One-Speed Point Kinetics Equations

Introduction

Reactor kinetics is the study of the time-dependent behavior of a nuclear core or system under both normal and off-normal transient conditions over a relatively short period of time. The variations in the system are usually caused by changes in control rod positioning, soluble boron concentration, coolant and/or fuel temperature changes, etc. These changes can, directly or indirectly, add positive or negative reactivity into the system. If the system was previously at the steady state critical condition, these variations perturb the neutron balance so that the neutron multiplication factor, k, is no longer unity, and the system enters a transient state, where the neutron density will increase or decrease until a new balance is reached.

The physics of both critical and subcritical systems undergoing a transient is described by the time-dependent neutron balance equation. In previous lectures (see Ref. 1, for example), we set the time derivative term in the multigroup balance equation to zero, since the focus there was on steady state operation. In kinetics studies however, the $\partial\phi_g(\vec{r},t)/\partial t$ term becomes important, and it must be included in the full description of the balance equation.

In the current development, we will focus on the one-speed (one energy group) approximation to the multigroup balance equation since both approaches lead to the same set of space-independent equations. This approximation is important, since it greatly simplifies the notation and the ease with which we can manipulate the equations. On the other hand, although the final equations are identical, there are important subtle differences in the definition of several kinetics parameters that are defined during the development process. The most important difference here is related to the fact that the delayed neutron emission spectrum is somewhat softer (i.e. lower average energy) than the prompt fission spectrum. In the detailed multigroup formulation, this important distinction is retained, leading to the definition of a series of "effective" kinetics parameters that explicitly account for the different emission spectra. Unfortunately, in the 1-group approximation, the prompt and delayed neutron spectra are identical (all neutrons appear in the single group) -- thus, the effects associated with the different emission spectra cannot be treated. Nevertheless, the gain in understanding and clarity associated with the one-speed approximation more than offsets its negative attributes. Also, since we know that the use of the effective kinetics parameters is important in real analysis, we can simply use the more rigorous definitions (from the literature) within the equations developed here.

Even with removal of the energy variable, the remaining neutron balance equation still contains full space and time dependence -- this is referred to as the **Space-Time Kinetics Formulation**. However, there are many applications of interest (especially during normal operations) where only small reactivity changes are made, and no significant change in the spatial distribution of the neutron population is observed. For these situations, only the time-dependence of the magnitude of the neutron density is important. When this approximation is appropriate, a full space-time formulation is not required, and one can simplify things considerably by using a space-independent (space-integrated) kinetics formulation -- what is commonly referred to as the **Point Kinetics Equations**. Thus, the focus of this set of Lecture Notes is to develop the theory and terminology of **One-Speed Point Kinetics**, along with the two primary formulations used in most applications: the **Lifetime Formulation** and the **Generation Time Formulation**.

Point Kinetics Equations

For this development, we will start with the one-speed time-dependent diffusion equation, since the relatively simple 1-group treatment of the neutron balance equation gives all the essential features of interest here (see Refs. 2-4 for more detailed treatments that include the energy variable). In words, the neutron balance equation states that

$$\begin{array}{c}\text{rate of change}\\ \text{of neutron density}\end{array} = \begin{array}{c}\text{production rate of}\\ \text{neutrons per unit volume}\end{array} - \begin{array}{c}\text{loss rate of}\\ \text{neutrons per unit volume}\end{array}$$

and, for the 1-group approximation, we have

$$\frac{\partial n}{\partial t} = \frac{1}{v}\frac{\partial \phi}{\partial t} = \left[(1-\beta)\nu\Sigma_f\phi + \sum_i \lambda_i C_i + Q\right] - \left[-\vec{\nabla}\bullet D\vec{\nabla}\phi + \Sigma_a\phi\right] \qquad (1)$$

where all the standard notation applies:

n	neutron density (neutrons/cm^3)
$\phi = nv$	neutron flux (neutrons/cm^2-s)
v	neutron speed (cm/s)
$\beta = \sum_{i=1}^{6}\beta_i$	total delayed neutron fraction
$\nu\Sigma_f\phi$	total fission source density (neutrons/cm^3-s)
$(1-\beta)\nu\Sigma_f\phi$	prompt fission source density (neutrons/cm^3-s)
λ_i	decay constant for delayed precursor group i (1/s)
C_i	fictitious delayed precursor concentration for group i (atoms/cm^3)
$\sum_i \lambda_i C_i$	delayed fission source density (neutrons/cm^3-s)
Q	external neutron source density (neutron/cm^3-s)
$-\vec{\nabla}\bullet D\vec{\nabla}\phi$	neutron leakage rate per unit volume (neutrons/cm^3-s)
$\Sigma_a\phi$	neutron absorption rate density (neutrons/cm^3-s)

Note that, in steady state, $\partial n/\partial t = 0$, and this implies that the prompt and delayed fission sources plus the external source must exactly balance the leakage and absorption loss terms.

