Point Source in a Moderating Medium

Overview

A classical problem that is often solved and discussed in some detail when first studying typical solutions of the 1-group diffusion equation involves a point source of neutrons in a pure moderating medium (i.e. with no fission source). Of particular interest here is the comparison of the point source problem in an infinite versus finite medium, where one can contrast, for example, the leakage across a spherical boundary of radius R (for both the infinite and finite geometries). In addition, it is also very instructive to observe how the flux profile, $\phi(r)$, changes for a variety of different moderator materials. A Matlab graphical user interface called **spheremm_gui** was designed to address this particular situation and to allow the student to easily explore different material options and finite region dimensions, with a focus on how the material's diffusion length and the finite geometry affect the flux distribution and leakage for this relatively simple point source problem. The goal here is to obtain a good understanding of the basic principles of neutron diffusion and geometric attenuation within a simple 1-group homogeneous spherical system.

Note: A similar Matlab GUI, called **slabmm_gui**, is also available (see Ref. 1). This planar source case shows similar behavior relative to the choice of moderating material (water, graphite, or beryllium), but it does not have the geometric attenuation as seen for the spherical geometry case. Thus, it is also very instructive to compare the behavior for the slab and spherical geometries.

The main user interface for **spheremm_gui** is shown in Fig. 1. The user can specify the desired moderating material (water, graphite, or beryllium), the radius, R, for the finite geometry case, and the plot scale for the subsequent flux plot (logarithmic or linear). With this information, the flux profiles for both the infinite and finite geometries are plotted over the range $0 \le r \le R$. In addition, a neutron balance table is given that contains the leakage and absorption rates (within a sphere of radius R) for each system with a point source of Q = 1 neutron/sec emitted at the center of the sphere (i.e. at r = 0). The user is encouraged to observe the flux profile and the neutron balance information as the material and system size are changed -- hopefully, this will help you better 'visualize' the neutron diffusion and geometric attenuation processes that are at work here.

The remainder of this report documents the equations programmed into the Matlab-based **spheremm_gui** code. The development here is relatively rigorous so that you can clearly see how the equations are derived, since this also gives additional insight into the application of the diffusion equation for other situations. And, you can never see too many examples!!!

Point Source in an Infinite Moderator

Consider an isotropic point source emitting Q neutrons/sec in an ***infinite moderating medium***. Our goal here is to **formally derive** a result for $\phi(r)$ for this system assuming 1-group theory, where r is measured relative to the source location. In addition, we want to develop analytical expressions for the net neutron leakage out of a sphere of radius R and for the absorption rate within a volume of radius R -- and, clearly, these should add to Q since, in steady state, the production and total loss rates must balance.

