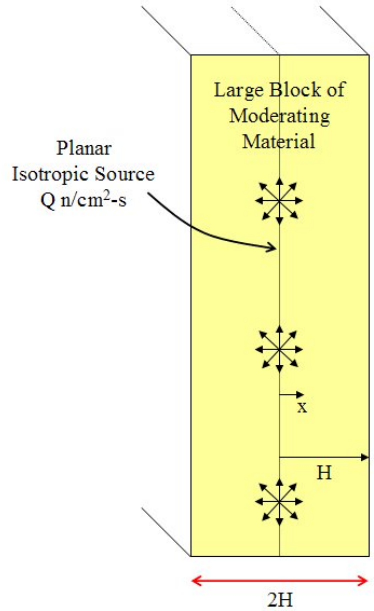
Planar Source in a Moderating Medium

Overview

A relatively simple application of the 1-group diffusion equation involves an infinite planar isotropic source of neutrons in a pure moderating medium (i.e. with no fissionable material present). This situation involves an isotropic source emitting Q neutrons/cm^2-sec at the center of a very large block of moderating material, as illustrated in the sketch. The block is essentially infinite in the y and z directions and has a half width, H, in the x direction (total width of 2H). Since the geometry is centered on the infinitesimally thin planar source region at x = 0 and is infinite in the y-z plane (perpendicular to the x-axis), the flux distribution will only vary in the x direction. This allows a simple 1-D Cartesian geometry (slab) treatment of the physical system, with a non-zero distributed source only at x = 0.



Mathematically, the source distribution can be formally described by

$$Q(x) = Q\,\delta(x)$$

where δ(x) is the Dirac delta function. The delta function is infinite at x = 0 and identically zero everywhere else. In addition, it has the following integral property:

$$\lim_{\varepsilon \to 0} \int_{0-\varepsilon}^{0+\varepsilon} Q\,\delta(x)dx = Q$$

This says that the integral of the discontinuous source gives the desired source strength of Q neutrons/cm^2-s. This formal treatment can be used if desired or, to keep things relatively simple, we will simply say that the source is only non-zero at exactly x = 0, and treat the discontinuous source at x = 0 with a special "source condition" (see below).

For this configuration, we are interested in how the neutrons diffuse throughout the medium for the case where H is finite and for the situation where H approaches infinity (i.e. the so-called ***finite*** and ***infinite medium planar source problems***). Of particular interest here is the comparison of the planar source problem in an infinite versus finite medium, where one can contrast, for example, the leakage across a planar boundary at x = H (for both the infinite and finite geometries). In addition, it is also quite instructive to observe how the flux profile, ϕ(x), changes for a variety of different moderator materials. A Matlab graphical user interface called **slabmm_gui** was designed to address this particular situation and to allow the student to easily explore different material options and finite region dimensions, with a focus on how the material's diffusion length, L, and the finite geometry affect the flux distribution and leakage for this relatively simple 1-group planar source problem. The goal here is to obtain a good understanding of the basic principles of neutron diffusion within a purely moderating homogeneous system.

Note: A similar Matlab GUI, called **spheremm_gui**, is also available (see Ref. 1). This point source case shows similar behavior relative to the choice of moderating material (water, graphite, or beryllium), but the spherical geometry case also has geometric attenuation as well as neutron diffusion. Thus, it is also very instructive to compare the behavior for the slab and spherical geometries.

The main user interface for **slabmm_gui** is shown in Fig. 1. The user can specify the desired moderating material (water, graphite, or beryllium), the full region width, 2H, for the finite geometry case, and the plot scale for the subsequent flux plot (logarithmic or linear). With this information, the flux profiles for both the infinite and finite geometry configurations are plotted over the range of positive x values given by $0 \le x \le H$ (note that the flux profile is symmetric for negative x values). In addition, a neutron balance table is given that contains the leakage and absorption rates within the right half of the block for each system with a planar source of Q = 1 neutron/cm^2-s emitted at the center of the block (i.e. at x = 0). Note also that, since the source is isotropic (neutrons are emitted in all directions with equal probability), only half of the neutrons move into the right half of the block. Since the neutron balance is performed over the range $0 \le x \le H$ and Q = 1 n/cm^2-s, then the total generation rate within the region of interest is only 0.5 n/cm^2-s.