The time-dependent balance for the delayed precursors in group i also represents a particle balance equation, where the

$$\begin{array}{c}\text{rate of change}\\ \text{of precursor density}\end{array} = \begin{array}{c}\text{precursor production}\\ \text{rate per unit volume}\end{array} - \begin{array}{c}\text{precursor loss rate}\\ \text{per unit volume}\end{array}$$

or

$$\frac{\partial C_i}{\partial t} = \beta_i \nu\Sigma_f \phi - \lambda_i C_i \qquad \text{for } i = 1, 2, \cdots 6 \tag{2}$$

with

$\beta_i \nu\Sigma_f \phi$ production rate of delayed precursors in group i (precursors/cm^3-s)

$\lambda_i C_i$ decay rate of the ith delayed precursors (precursors/cm^3-s)

and, for steady state, these two terms must balance.

Equations (1) and (2) represent a set of seven coupled partial differential equations in space and time, where all the cross sections, fluxes, and source terms are functions of both space and time. In general, these equations are rather difficult to solve!!!

However, there are many applications in reactor operations when the spatial flux shape does not change significantly with time. For these cases, the general space-time description can be reduced to a point model (spatially integrated model) that includes time as the only independent variable. This procedure reduces the system to seven ordinary differential equations – which are significantly easier to solve, especially with the modern ODE solvers that are available in a variety of computational packages.

There is a formal procedure for doing this reduction, during which, the so-called effective kinetics parameters are defined precisely. The most general procedure usually starts with the energy-dependent (or multigroup) neutron balance equation, instead of the simple 1-group formulation given here. Our specialization allows a more straightforward development that gives identical point kinetics equations, with slightly less rigor in the definition of some parameters. Since the resultant differences in definition do not affect our present discussion and application of the final equations, we will proceed here with the 1-group formulation (since the notation is much easier to follow).

Thus, starting with eqns. (1) and (2), we assume that the flux can be separated into a slowly varying spatial distribution and a more rapidly varying amplitude function, where

$$\phi(\vec{r}, t) = \psi(\vec{r}, t)T(t) \approx \psi_o(\vec{r})T(t) \tag{3}$$

where the spatial distribution with the 'o' subscript represents the initial steady state value and T(t) represents the time-dependent amplitude of the neutron flux. Although the assumption that $\psi(\vec{r}, t) \approx \psi_o(\vec{r})$, is not really necessary during the formal derivation, we will make this common approximation here just to simplify some subsequent manipulations -- and this simplification is indeed often used in most practical applications anyway.

Now, we substitute eqn. (3) into the neutron and precursor balance equations and integrate the resultant equations over the spatial domain of interest to give

$$\frac{1}{v}\langle \psi_o \rangle \frac{dT}{dt} = \left[(1-\beta)\langle \nu\Sigma_f \psi_o \rangle T + \sum_i \lambda_i \langle C_i \rangle + \langle Q \rangle \right] - \left[\left\langle -\vec{\nabla} \bullet D\vec{\nabla}\psi_o \right\rangle + \langle \Sigma_a \psi_o \rangle \right] T \tag{4}$$

$$\frac{d}{dt}\langle C_i \rangle = \beta_i \langle \nu\Sigma_f \psi_o \rangle T - \lambda_i \langle C_i \rangle \qquad \text{for } i = 1, 2, \cdots 6 \tag{5}$$

where the angle bracket notation, $\langle \cdots \rangle$, implies the spatial integration over the given domain. Note, however, that all the quantities, in general, are still functions of time. These equations represent one form of the so-called **Point Kinetics Equations**.

Although eqns. (4) and (5) appear even more complicated than the original space-time neutron balance equations, it is important to recognize that they are indeed much simpler -- they represent a system of ordinary differential equations with time as the only independent variable. Since each term includes a spatial integral (i.e. the terms in the angle brackets), these equations represent the spatially integrated or point model that is used widely to describe a variety of reactor dynamics problems.