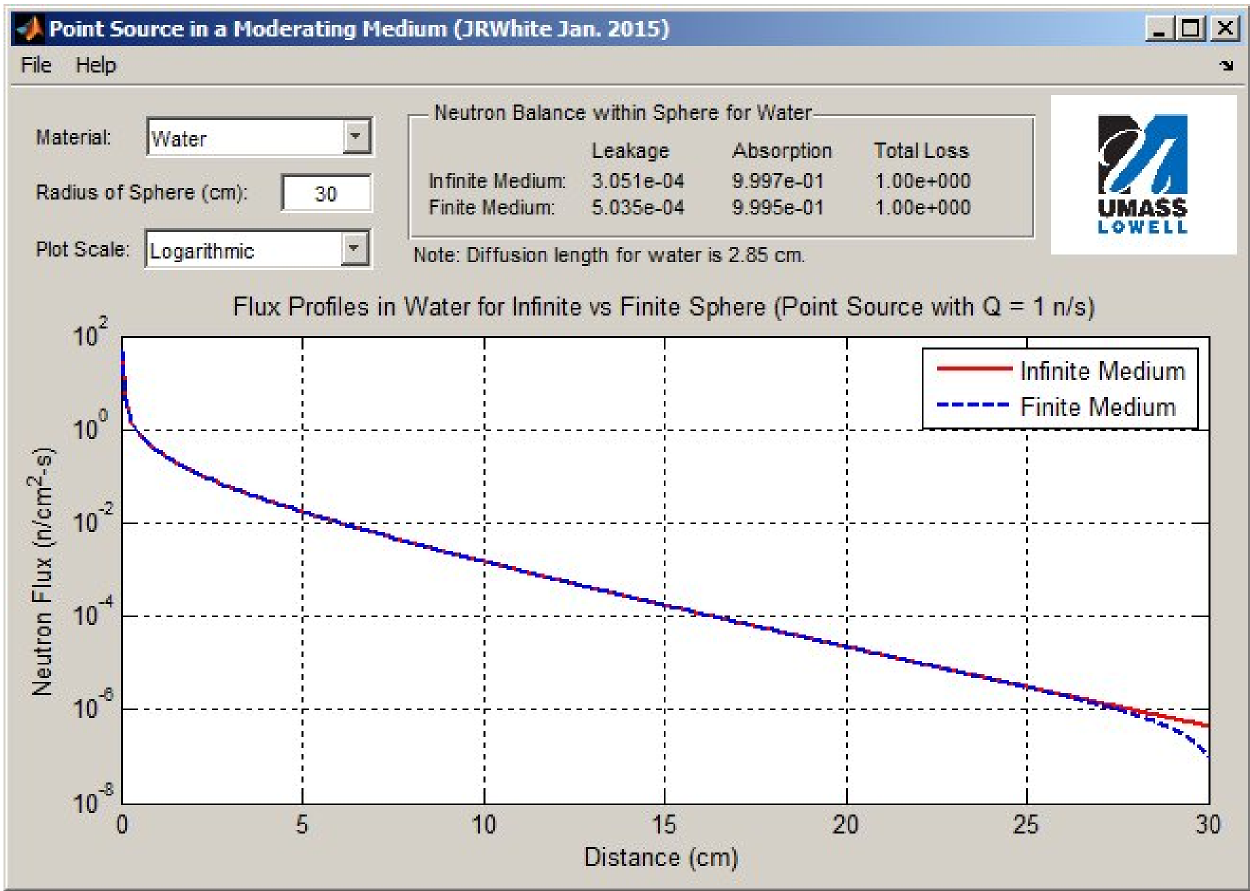

Fig. 1 User interface for the spheremm GUI.

The 1-group diffusion equation for the case of no fission is given by (see Ref. 2 or Ref. 3 for example)

$$\nabla^2\phi - \frac{1}{L^2}\phi = -\frac{Q}{D} \tag{1}$$

where $L^2 = D/\Sigma_a$ is the diffusion area. For 1-D spherical geometry, the Laplacian becomes

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) \tag{2}$$

In addition, since the source is only non-zero at r = 0, we can write eqn. (1) as a homogeneous equation for r > 0, or

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) - \frac{1}{L^2}\phi = 0 \qquad \text{for } r > 0 \qquad \textbf{[this is a variable coefficient ODE]} \tag{3}$$

Since this is a variable coefficient ODE, let's first make the substitution $\omega = r\phi$ to put this in a more manageable form. Thus, letting

$$\omega = r\phi \qquad \text{or} \qquad \phi = \frac{\omega}{r} \tag{4}$$

gives

$$\frac{d\phi}{dr} = -\frac{\omega}{r^2} + \frac{1}{r}\frac{d\omega}{dr}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(-\omega + r\frac{d\omega}{dr}\right) = \frac{1}{r^2}\left[-\frac{d\omega}{dr} + r\frac{d^2\omega}{dr^2} + \frac{d\omega}{dr}\right] = \frac{1}{r}\frac{d^2\omega}{dr^2}$$

and, after formal substitution of this latter result into eqn. (3), we get the following balance equation for $\omega(r)$,

$$\frac{d^2}{dr^2}\omega - \frac{1}{L^2}\omega = 0 \qquad \text{for } r > 0 \qquad \textbf{[this is a constant coefficient ODE]} \tag{5}$$

The general solution to this linear constant coefficient homogeneous ODE is of the form $e^{\alpha r}$, which, upon substitution, gives the characteristic equation

$$\alpha^2 - \frac{1}{L^2} = 0 \qquad \text{or} \qquad \alpha_{1,2} = \pm\frac{1}{L} \tag{6}$$

and the general solution for $\omega(r)$ as

$$\omega(r) = C_1 e^{-r/L} + C_2 e^{r/L} \tag{7}$$

Now, using eqn. (4), the general solution for the flux distribution is simply

$$\phi(r) = C_1\frac{e^{-r/L}}{r} + C_2\frac{e^{r/L}}{r} \tag{8}$$

To obtain a unique solution, we must apply two boundary conditions to eqn. (8). In the case of the infinite system, where r can become large, we require that the flux must remain finite as $r \to \infty$. Therefore, the growing exponential term in eqn. (8) immediately forces us to set C_2 to zero, or

$$C_2 = 0 \tag{9}$$

This condition reduces the flux and net current ($\vec{J} = -D\vec{\nabla}\phi$) for this case to the following expressions,

$$\phi(r) = C_1\frac{e^{-r/L}}{r} \tag{10}$$

and $$J(r) = -D\frac{d}{dr}\left(C_1\frac{1}{r}e^{-r/L}\right) = -DC_1\left[-\frac{1}{rL} - \frac{1}{r^2}\right]e^{-r/L} = DC_1\left[\frac{1}{rL} + \frac{1}{r^2}\right]e^{-r/L} \tag{11}$$

where J(r) is the radially-directed neutron current (this is the only non-zero current component in this 1-D system).