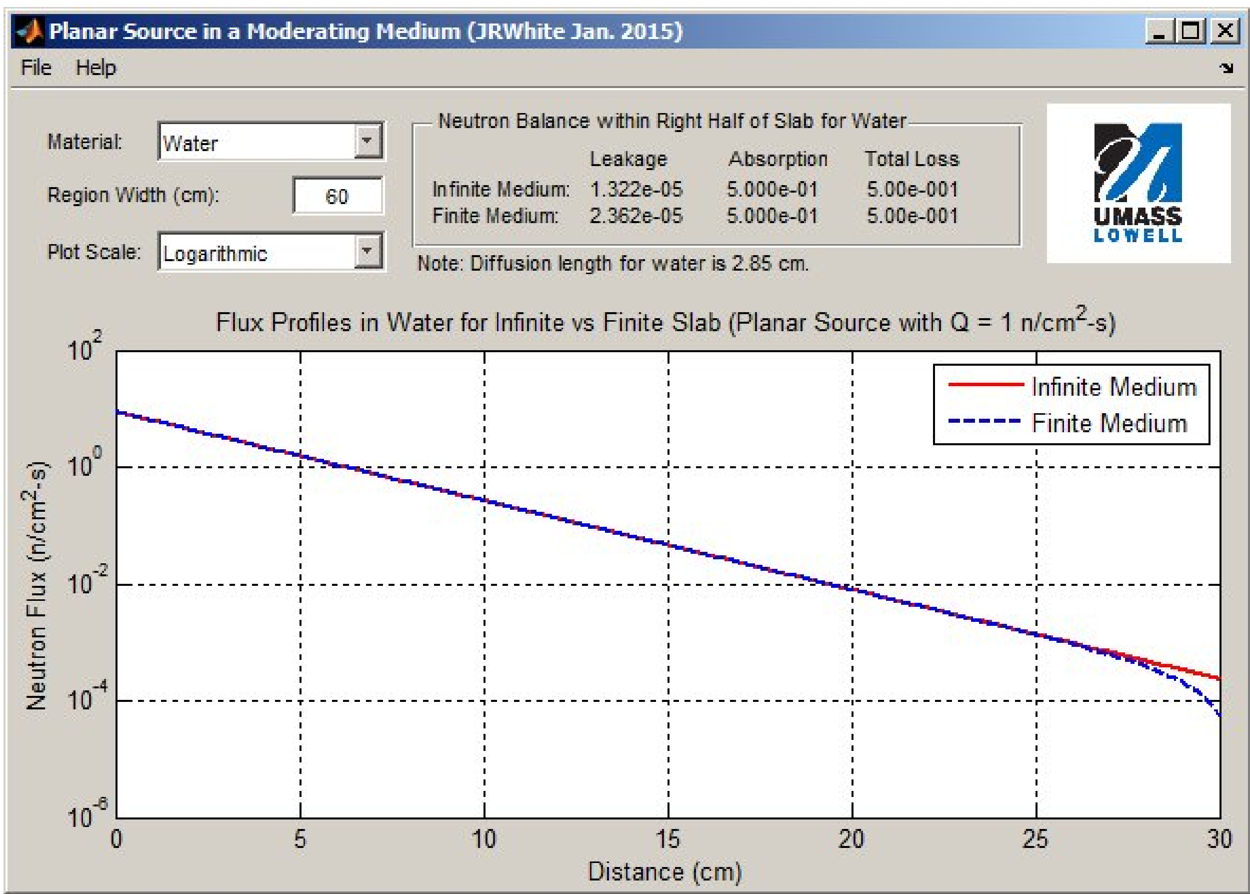


Fig. 1 User interface for the slabmm GUI.

The user is encouraged to observe the flux profiles and the neutron balance information as the material and system size are changed. The goal, of course, is to better 'visualize' the neutron diffusion processes that are at work here.

The remainder of this report documents the equations programmed into **slabmm_gui**. The development here is quite formal, since this development also gives additional insight into the application of the diffusion equation for other situations of interest.

Planar Source in an Infinite Moderator

Consider an isotropic planar source emitting Q neutrons/cm^2-sec into an ***infinite moderating medium*** (here the actual medium extends to infinity in the positive and negative x-directions). Our goal here is to **formally derive** a result for $\phi(x)$ for this system assuming 1-group theory, where x is measured relative to the planar source location. In addition, we want to develop analytical expressions for the net neutron leakage out of the right-half plane at a distance x = H from the source and for the absorption rate within this same volume. Clearly, from a neutron balance perspective, these should add to Q/2 since, in steady state, the production and total loss rates must balance (remember that only half of the original source neutrons enter the x > 0 region).

