A 2-Group Example: Point Source of Fast Neutrons in an Infinite Moderating Medium

Most of the examples involving non-multiplying media applications up to now have used 1-group theory. In this set of Lecture Notes, we will treat a simple 2-group application to illustrate the basic mathematical procedure that is needed, and then delay further applications involving the use of 2-group theory until the critical reactor problem has been examined. This approach makes sense because 2-group diffusion theory is not usually applied to the pure shielding problem -- but it is indeed the primary workhorse for the core physics problem. Thus, the goal here is to get a qualitative view of neutron diffusion within the 2-group approximation, and to overview the sequential solution scheme for each group balance equation (not to accurately predict the actual flux level in such systems).

The 2-group diffusion equation for a homogeneous non-multiplying medium can be written as follows (see the general multigroup diffusion equation from Ref. 1 with constant material properties, no fission, and no upscatter):

$$-D_1\nabla^2\phi_1 + (\Sigma_{a1} + \Sigma_{1\to 2})\phi_1 = Q_1 \qquad (1a)$$

$$-D_2\nabla^2\phi_2 + \Sigma_{a2}\phi_2 - \Sigma_{1\to 2}\phi_1 = Q_2 \qquad (1b)$$

Since the coefficients are constant, division by the diffusion coefficient gives

$$\nabla^2\phi_1 - \frac{1}{L_1^2}\phi_1 = -\frac{Q_1}{D_1} \qquad (2a)$$

$$\nabla^2\phi_2 - \frac{1}{L_2^2}\phi_2 = -\frac{Q_2}{D_2} - \frac{\Sigma_{1\to 2}}{D_2}\phi_1 \qquad (2b)$$

where L_1^2 and L_2^2 are the fast and thermal diffusion areas.

Now, to illustrate the application of the 2-group diffusion equation in a diffusing medium, let's assume that we have a point source of fast neutrons (Q_1 neutrons/sec) in an infinite homogeneous non-multiplying medium where it is valid to assume that $\Sigma_{1\to 2} >> \Sigma_{a1}$ (which allows replacing the fast diffusion area with the neutron age, $\tau_T = D_1/\Sigma_{1\to 2}$). In this case, eqn. (2) becomes

$$\nabla^2\phi_1 - \frac{1}{\tau_T}\phi_1 = 0 \qquad r > 0 \qquad (3a)$$

$$\nabla^2\phi_T - \frac{1}{L_T^2}\phi_T = -\frac{\Sigma_{1\to 2}}{D_2}\phi_1 \qquad (3b)$$

Note here that we have decided to use a subscript T to denote the cross sections and flux within the thermal group (except for D_2). Also notice that since the fast fixed source is only non-zero at r = 0, we will treat this as a source condition and exclude the point r = 0 when solving the fast group equation. One final, but important, observation is that eqns. (3a) and (3b) are only sequentially coupled. This means that the fast group equation is independent of ϕ_T, and it can be solved without consideration of the thermal equation. Then, once ϕ_1 is known, ϕ_T can be obtained from the solution of eqn. (3b).

In performing this procedure, one should note that the ϕ_1 equation is identical in form to the 1-group point source problem that was solved previously [see Refs. 2 or 3]. Therefore, by analogy, we can immediately write the solution to eqn. (3a) as

$$\phi_1(r) = \frac{Q_1}{4\pi D_1}\frac{1}{r}e^{-r/\sqrt{\tau_T}} \tag{4}$$

With $\phi_1(r)$ known, eqn. (3b) for $\phi_T(r)$ in a 1-D spherical coordinates' system becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\phi_T\right) - \frac{1}{L_T^2}\phi_T = -\frac{\Sigma_{1\to 2}}{D_2}\frac{Q_1}{4\pi D_1}\frac{1}{r}e^{-r/\sqrt{\tau_T}} = -\frac{Q_1}{4\pi\tau_T D_2}\frac{1}{r}e^{-r/\sqrt{\tau_T}} \tag{5}$$

This expression is a linear non-homogeneous 2nd order ***variable-coefficient*** differential equation. As detailed in Ref. 3, the substitution $\phi_T = \omega/r$ can simplify this expression considerably, giving

$$\frac{d^2}{dr^2}\omega - \frac{1}{L_T^2}\omega = -\frac{Q_1}{4\pi\tau_T D_2}e^{-r/\sqrt{\tau_T}} \tag{6}$$

which is now a linear non-homogeneous 2nd order ***constant-coefficient*** ODE.