However, the equations given here are usually not used in this form for practical application. In particular, since the cross sections can be time dependent and under operator control (i.e. movement of a control rod affects Σ_a, etc.) almost every term in these equations can be modified to initiate a transient case. However, from an operational perspective, the effect of a change in the cross sections manifests itself as a change in the multiplication factor, k, (or in the reactivity, ρ). The manipulation of eqns. (4) and (5) to incorporate k or ρ directly leads to the traditional **Lifetime Formulation** (uses k) and **Generation Time Formulation** (uses ρ) of **Point Kinetics**. In these formulations, the multiplication factor, k(t), and reactivity, $\rho(t)$, become the driving force for initiating most transient analyses.

Note: In the context of the 1-group diffusion equation, k and ρ have the following formal definitions (which will be useful in subsequent manipulations):

$$k = \frac{\text{neutron production rate from fission}}{\text{loss rate}} = \frac{\text{production}}{\text{loss}} = \frac{\langle \nu\Sigma_f \psi_o \rangle}{\langle -\vec{\nabla}\bullet D\vec{\nabla}\psi_o \rangle + \langle \Sigma_a \psi_o \rangle} \tag{6}$$

$$\rho = \frac{k-1}{k} = \frac{\text{production - loss}}{\text{production}} = \frac{\langle \nu\Sigma_f \psi_o \rangle - \left[\langle -\vec{\nabla}\bullet D\vec{\nabla}\psi_o \rangle + \langle \Sigma_a \psi_o \rangle \right]}{\langle \nu\Sigma_f \psi_o \rangle} \tag{7}$$

The Lifetime Formulation

For this representation of point kinetics, one first defines the prompt neutron lifetime. This can be done in an intuitive fashion by arguing that, at steady state, the total neutron loss rate is given by the total neutron population divided by the neutron lifetime. In equation form, this can be written as

$$\text{loss rate} = \frac{\text{neutron population}}{\text{lifetime}} \qquad \text{or} \qquad \text{lifetime} = \frac{\text{neutron population}}{\text{loss rate}}$$

Defining l_p as the prompt neutron lifetime, the one-speed approximation gives

$$l_p = \frac{\frac{1}{v}\langle \psi_o \rangle}{\langle -\vec{\nabla}\bullet D\vec{\nabla}\psi_o \rangle + \langle \Sigma_a \psi_o \rangle} \tag{8}$$

where the numerator is just the total neutron population at steady state conditions (recall that $\phi = nv$ or $n = \phi/v$).

We now divide every term in eqn. (4) by the neutron loss rate to give

$$\frac{\frac{1}{v}\langle\psi_o\rangle}{\langle-\vec{\nabla}\bullet D\vec{\nabla}\psi_o\rangle+\langle\Sigma_a\psi_o\rangle}\frac{dT}{dt}=(1-\beta)\frac{\langle\nu\Sigma_f\psi_o\rangle}{\langle-\vec{\nabla}\bullet D\vec{\nabla}\psi_o\rangle+\langle\Sigma_a\psi_o\rangle}T \;+\; \sum_i\lambda_i\frac{1}{\langle-\vec{\nabla}\bullet D\vec{\nabla}\psi_o\rangle+\langle\Sigma_a\psi_o\rangle}\langle C_i\rangle$$
$$+\quad\frac{1}{\langle-\vec{\nabla}\bullet D\vec{\nabla}\psi_o\rangle+\langle\Sigma_a\psi_o\rangle}\langle Q\rangle\;-\;\frac{\langle-\vec{\nabla}\bullet D\vec{\nabla}\psi_o\rangle+\langle\Sigma_a\psi_o\rangle}{\langle-\vec{\nabla}\bullet D\vec{\nabla}\psi_o\rangle+\langle\Sigma_a\psi_o\rangle}T$$

and use the definitions of k and l_p to simplify to

$$l_p\frac{dT}{dt}=\left[(1-\beta)k-1\right]T\quad+\quad\sum_i\lambda_i\frac{1}{\langle-\vec{\nabla}\bullet D\vec{\nabla}\psi_o\rangle+\langle\Sigma_a\psi_o\rangle}\langle C_i\rangle\quad+\quad\frac{1}{\langle-\vec{\nabla}\bullet D\vec{\nabla}\psi_o\rangle+\langle\Sigma_a\psi_o\rangle}\langle Q\rangle$$

Dividing by l_p gives

$$\frac{dT}{dt}=\frac{\left[(1-\beta)k-1\right]}{l_p}T\quad+\quad\sum_i\lambda_i\frac{1}{\frac{1}{v}\langle\psi_o\rangle}\langle C_i\rangle\quad+\quad\frac{1}{\frac{1}{v}\langle\psi_o\rangle}\langle Q\rangle \tag{9}$$