To find an explicit expression for C_1, we apply a second boundary condition -- this time at r = 0. At the center of the system we require that the leakage out of a small sphere of radius r approach the source strength Q as r approaches zero. Mathematically, this can be written as

$$\lim_{r\to 0}\left\{\text{leakage out of sphere of radius r}\right\} = \lim_{r\to 0}\left\{4\pi r^2 J(r)\right\} = Q \tag{12}$$

or $$\lim_{r\to 0}\left\{4\pi DC_1\left[\frac{r}{L}+1\right]e^{-r/L}\right\} = 4\pi DC_1 = Q \quad \text{or} \quad C_1 = \frac{Q}{4\pi D} \tag{13}$$

Finally, substituting this expression for C_1 into eqns. (10) and (11) gives

$$\phi(r) = \frac{Q}{4\pi D}\frac{1}{r}e^{-r/L} \qquad \textbf{\textit{flux due to a point source in an infinite medium}} \tag{14}$$

and $$J(r) = \frac{Q}{4\pi}\left[\frac{1}{rL}+\frac{1}{r^2}\right]e^{-r/L} \qquad \textbf{\textit{current due to a point source in an infinite medium}} \tag{15}$$

where one should note that the 1/r term in the flux expression is associated with the geometric attenuation inherent in spherical geometry problems (i.e. the surface area increases with increasing radius which causes the leakage per unit area to decrease with increasing r), and the $e^{-r/L}$ term is related to the diffusion of neutrons within a material characterized by the diffusion length, L.

To address the neutron balance within the sphere, we need to compute the leakage out of a sphere of radius R and the absorption rate within the sphere, and then add these to show that they sum to the total source within the sphere. Treating these individually, we have

Leakage out of a sphere of radius R:

$$\text{leakage} = \int_A \vec{J}\cdot\hat{n}\,dA = J(R)\,4\pi R^2 = Q\left[\frac{R}{L}+1\right]e^{-R/L} \tag{16}$$

Absorption rate within sphere of radius R:

$$\text{absorption rate} = \int_0^R \Sigma_a\phi(r)\,4\pi r^2 dr = \frac{Q\Sigma_a}{D}\int_0^R re^{-r/L}dr = \frac{Q}{L^2}\int_0^R re^{-r/L}dr \tag{17}$$

but, using a standard integration by parts technique, gives

$$\int udv = uv - \int vdu$$

$$u = r \qquad du = dr \qquad \text{and} \qquad dv = e^{-r/L}dr \qquad v = -Le^{-r/L}$$

and, upon substitution,

$$\text{absorption rate} = \frac{Q}{L^2}\left[-rLe^{-r/L}\Big|_0^R + L\int_0^R e^{-r/L}dr\right] = \frac{Q}{L^2}\left[-RLe^{-R/L} - L^2 e^{-r/L}\Big|_0^R\right]$$

or $$\text{absorption rate} = \frac{Q}{L^2}\left[-RLe^{-R/L} - L^2e^{-R/L} + L^2\right] = Q - Q\left[\frac{R}{L}+1\right]e^{-R/L} \tag{18}$$

Source within sphere of radius R [where $\delta(r)$ is the Dirac delta function]***:***

$$\int_V Q\,\delta(r)\,d\vec{r} = Q \tag{19}$$

Overall balance equation [leakage + absorption = source]***:***

$$\left[Q\left(\frac{R}{L}+1\right)e^{-R/L}\right]+\left[Q-Q\left(\frac{R}{L}+1\right)e^{-R/L}\right]=Q \tag{20}$$

or, Q = Q as expected.

In particular, the equations actually implemented into **spheremm_gui** include eqn. (14) for the flux profile, and eqns. (16) and (18), respectively, for the leakage and absorption components of the overall neutron balance. These give the appropriate relationships for the infinite medium model.

Point Source in a Finite Moderator

Consider the same isotropic point source from above placed at the center of a ***finite sphere*** of moderator of radius R. If the external boundary condition for this case is such that the flux goes to zero at the extrapolated boundary (i.e. at R+d), our goal is to again **formally derive** a result for $\phi(r)$ for this system. Also, as above, we want to develop analytical expressions for the net neutron leakage out of the sphere and for the absorption rate within the spherical volume of interest -- and, again, this should equal to the neutron generation rate within the sphere.