The standard solution technique for solving eqn. (6) is to write the general solution as the linear combination of the homogeneous and particular solutions. From the 1-group point source problem treated earlier (Ref. 3), we know that the homogeneous solution is simply

$$\omega_h(r) = A_1 e^{-r/L_T} + A_2 e^{r/L_T} \tag{7}$$

Now, if we assume that the particular solution has the same functional behavior as the down-scatter source, we might try

$$\omega_p(r) = Ce^{-r/\sqrt{\tau_T}} \tag{8}$$

as a potential particular solution [notice that this has the same form as the forcing function in eqn. (6), including all of its linearly independent derivatives -- and this follows the general rule for finding the particular solution within the Method of Undetermined Coefficients]. Putting this assumed solution into the defining source-driven ODE [i.e. eqn. (6)] gives

$$\frac{C}{\tau_T}e^{-r/\sqrt{\tau_T}} - \frac{C}{L_T^2}e^{-r/\sqrt{\tau_T}} = -\frac{Q_1}{4\pi\tau_T D_2}e^{-r/\sqrt{\tau_T}}$$

and, solving for C gives

$$C = -\frac{Q_1}{4\pi\tau_T D_2}\left[\frac{1}{1/\tau_T - 1/L_T^2}\right] = -\frac{Q_1 L_T^2}{4\pi D_2(L_T^2 - \tau_T)} \tag{9}$$

Thus, our assumed solution was correct and we can now write the general solution as

$$\phi_T(r) = A_1\frac{e^{-r/L_T}}{r} + A_2\frac{e^{r/L_T}}{r} - \frac{Q_1 L_T^2}{4\pi D_2(L_T^2 - \tau_T)}\frac{e^{-r/\sqrt{\tau_T}}}{r} \tag{10}$$

The next step in the standard solution procedure is to apply appropriate boundary conditions so as to uniquely determine the A_1 and A_2 coefficients within the general solution. In this case, the fact that the flux must remain finite as r → ∞, immediately forces $A_2 = 0$. With this constraint, eqn. (10) becomes

$$\phi_T(r) = A_1 \frac{e^{-r/L_T}}{r} - \frac{Q_1 L_T^2}{4\pi D_2 (L_T^2 - \tau_T)} \frac{e^{-r/\sqrt{\tau_T}}}{r} \tag{11}$$

Also, since there must be symmetry around the point r = 0 and there is no discrete or discontinuous source of thermal neutrons at r = 0, the second boundary condition is simply that the thermal leakage out of a sphere of radius r is zero as r → 0. We can impose this constraint as a mathematical limiting expression, or

$$\underset{\text{as } r\to 0}{\text{leakage}} = \lim_{r\to 0} J_T(r)\, 4\pi r^2 = 0 \tag{12}$$

where

$$\begin{aligned} J_T(r) &= -D_2 \frac{d}{dr}\left[A_1 \frac{e^{-r/L_T}}{r} + C \frac{e^{-r/\sqrt{\tau_T}}}{r} \right] \\ &= D_2 \left[A_1 \left(\frac{1}{rL_T} + \frac{1}{r^2} \right) e^{-r/L_T} + C \left(\frac{1}{r\sqrt{\tau_T}} + \frac{1}{r^2} \right) e^{-r/\sqrt{\tau_T}} \right] \end{aligned} \tag{13}$$

Substituting eqn. (13) into eqn. (12) and taking the limit as r → 0 gives

$$\lim_{r\to 0} 4\pi D_2 \left[A_1 \left(\frac{r}{L_T} + 1 \right) e^{-r/L_T} + C \left(\frac{r}{\sqrt{\tau_T}} + 1 \right) e^{-r/\sqrt{\tau_T}} \right] = 4\pi D_2 \left(A_1 + C \right) = 0 \tag{14}$$

or $$A_1 = -C \tag{15}$$

Therefore, the final solution for the thermal flux is

$$\phi_T(r) = \frac{Q_1 L_T^2}{4\pi D_2 (L_T^2 - \tau_T)} \frac{1}{r} \left(e^{-r/L_T} - e^{-r/\sqrt{\tau_T}} \right) \tag{16}$$