Now, if we define the time dependent ***normalized*** precursor and external source amplitudes as

$$c_i(t)=\frac{1}{\frac{1}{v}\langle\psi_o\rangle}\langle C_i(t)\rangle \tag{10}$$

$$q(t)=\frac{1}{\frac{1}{v}\langle\psi_o\rangle}\langle Q(t)\rangle \tag{11}$$

we get the standard form of the neutron balance equation for the lifetime formulation of point kinetics, or

$$\frac{dT}{dt}=\frac{\left[(1-\beta)k-1\right]}{l_p}T\quad+\quad\sum_i\lambda_i c_i\quad+\quad q \tag{12}$$

To complete the derivation, we need to manipulate the precursor balance equation in a similar fashion. In particular, dividing eqn. (5) by the total neutron population, $\langle\psi_o\rangle/v$, gives

$$\frac{1}{\frac{1}{v}\langle\psi_o\rangle}\frac{d}{dt}\langle C_i\rangle=\beta_i\frac{\langle\nu\Sigma_f\psi_o\rangle}{\frac{1}{v}\langle\psi_o\rangle}T\quad-\quad\lambda_i\frac{1}{\frac{1}{v}\langle\psi_o\rangle}\langle C_i\rangle\qquad\text{for } i=1,2,\cdots 6$$

Using eqns. (8) and (10), this expression for the precursor balance can be converted into standard form, giving

$$\frac{dc_i}{dt}=\beta_i\frac{k}{l_p}T\quad-\quad\lambda_i c_i\qquad\text{for } i=1,2,\cdots 6 \tag{13}$$

Equations (12) and (13) represent the Lifetime Formulation of the point kinetics equations. Once l_p has been defined, the conversion of eqns. (4) and (5) into this form follows a fairly straightforward algebraic procedure. The process, however, forces one to think about the meaning of each term and, in particular, it emphasizes that the precursor amplitudes and external source magnitude in the final equations really represent normalized spatially integrated quantities -- and this fact is important when these equations are used in practical applications. In fact, the main reason for deriving the equations here (rather than simply using the final equations from an appropriate reference) was to emphasize this point. Thus, when using eqns. (12) and (13), one always needs to be aware of the real meaning of each term.

The Generation Time Formulation

The development of the generation time formulation of point kinetics follows the same general procedure as above. Here, we first define the prompt generation time and then use this, and the definition of reactivity, to formally convert eqns. (4) and (5) into standard form. In particular, in an analogous fashion to the above development, we define the neutron production rate from fission in a steady state critical system as the total neutron population divided by the neutron generation time, or

$$\text{production rate} = \frac{\text{neutron population}}{\text{generation time}} \qquad \text{or} \qquad \text{generation time} = \frac{\text{neutron population}}{\text{production rate}}$$

Defining Λ as the prompt neutron generation time, the one-speed approximation gives

$$\Lambda = \frac{\frac{1}{v}\langle \psi_o \rangle}{\langle \nu\Sigma_f \psi_o \rangle} \tag{14}$$

Now we divide every term in eqn. (4) by the neutron production rate from fission to give

$$\frac{\frac{1}{v}\langle \psi_o \rangle}{\langle \nu\Sigma_f \psi_o \rangle}\frac{dT}{dt} = (1-\beta)\frac{\langle \nu\Sigma_f \psi_o \rangle}{\langle \nu\Sigma_f \psi_o \rangle}T \quad + \quad \sum_i \lambda_i \frac{1}{\langle \nu\Sigma_f \psi_o \rangle}\langle C_i \rangle$$
$$+ \quad \frac{1}{\langle \nu\Sigma_f \psi_o \rangle}\langle Q \rangle \quad - \quad \frac{\langle -\vec{\nabla}\bullet D\vec{\nabla}\psi_o \rangle + \langle \Sigma_a \psi_o \rangle}{\langle \nu\Sigma_f \psi_o \rangle}T$$

and use the definitions of ρ and Λ to simplify to

$$\Lambda\frac{dT}{dt} = \left[\frac{\langle \nu\Sigma_f \psi_o \rangle - \left[\langle -\vec{\nabla}\bullet D\vec{\nabla}\psi_o \rangle + \langle \Sigma_a \psi_o \rangle\right]}{\langle \nu\Sigma_f \psi_o \rangle} - \beta\right]T \quad + \quad \sum_i \lambda_i \frac{1}{\langle \nu\Sigma_f \psi_o \rangle}\langle C_i \rangle \quad + \quad \frac{1}{\langle \nu\Sigma_f \psi_o \rangle}\langle Q \rangle$$

or

$$\Lambda\frac{dT}{dt} = (\rho-\beta)T \quad + \quad \sum_i \lambda_i \frac{1}{\langle \nu\Sigma_f \psi_o \rangle}\langle C_i \rangle \quad + \quad \frac{1}{\langle \nu\Sigma_f \psi_o \rangle}\langle Q \rangle \tag{15}$$