The 1-group diffusion equation for the case of no fission is given by (see Ref. 2 or Ref. 3 for example)

$$\nabla^2\phi - \frac{1}{L^2}\phi = -\frac{Q}{D} \tag{1}$$

where $L^2 = D/\Sigma_a$ is the diffusion area. For 1-D slab geometry, the Laplacian simply becomes the 2nd derivative of the flux (i.e. $\nabla^2\phi = d^2\phi/dx^2$). In addition, since the source is only non-zero at the centerline of the block at x = 0, we can write eqn. (1) as a homogeneous equation for x > 0, or

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi = 0 \qquad x > 0 \tag{2}$$

The general solution to this simple 2nd order constant coefficient homogeneous ODE is

$$\phi(x) = C_1 e^{-\alpha x} + C_2 e^{\alpha x} = C_1 e^{-x/L} + C_2 e^{x/L} \tag{3}$$

where $\alpha = 1/L$ and C_1 and C_2 are arbitrary coefficients. Equation (3) represents the general solution to eqn. (2).

As usual, to obtain a unique solution to a 2nd order equation, we must apply two boundary conditions to the general solution. In the case of the infinite system, where x can become large, we require that the flux must remain finite as $x \to \infty$. Therefore, the growing exponential term in eqn. (3) immediately forces us to set C_2 to zero, or

$$C_2 = 0 \tag{4}$$

This condition reduces the flux and net current ($\vec{J} = -D\vec{\nabla}\phi$) for this case to the following expressions,

$$\phi(x) = C_1 e^{-x/L} \tag{5}$$

and $$\vec{J}(x) = J_x(x)\,\hat{i} = -D\frac{d\phi}{dx}\hat{i} = \frac{DC_1}{L}e^{-x/L}\,\hat{i} \qquad (6)$$

where, for convenience, we will refer to the x-directed current, J_x, in all subsequent usage simply as J, since this is the only nonzero component for the 1-D slab problem of interest here (that is, $J_y = J_z = 0$ for the current problem).

To find an explicit expression for C_1, we apply a second boundary condition -- the so-called ***source condition*** -- at x = 0 (i.e. the discontinuous source at x = 0 requires that a special source condition be applied in the current situation). For 1-D Cartesian geometry, this source condition can be written at x = 0 as

$$\lim_{\Delta x \to 0}\left\{\begin{matrix}\text{leakage per unit area from}\\ \text{left side of a thin box}\end{matrix} + \begin{matrix}\text{leakage per unit area from}\\ \text{right side of a thin box}\end{matrix}\right\} = \begin{matrix}\text{source contained in the thin box of}\\ \text{width } \Delta x \text{ and unit cross sectional area}\end{matrix}$$

In mathematical terms, this statement translates to the following equation

$$\lim_{x\to 0}\left[\vec{J}(x<0)\cdot(-\hat{i}) + \vec{J}(x>0)\cdot(\hat{i})\right] = \lim_{x\to 0}\left[-J(x<0) + J(x>0)\right] = Q \qquad (7)$$

or, so we don't have to treat the x < 0 case, for a symmetric geometry we can simply write this condition as

$$\lim_{x\to 0} J(x>0) = \frac{Q}{2} \qquad \text{(for a symmetric block)} \qquad (8)$$

where this is consistent with the above discussion that argued that only half of the original source neutrons enter the right half of the homogeneous block centered at x = 0.

Using eqn. (6), we can apply the appropriate source condition in eqn. (8) to give an explicit expression for C_1, or

$$\lim_{x\to 0}\left\{\frac{DC_1}{L}e^{-x/L}\right\} = \frac{DC_1}{L} = \frac{Q}{2} \qquad \text{or} \qquad C_1 = \frac{QL}{2D} \qquad (9)$$

Finally, substituting this expression for C_1 into eqns. (5) and (6) gives

$$\phi(x) = \frac{QL}{2D}e^{-x/L} \qquad \textit{flux due to planar source in an infinite medium} \qquad (10)$$

and $$J(x) = \frac{Q}{2}e^{-x/L} \qquad \textit{current due to planar source in an infinite medium} \qquad (11)$$

Thus, the neutron flux decreases with distance from the source location in a simple exponential manner. However, of note, is that the rate of decrease is directly related to the material's diffusion length, L. In fact, we see that, for any material, at a distance of one diffusion length from the source (i.e. x = L in the material of interest), the flux is attenuated by a factor of $e^{-1} \approx 0.368$. Or, said differently, the flux attenuates at a greater rate for a material with a small diffusion length (using the Matlab GUI to compare the flux profiles for water, graphite, and/or beryllium shows this effect quite nicely).