Thus, we have solved our first 2-group problem! This was done for the specific case where there is a point isotropic source of fast neutrons in an infinite non-multiplying medium. The spatial distribution of the fast flux is given by eqn. (4) and the thermal flux profile is given by eqn. (16). The ratio of these two flux profiles -- referred to as the fast-to-thermal flux ratio, ϕ_1/ϕ_T -- is also of interest, and this can be written as

$$\frac{\phi_1(r)}{\phi_T(r)} = \frac{D_2}{D_1} \frac{(L_T^2 - \tau_T)}{L_T^2} \frac{e^{-r/\sqrt{\tau_T}}}{e^{-r/L_T} - e^{-r/\sqrt{\tau_T}}} \tag{17}$$

To help visualize the spatial behavior of these profiles, a short Matlab program called **ptsrc2g.m** was written to evaluate and plot $\phi_1(r)$, $\phi_T(r)$, and ϕ_1/ϕ_T for various moderators with $Q_1 = 1$ neutron/sec (i.e. a unit source). The fast and thermal material data were obtained from Table 5.3 and Table 5.2 in Lamarsh (Ref. 2), respectively, and the resultant profiles are contained in Figs. 1 and 2. Note that the material options within the code (see code listing in Table 1) include

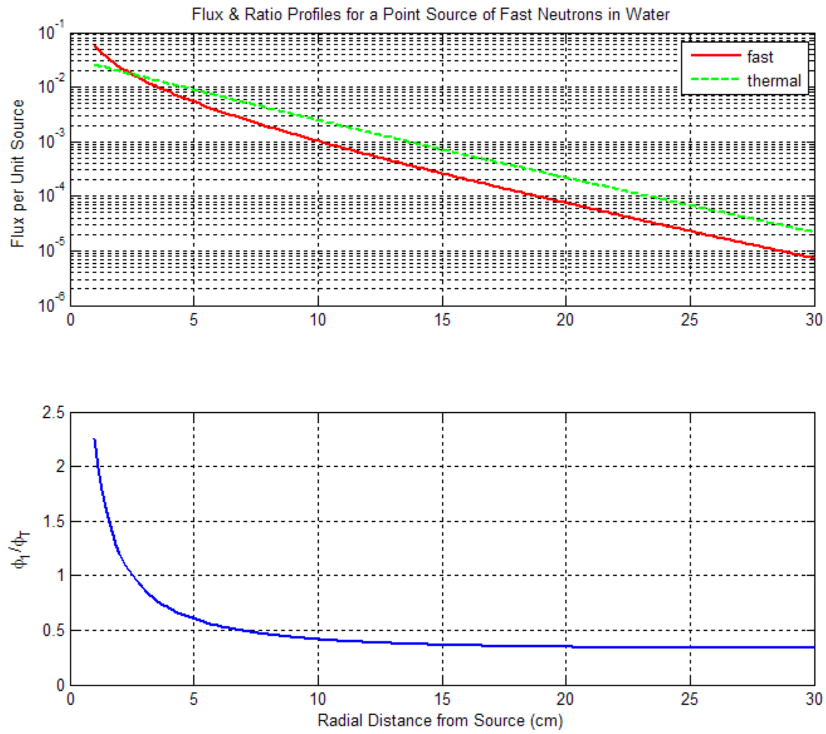

Fig. 1 Results from ptsrc2g.m for an infinite water medium.