The development of the appropriate equations for this case follows the same procedure as above. In particular, the derivation is identical up to eqn. (6). At this point, however, it is convenient, for a finite geometry, to write the general solution to eqn. (5) as

$$\omega(r) = A_1 \sinh r/L + A_2 \cosh r/L \tag{21}$$

where it should be emphasized that eqn. (21) and eqn. (7) are equivalent representations for the general solution to the balance equation for $\omega(r)$. The form given in eqn. (7) is often more convenient for situations where the independent variable can become large, and eqn. (21) is usually better suited for finite geometry cases. Now, using eqn. (4), we can write the general solution for $\phi(r)$ for this case as

$$\phi(r) = A_1 \frac{\sinh r/L}{r} + A_2 \frac{\cosh r/L}{r} \tag{22}$$

As before, to find A_1 and A_2 to give a unique solution, we need to apply two independent boundary constraints. At the center, where r = 0, we apply the same source condition as above that says "The leakage out of a sphere of radius r, as $r \to 0$, must be Q." This is stated in mathematical terms by eqn. (12). For the present case

$$J(r) = -DA_1\left(-\frac{\sinh r/L}{r^2}+\frac{1}{rL}\cosh r/L\right) - DA_2\left(-\frac{\cosh r/L}{r^2}+\frac{1}{rL}\sinh r/L\right) \tag{23}$$

Thus, evaluating eqn. (12) using this expression for the net neutron current gives

$$\lim_{r\to 0}\left\{4\pi r^2 J(r)\right\} = -4\pi DA_1\left(-0+0\right) - 4\pi DA_2\left(-1+0\right) = 4\pi DA_2 = Q$$

or $$A_2 = \frac{Q}{4\pi D} \tag{24}$$

At the outer boundary (i.e. at r = R + d), we say that the flux goes to zero at the extrapolated boundary, where d is the extrapolation distance. Mathematically this statement is written as

$$\phi(R+d) = 0 \tag{25}$$

Therefore, from eqn. (22), we have

$$0 = A_1 \frac{\sinh (R+d)/L}{R+d} + A_2 \frac{\cosh (R+d)/L}{R+d}$$

or

$$A_1 = -A_2 \frac{\cosh (R+d)/L}{\sinh (R+d)/L} \tag{26}$$

Putting the expressions for A_1 and A_2 back into the expression for $\phi(r)$ gives

$$\phi(r) = \frac{Q}{4\pi D}\left[\frac{\cosh r/L}{r} - \frac{\cosh (R+d)/L}{\sinh (R+d)/L}\frac{\sinh r/L}{r}\right]$$

or $$\phi(r) = \frac{Q}{4\pi D \sinh (R+d)/L}\frac{1}{r}\left[\sinh\frac{R+d}{L}\,\cosh\frac{r}{L} - \cosh\frac{R+d}{L}\,\sinh\frac{r}{L}\right] \tag{27}$$

But, a useful identity that is often used to simplify expressions containing hyperbolic functions is

$$\sinh (u-v) = \sinh u \cosh v - \cosh u \sinh v \tag{28}$$

Using this identity with eqn. (27) gives the ***flux due to a point source in a finite medium***

$$\phi(r) = \frac{Q}{4\pi D \sinh (R+d)/L}\frac{1}{r}\sinh\left(\frac{R+d-r}{L}\right) \tag{29}$$

And, from the definition of the net current, we can also write the ***current due to a point source in a finite medium*** as

$$J(r) = -D\left(\frac{Q}{4\pi D \sinh (R+d)/L}\right)\left[\left(-\frac{1}{Lr}\right)\cosh\left(\frac{R+d-r}{L}\right) - \frac{1}{r^2}\sinh\left(\frac{R+d-r}{L}\right)\right]$$

or

$$J(r) = \frac{Q}{4\pi \sinh (R+d)/L}\left[\frac{1}{Lr}\cosh\left(\frac{R+d-r}{L}\right) + \frac{1}{r^2}\sinh\left(\frac{R+d-r}{L}\right)\right] \tag{30}$$

where, of course, the net current is pointed in the radially outward direction.

Note here that we again see a 1/r term in the flux expression that is associated with the geometric attenuation inherent in spherical geometry and a hyperbolic sine term containing the diffusion length, L, that accounts for the diffusion of neutrons within the finite spherical medium.