To address the neutron balance within the infinite slab, we need to compute the leakage out of the right side of the block at x = H, the absorption rate within this portion of the block, and then add these to show that they sum to the total source within this region (i.e. Q/2). Treating these individually, we have

Leakage out of right side of the block at x = H:

$$\begin{matrix}\text{leakage}\\ \text{per unit area}\end{matrix} = \int_A \vec{J}\cdot\hat{n}\,dA = J(H)\,(1) = \frac{Q}{2}e^{-H/L} \tag{12}$$

Absorption rate within right side of block up to x = H:

$$\begin{matrix}\text{absorption rate}\\ \text{per unit area}\end{matrix} = \int_0^H \Sigma_a\phi(x)\,(1)dx = \frac{QL\Sigma_a}{2D}\int_0^H e^{-x/L}dx = \frac{Q}{2L}\left[-Le^{-x/L}\Big|_0^H\right]$$

or $$\begin{matrix}\text{absorption rate}\\ \text{per unit area}\end{matrix} = \frac{Q}{2}\left(1-e^{-H/L}\right) \tag{13}$$

Source within right side of block:

$$\int_V Q\,\delta(x)\,d\vec{r} = \int_0^H Q\,\delta(x)\,dx = \frac{Q}{2} \quad \text{(where half of the neutrons go to the left side)} \tag{14}$$

Overall balance equation [leakage + absorption = source]***:***

$$\left[\frac{Q}{2}e^{-H/L}\right] + \left[\frac{Q}{2}\left(1-e^{-H/L}\right)\right] = \frac{Q}{2} \tag{15}$$

or, Q/2 = Q/2 as expected.

The equations actually implemented into theMatlab-based **slabmm_gui** code include eqn. (10) for the flux profile, and eqns. (12) and (13), respectively, for the leakage and absorption rate components of the overall neutron balance. These give the appropriate relationships for the infinite medium model.

Planar Source in a Bare Finite Slab of Moderator

Consider the same isotropic planar source from above placed along the centerline of a ***finite slab*** of moderator of thickness 2H. If the external boundary condition for the bare slab is such that the flux goes to zero at the extrapolated boundary (i.e. at H+d), our goal is to again **formally derive** a result for $\phi(x)$ for this system. Also, as above, we want to develop analytical expressions for the net neutron leakage out of the right half of the block and for the absorption rate within the volume of interest -- and, again, this should equal to the neutron generation rate within the volume.

The development of the appropriate equations for this case follows the same procedure as above. However, although the defining balance equations are the same, it is convenient to write the general solution for a finite geometry in terms of hyperbolic sine and cosine functions [instead of the real exponentials given in eqn. (3)]. Thus, the general solution to eqn. (2) for the finite geometry case is usually written as

$$\phi(x) = A_1\sinh\alpha x + A_2\cosh\alpha x = A_1\sinh x/L + A_2\cosh x/L \tag{16}$$

where $\alpha = 1/L$. Also, we emphasize that eqn. (16) and eqn. (3) are equivalent representations for the general solution. However, the form given in eqn. (3) is simply more convenient for situations where the independent variable can become large, and eqn. (16) is better suited for finite geometry cases.

As before, to find A_1 and A_2 to give a unique solution, we need to apply two independent boundary constraints. At the right boundary of the bare slab (i.e. at x = H + d), we say that the flux goes to zero at the extrapolated boundary, where d is the extrapolation distance (this is the standard vacuum boundary condition used in the diffusion theory approximation to neutron transport). Mathematically this statement is written as

$$\phi\left(H+d\right)=0 \tag{17}$$

Therefore, from eqn. (16), we have

$$0 = A_1 \sinh(H+d)/L + A_2 \cosh(H+d)/L$$

or $$A_1 = -A_2 \frac{\cosh(H+d)/L}{\sinh(H+d)/L} \tag{18}$$

For the second boundary constraint, we apply the same source condition as above [see eqn. (8)] at the centerline of the block. For the present case, the current is given by

$$\vec{J}(x) = -D\frac{d\phi}{dx}\hat{i} = -\frac{D}{L}\left(A_1 \cosh x/L + A_2 \sinh x/L\right)\hat{i}$$

or $$J(x) = -\frac{DA_2}{L}\left(-\frac{\cosh(H+d)/L}{\sinh(H+d)/L}\cosh x/L + \sinh x/L\right) \tag{19}$$

Thus, evaluating eqn. (8) using this expression for the net neutron current gives

$$\lim_{x\to 0}\left\{-\frac{DA_2}{L}\left(-\frac{\cosh(H+d)/L}{\sinh(H+d)/L}\cosh x/L + \sinh x/L\right)\right\} = \frac{DA_2}{L}\frac{\cosh(H+d)/L}{\sinh(H+d)/L} = \frac{Q}{2}$$

or $$A_2 = \frac{QL}{2D}\frac{\sinh(H+d)/L}{\cosh(H+d)/L} \tag{20}$$

Note that substitution of this expression into eqn. (18) gives a simple result for A_1, or

$$A_1 = -\frac{QL}{2D} \tag{21}$$

Now putting the expressions for A_1 and A_2 back into the expression for $\phi(x)$ gives the ***flux due to a planar source in a finite medium*** as

$$\phi(x) = \frac{QL}{2D}\left[\frac{\sinh(H+d)/L}{\cosh(H+d)/L}\cosh x/L - \sinh x/L\right] \tag{22}$$