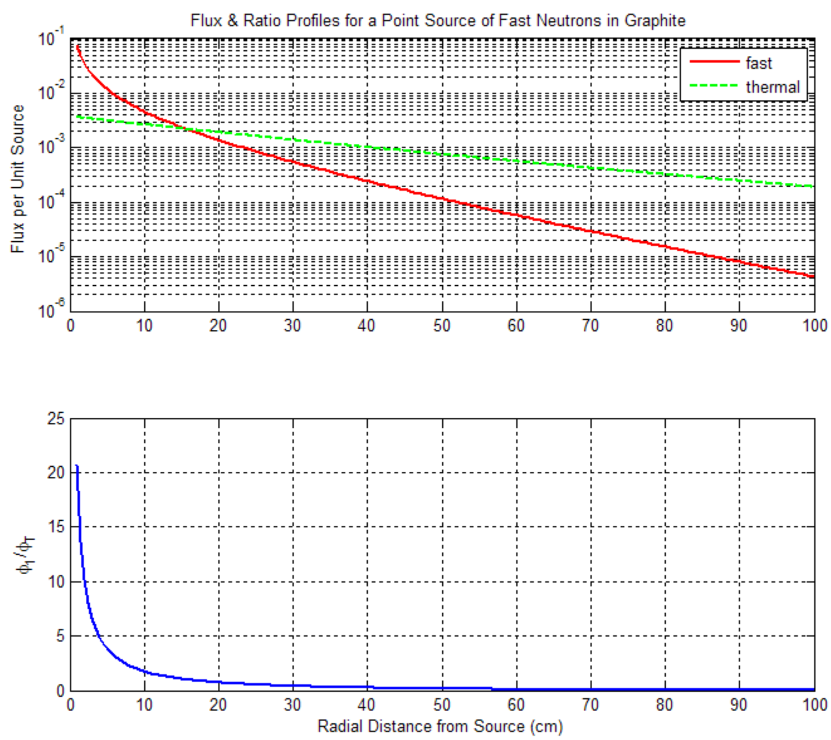

Fig. 2 Results from ptsrc2g.m for an infinite graphite medium.

water, graphite, and beryllium, but only the water and graphite profiles are shown here. In addition, a user option to select either a linear of logarithmic y-axis for the flux profiles is available since substantial attenuation over several orders of magnitude can occur. Again, only one of the options (logarithmic axis) is shown here, but the reader is encouraged to actually run the **ptsrc2g** code to explore other possibilities.

Concerning the results, first we note that the decrease in the flux level in both groups is faster in water than in graphite since, as we have seen before in Ref. 3, the diffusion lengths in graphite are much larger than for water (note that, for the plots, we let r go to 100 cm for the graphite case and only to 30 cm for the water moderator case). Also of interest in this 2-group model is the relative behavior of the fast and thermal fluxes for a given material. In particular, note that, after a short distance, the fast-to-thermal flux ratio approaches a constant in water but, in graphite, the fast flux continues to attenuate at a faster rate than the thermal flux, with the ϕ_1/ϕ_T ratio eventually approaching zero. Thus, for example, a large thickness of graphite could be used within an experimental facility to selectively filter out the high energy neutrons, leaving a nearly pure population of thermal neutrons -- such as in the graphite thermal column in the UMass-Lowell research reactor (UMLRR)…

To see why graphite behaves differently from water, we only need to look at the diffusion properties of these two materials (from Ref. 2), as shown in the short table below:

Material/Property	D_1 (cm)	τ_T (cm^2)	$1/\sqrt{\tau_T}$ (cm^{-1})	D_2 (cm)	L_T^2 (cm^2)	$1/L_T$ (cm^{-1})
water	1.13	27	0.192	0.16	8.1	0.351
graphite	1.02	368	0.052	0.84	3500	0.017

From here we see that, ***for water***, $1/\sqrt{\tau_T} < 1/L_T$, which says that e^{-r/L_T} decreases faster than $e^{-r/\sqrt{\tau_T}}$. Thus, as $r \to \infty$, we have

$$\lim_{r\to\infty} \frac{e^{-r/\sqrt{\tau_T}}}{e^{-r/L_T} - e^{-r/\sqrt{\tau_T}}} \to -1$$

and

$$\left.\frac{\phi_1}{\phi_T}\right|_\infty = -\frac{D_2}{D_1}\frac{(L_T^2 - \tau_T)}{L_T^2} = -\left(\frac{0.16}{1.13}\right)\left(\frac{8.1-27}{8.1}\right) \approx 0.33 \qquad (\textbf{for water})$$

But, ***for graphite***, $1/\sqrt{\tau_T} > 1/L_T$, which says that $e^{-r/\sqrt{\tau_T}}$ decreases faster than e^{-r/L_T}. Thus, as $r \to \infty$, we have

$$\lim_{r\to\infty} \frac{e^{-r/\sqrt{\tau_T}}}{e^{-r/L_T} - e^{-r/\sqrt{\tau_T}}} \to 0$$

and $\left.\frac{\phi_1}{\phi_T}\right|_\infty \approx 0$ (**for graphite**)

Of course, these quantitative results agree with the plots of ϕ_1/ϕ_T from above, and these help us rationalize the observed trends -- and the particular observation made here is extremely useful for the design of experimental facilities with relatively high thermal fluxes and relatively low fast neutron contamination.

