{dt} = \frac{\left[(1-\beta)k-1\right]}{l_p} T + \sum_i \lambda_i c_i + q \tag{A.1}$$

$$\frac{dc_i}{dt} = \beta_i \frac{k}{l_p} T - \lambda_i c_i \qquad \text{for } i = 1, 2, \cdots 6 \tag{A.2}$$

where T(t) is the neutron flux amplitude and $c_i(t)$ and q(t) are normalized precursor and external source amplitudes defined by

$$c_i(t) = \frac{1}{\frac{1}{v}\left\langle \psi_o \right\rangle} \left\langle C_i(t) \right\rangle \tag{A.3}$$

$$q(t) = \frac{1}{\frac{1}{v}\left\langle \psi_o \right\rangle} \left\langle Q(t) \right\rangle \tag{A.4}$$

and $\left\langle \psi_o \right\rangle$ is the spatially integrated steady state flux distribution, and all the remaining terms and symbols are defined in detail in Ref. A.1.

For steady state subcritical operation, a steady state flux distribution will result from the subcritical multiplication of the source neutrons. Clearly, the resultant total neutron population is related to the total external neutron source, and a formal mathematical representation for this relationship can be derived from the above equations. In particular, in steady state, the derivative terms vanish and the production and loss terms just balance each other. From the precursor balance we have

$$0 = \beta_i \frac{k}{l_p} T(0) - \lambda_i c_i(0) \qquad \text{or} \qquad \sum_i \lambda_i c_i(0) = \beta \frac{k}{l_p} T(0)$$

and, inserting this into the neutron balance gives

$$0 = \frac{[(1-\beta)k - 1]}{l_p} T(0) + \beta \frac{k}{l_p} T(0) + q(0)$$

or

$$\frac{1-k}{l_p} T(0) = q(0) \qquad \text{and} \qquad T(0) = \frac{l_p}{1-k} q(0)$$

Using the 1-group representation for the prompt neutron lifetime (from Ref. A.1),

$$l_p = \frac{\frac{1}{v}\langle \psi_o \rangle}{\langle -\vec{\nabla} \bullet D \vec{\nabla} \psi_o \rangle + \langle \Sigma_a \psi_o \rangle} \qquad \text{(A.5)}$$

and the definition of the normalized external source from eqn. (A.4) in this last expression gives

$$T(0) = \frac{1}{1-k} \frac{\frac{1}{v}\langle \psi_o \rangle}{\left[\langle -\vec{\nabla} \bullet D \vec{\nabla} \psi_o \rangle + \langle \Sigma_a \psi_o \rangle\right]} \frac{1}{\frac{1}{v}\langle \psi_o \rangle} \langle Q(0) \rangle$$

Finally, writing the loss rate terms (leakage plus absorption) as the production rate divided by k, gives

$$T(0) = \frac{1}{1-k} \frac{k}{\langle \nu \Sigma_f \psi_o \rangle} \langle Q(0) \rangle$$

Now, knowing T(0), we can write the fission source as

$$S_{fis} = \langle \nu \Sigma_f \psi_o \rangle T(0) = \frac{k}{1-k} \langle Q(0) \rangle = \frac{k}{1-k} S_{ext} \qquad \text{(A.6)}$$

where S_{fis} is the total fission source and S_{ext} is the external neutron source strength -- both with units of neutrons per second.

As a last step, we let the total neutron source, N, be the sum of the fission source, S_{fis}, and the input source, S_{ext}, or

$$N = S_{fis} + S_{ext} = \frac{k}{1-k} S_{ext} + S_{ext} = \left(\frac{k}{1-k} + 1\right) S_{ext} = \frac{1}{1-k} S_{ext} = M\, S_{ext} \qquad \text{(A.7)}$$

where M = 1/(1-k) is the desired subcritical multiplication factor. This says that, at steady state subcritical, the total neutron source is simply M, the subcritical multiplication factor, times the input external source strength, S_{ext} -- and, of course, this is the same result obtained in the main body of these lecture notes.

The Generation Time Formulation

Another popular form of the point kinetics equations is the Generation Time Formulation which, using the notation from Ref. A.1, can be written as

$$\frac{dT}{dt} = \left(\frac{\rho - \beta}{\Lambda}\right) T + \sum_i \lambda_i c_i + q \qquad \text{(A.8)}$$

$$\frac{dc_i}{dt} = \frac{\beta_i}{\Lambda} T - \lambda_i c_i \qquad \text{for } i = 1, 2, \cdots 6 \qquad \text{(A.9)}$$

where the normalized precursor concentration and external source have already been defined in eqns. (A.3) and (A.4).

Now, to develop an expression for the subcritical multiplication factor, we again set the derivative terms to zero and use the precursor balance to simplify the neutron balance equation. In particular, at steady state, eqn. (A.9) gives

$$0 = \frac{\beta_i}{\Lambda} T(0) - \lambda_i c_i(0) \qquad \text{or} \qquad \sum_i \lambda_i c_i(0) = \frac{\beta}{\Lambda} T(0)$$

and putting this into the neutron balance in eqn. (A.8) yields

$$0 = \left(\frac{\rho - \beta}{\Lambda}\right) T(0) + \frac{\beta}{\Lambda} T(0) + q(0)$$

or

$$0 = \frac{\rho}{\Lambda} T(0) + q(0) \qquad \text{and} \qquad T(0) = -\frac{1}{\rho} \Lambda q(0) = -\frac{k}{k-1} \Lambda q(0) = \frac{k}{1-k} \Lambda q(0)$$

Now, using the definition of the normalized input source from eqn. (A.4) and the definition of the prompt generation time, Λ, from Ref. A.1, we have

$$T(0) = \frac{k}{1-k} \frac{\frac{1}{v}\langle \psi_o \rangle}{\langle \nu \Sigma_f \psi_o \rangle} \frac{1}{\frac{1}{v}\langle \psi_o \rangle} \langle Q(0) \rangle = \frac{k}{1-k} \frac{1}{\langle \nu \Sigma_f \psi_o \rangle} \langle Q(0) \rangle$$

and, multiplying both sides by $\langle \nu \Sigma_f \psi_o \rangle$ gives

$$S_{fis} = \langle \nu \Sigma_f \psi_o \rangle T(0) = \frac{k}{1-k} \langle Q(0) \rangle = \frac{k}{1-k} S_{ext} \qquad \text{(A.10)}$$

which is identical to eqn. (A.6) from the above development using the Lifetime Formulation as the starting point. The last step in the current derivation for the Generation Time Formulation, of course, is the same as above, and this leads to eqn. (A.7) from above. Thus, here again, we get the result that M = 1/(1-k) is the final formula for the desired subcritical multiplication factor.

References for Appendix

1. J. R. White, "One-Speed Point Kinetics Equations," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.