And, from the definition of the net current, we can also write the ***current due to a planar source in a finite medium*** as

$$J(x) = \frac{Q}{2}\left[\cosh x/L - \frac{\sinh(H+d)/L}{\cosh(H+d)/L}\sinh x/L\right] \tag{23}$$

where, of course, the net current for x > 0 is pointed in the +x direction.

To address the neutron balance within the finite bare slab, we again need to compute the leakage and absorption rates within the right half of the block using the flux and current expressions given respectively in eqns. (22) and (23). Thus, for the finite geometry case, we have

Leakage out of right side of the block of half width H:

$$\begin{matrix}\text{leakage} \\ \text{per unit area}\end{matrix} = \int_A \vec{J}\cdot\hat{n}\,dA = J(H)(1) = \frac{Q}{2}\left[\cosh H/L - \frac{\sinh(H+d)/L}{\cosh(H+d)/L}\sinh H/L\right] \tag{24}$$

Absorption rate within the block of half width H:

$$\begin{matrix}\text{absorption rate} \\ \text{per unit area}\end{matrix} = \int_0^H \Sigma_a \phi(x)(1)dx = \frac{QL\Sigma_a}{2D}\int_0^H\left[\frac{\sinh(H+d)/L}{\cosh(H+d)/L}\cosh x/L - \sinh x/L\right]dx$$

or
$$\begin{matrix}\text{absorption rate} \\ \text{per unit area}\end{matrix} = \frac{Q}{2L}\left[\frac{\sinh(H+d)/L}{\cosh(H+d)/L}L\sinh x/L - L\cosh x/L\right]_0^H$$

or
$$\begin{matrix}\text{absorption rate} \\ \text{per unit area}\end{matrix} = \frac{Q}{2}\left[\frac{\sinh(H+d)/L}{\cosh(H+d)/L}\sinh H/L - \cosh H/L\right] + \frac{Q}{2} \tag{25}$$

Source within right side of block:

$$\int_V Q\,\delta(x)\,d\vec{r} = \int_0^H Q\,\delta(x)\,dx = \frac{Q}{2} \quad \text{(where half of the neutrons go to the left side)} \tag{26}$$

Overall balance equation [leakage + absorption = source]***:***

$$\left\{\frac{Q}{2}\left[\cosh H/L - \frac{\sinh(H+d)/L}{\cosh(H+d)/L}\sinh H/L\right]\right\} + \left\{\frac{Q}{2}\left[\frac{\sinh(H+d)/L}{\cosh(H+d)/L}\sinh H/L - \cosh H/L\right] + \frac{Q}{2}\right\} = \frac{Q}{2} \tag{27}$$

or, Q/2 = Q/2 as expected.

In particular, the equations actually implemented into the **slabmm_gui** code for the finite bare slab geometry case include eqn. (22) for the flux profile, and eqns. (24) and (25), respectively, for the leakage and absorption rate components of the overall neutron balance. These give the appropriate 1-group relationships for the finite medium bare slab geometry model.

Summary

This document provides a detailed derivation of the 1-group flux, current, and neutron balance components (leakage and absorption) for the case of a planar source in a pure moderating medium -- with developments for both infinite and finite slab geometries centered on the source location. The resultant equations have been implemented into the **slabmm_gui** code, which provides an easy-to-use Matlab-based GUI where one can explore and contrast the use of different moderating materials and geometric dimensions. The formal development of the appropriate equations, coupled with the user-friendly computational tool, should allow the user to get a better understanding of the fundamental physics at play here, and also get a good feel for both the qualitative and quantitative aspects of 1-group neutron diffusion in a simple 1-D Cartesian geometry.

Have fun using the **slabmm** GUI!!! We hope that it helps in the visualization/understanding of the basic processes associated with the diffusion of neutrons originating from a planar source…

References

1. J. R. White, "Point Source in a Moderating Medium," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***spheremm_gui*** Matlab program.
2. J. R. White, "The Multigroup Neutron Balance Equation," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.
3. J. R. Lamarsh and A. J. Baratta, ***Introduction to Nuclear Engineering***, 3rd Edition, Prentice Hall (2001).