Finally, we note once again that the 2-group diffusion theory model used here only gives a qualitative perspective on the behavior of neutron diffusion in non-multiplying media. Since there is indeed a strong angular dependence of the flux in the direction away from the source, the original approximations made to reduce the general neutron balance equation to the diffusion equation are not really valid for these type of analyses -- and, in practice, transport theory is usually used to get much better quantitative estimates of the neutron transport in these systems. However, our simple 2-group diffusion theory example still gives us a good qualitative understanding of these systems, and this was the primary physics-oriented goal of this example. Thus, along with the demonstration of how to mathematically solve a multigroup source-driven problem (with no upscatter), we have also gained a little physical insight into the neutronic behavior of general 2-group non-multiplying systems -- and both accomplishments should add significantly to your growing inventory of tools and experiences for understanding general steady state problems in reactor theory…

Table 1 Listing of the ptsrc.m code.

```
%
%   PTSRC2G.M  Plots the flux profiles associated with a Point Source of
%              Fast Neutrons in an Infinite Moderator
%
%   This file plots the fast and thermal flux profiles associated with a point
%   source of fast neutrons diffusing in an infinite moderating medium. Data
%   for water, graphite, and beryllium are available (from Lamarsh) for comparison
%   of how the diffusion parameters of different materials affect the diffusion and
%   attenuation of neutrons.
%
%   A plot of the fast-to-thermal flux ratio is also generated.
%
%   File prepared by J. R. White, UMass-Lowell (Jan. 2015)
%

      clear all,  close all,  nfig = 0;
%
%   define material properties with data from Lamarsh 3rd Ed.(Tables 5.2 and 5.3)
%   --> order: water, graphite, and beryllium
      DD1 = [1.13 1.016 0.562];  % fast diffusion coeff (cm)
      TT = [27 368 102];         % thermal neutron age (cm^2)
      TTSR = sqrt(TT);           % square root of neutron age (cm)
      DD2 = [0.16 0.84 0.50];    % thermal diffusion coeff (cm)
      LLS = [8.1 3500 480];      % diffusion area (cm^2)
      LL = sqrt(LLS);            % diffusion length (cm)
      TITL = ['Water    '; 'Graphite '; 'Beryllium'];
%
%   select moderator material
      K = menu('Choose Moderator Matl', ...
               '           Water           ', ...
               '         Graphite          ', ...
               '         Beryllium         ');
```

```
      D1 = DD1(K);  T = TT(K);    TSR = TTSR(K);
      D2 = DD2(K);  LS = LLS(K);  L = LL(K);   titl = TITL(K,:);
%
%   select type of plot
      IPLT = menu('Type of Flux Plot?', '   Linear Y-Axis   ','   Log Y-Axis   ');
%
%   define radial grid  (use R approx 5*fast diffusion length)
%   Note: start at about 1 cm to avoid the singularity at r = 0
      N = 200;   R = ceil(5*TSR/10)*10;    r = linspace(1,R,N);
%
%   compute fluxes and fast to thermal flux ratio (for unit source strength)
      c1 = 1/(4*pi*D1);   c2 = LS/(4*pi*D2*(LS-T));
      phi1 = c1*exp(-r/TSR)./r;
      phi2 = c2*(exp(-r/L) - exp(-r/TSR))./r;
      ratio = phi1./phi2;
%
%   plot flux profiles
      nfig = nfig+1;  figure(nfig)
      if  IPLT == 1,
        subplot(2,1,1),plot(r,phi1,'r-',r,phi2,'g--','LineWidth',2),grid
      end
      if  IPLT == 2,
        subplot(2,1,1),semilogy(r,phi1,'r-',r,phi2,'g--','LineWidth',2),grid
      end
      title(['Flux & Ratio Profiles for a Point Source of Fast Neutrons in ',titl])
      ylabel('Flux per Unit Source '),legend('fast','thermal')
%
%   plot fast-to-thermal flux ratio
      subplot(2,1,2),plot(r,ratio,'b-','LineWidth',2),grid
      xlabel('Radial Distance from Source (cm)'),ylabel('\phi_1/\phi_T')
%
%   end of problem
```

References

1. J. R. White, “The Multigroup Neutron Balance Equation,” part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.

2. J. R. Lamarsh and A. J. Baratta, ***Introduction to Nuclear Engineering***, 3rd Edition, Prentice Hall (2001).

3. J. R. White, “Point Source in a Moderating Medium,” part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***spheremm_gui*** Matlab program.