To address the neutron balance within the finite sphere, we again need to compute the leakage and absorption rates within the sphere using the flux and current expressions given respectively in eqns. (29) and (30). Thus, for the finite geometry case, we have

Leakage out of a finite sphere of radius R:

$$\text{leakage } = \int_A \vec{J}\cdot\hat{n}\,dA = J(R)\,4\pi R^2 = \frac{Q}{\sinh(R+d)/L}\left[\frac{R}{L}\cosh\left(\frac{d}{L}\right)+\sinh\left(\frac{d}{L}\right)\right] \qquad (31)$$

Absorption rate within a finite sphere of radius R:

$$\text{absorption rate} = \int_0^R \Sigma_a\phi(r)\,4\pi r^2 dr = \frac{Q}{L^2\sinh(R+d)/L}\int_0^R \sinh\left(\frac{R+d-r}{L}\right) r\,dr \qquad (32)$$

Again, using a standard integration by parts technique, gives

$$\int u dv = uv - \int v du$$

$$u = r \qquad du = dr \qquad \text{and} \qquad dv = \sinh\left(\frac{R+d-r}{L}\right)dr \qquad v = -L\cosh\left(\frac{R+d-r}{L}\right)$$

and, upon substitution,

$$\text{absorption rate} = \frac{Q}{L^2\sinh(R+d)/L}\left[-rL\cosh\left(\frac{R+d-r}{L}\right)\Big|_0^R + L\int_0^R \cosh\left(\frac{R+d-r}{L}\right)dr\right]$$

or $$\text{absorption rate} = \frac{Q}{L^2}\frac{1}{\sinh(R+d)/L}\left[-RL\cosh\left(\frac{d}{L}\right)+L(-L)\sinh\left(\frac{R+d-r}{L}\right)\Big|_0^R\right]$$

or $$\text{absorption rate} = \frac{Q}{L^2}\frac{1}{\sinh(R+d)/L}\left[-RL\cosh\left(\frac{d}{L}\right)-L^2\sinh\left(\frac{d}{L}\right)+L^2\sinh\left(\frac{R+d}{L}\right)\right]$$

or, in final form, we can write the absorption rate within the finite sphere as

$$\text{absorption rate} = \frac{Q}{\sinh(R+d)/L}\left[-\frac{R}{L}\cosh\left(\frac{d}{L}\right)-\sinh\left(\frac{d}{L}\right)\right]+Q \qquad (33)$$

Source within a finite sphere of radius R [where $\delta(r)$ is the Dirac delta function]***:***

$$\int_V Q\,\delta(r)\,d\vec{r} = Q \qquad (34)$$

Overall balance equation [leakage + absorption = source]***:***

$$\left\{\frac{Q}{\sinh(R+d)/L}\left[\frac{R}{L}\cosh\left(\frac{d}{L}\right)+\sinh\left(\frac{d}{L}\right)\right]\right\}+\left\{\frac{Q}{\sinh(R+d)/L}\left[-\frac{R}{L}\cosh\left(\frac{d}{L}\right)-\sinh\left(\frac{d}{L}\right)\right]+Q\right\}=Q \qquad (35)$$

or, Q = Q as expected.

In particular, the equations actually implemented into the **spheremm_gui** code for the finite geometry case include eqn. (29) for the flux profile, and eqns. (31) and (33), respectively, for the leakage and absorption components of the overall neutron balance. These give the appropriate 1-group relationships for the finite medium spherical geometry model.

Summary

This document provides a detailed derivation of the 1-group flux, current, and neutron balance components (leakage and absorption) for the case of an isotropic point source in a pure moderating medium -- with developments for both infinite and finite spherical geometries centered on the source location. The resultant equations have been implemented into the Matlab-based **spheremm_gui** code, which provides an easy-to-use GUI where one can explore and contrast the use of different moderating materials and geometric dimensions. The formal development of the appropriate equations, coupled with the user-friendly computational tool, should allow the user to get a better understanding of the fundamental physics at play here, and also get a good feel for both the qualitative and quantitative aspects of 1-group neutron diffusion.

Have fun using the **spheremm** GUI!!! We hope that it helps in the visualization/understanding of the basic processes associated with the diffusion of neutrons originating from a point source…

References

1. J. R. White, “Planar Source in a Moderating Medium,” part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***slabmm_gui*** Matlab program.
2. J. R. White, “The Multigroup Neutron Balance Equation,” part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.
3. J. R. Lamarsh and A. J. Baratta, ***Introduction to Nuclear Engineering***, 3rd Edition, Prentice Hall (2001).