Now, if we use the ***normalized*** precursor and external source amplitudes as defined in eqns. (10) and (11), eqn. (15) becomes

$$\Lambda \frac{dT}{dt} = (\rho - \beta) T \quad + \quad \sum_i \lambda_i \frac{\frac{1}{v}\langle \psi_o \rangle}{\langle \nu \Sigma_f \psi_o \rangle} c_i \quad + \quad \frac{\frac{1}{v}\langle \psi_o \rangle}{\langle \nu \Sigma_f \psi_o \rangle} q \tag{16}$$

But, with the definition of the prompt generation time, Λ, in eqn. (14), the final neutron balance equation results,

$$\frac{dT}{dt} = \frac{(\rho - \beta)}{\Lambda} T \quad + \quad \sum_i \lambda_i c_i \quad + \quad q \tag{17}$$

To complete the generation time formulation, we simply divide eqn. (5) by the total neutron population,

$$\frac{1}{\frac{1}{v}\langle \psi_o \rangle} \frac{d}{dt} \langle C_i \rangle = \beta_i \frac{\langle \nu \Sigma_f \psi_o \rangle}{\frac{1}{v}\langle \psi_o \rangle} T - \lambda_i \frac{\langle C_i \rangle}{\frac{1}{v}\langle \psi_o \rangle} \qquad \text{for } i = 1, 2, \cdots 6$$

and introduce the normalized precursor amplitude and definition of Λ to give

$$\frac{dc_i}{dt} = \frac{\beta_i}{\Lambda} T \quad - \quad \lambda_i c_i \qquad \text{for } i = 1, 2, \cdots 6 \tag{18}$$

Equations (17) and (18) represent the "standard" form of the Generation Time Formulation of the point kinetics equations.

Personal Note: Previous versions of these Lecture Notes used a slightly different formulation for eqns. (17) and (18). The earlier versions followed the suggestions made in Ref. 2 (which is slightly different from most other treatments), and defined the ***normalized*** precursor and external source amplitudes for the **Generation Time Formulation** as

$$c_i(t) = \frac{1}{\langle \nu \Sigma_f \psi_o \rangle} \langle C_i(t) \rangle \qquad \text{and} \qquad q(t) = \frac{1}{\langle \nu \Sigma_f \psi_o \rangle} \langle Q(t) \rangle$$

which then lead to a slightly different set of final equations, as follows:

$$\Lambda \frac{dT}{dt} = (\rho - \beta) T \quad + \quad \sum_i \lambda_i c_i \quad + \quad q$$

$$\frac{dc_i}{dt} = \beta_i T \quad - \quad \lambda_i c_i \qquad \text{for } i = 1, 2, \cdots 6$$

Ott's Generation Time Formulation

Although I personally like this formulation from Ott's text (Ref. 2), this approach caused more confusion with my previous students than was warranted -- because it was somewhat different from most of the readily available references. Thus, to avoid any unnecessary agony for future students, I have decided to modify all my Lecture Notes that use the Generation Time Formulation of Point Kinetics to conform to the "standard" form (as in Refs. 3 and 4, for example). Thus, all future reference to the Generation Time Formulation will use eqns. (17) and (18) as given above in the main development and **not** Ott's formulation as suggested in Ref. 2.

(JRWhite, 12/30/11)

Summary

The goal here was to derive the equations that are commonly utilized to represent and model the dynamics of nuclear systems and to get a good understanding of the meaning of each term. The one-speed approximation simplified the development (needed to integrate over only space instead of both space and energy) and the resultant expressions are identical to the ones developed using the multigroup formulation. The reader is cautioned, however, that one should always use the most accurate information available for a given system when doing actual analysis -- thus, if β_{eff} and Λ_{eff} are available for a particular system, for example, they definitely should be used!!! Also note that it was assumed that only one isotope contributes to the fission process whereas, in practice, β and λ would be isotope dependent in a detailed computational study. All these details were deliberately omitted here, so that only the basic elements of the balance equations (and their subsequent manipulation) could be highlighted. The goal, of course, was to keep things as simple as possible so that one does not get too lost in all the details. Hopefully this development was successful in achieving this goal -- since, once a good foundation is established, one can always go back and add further details as needed for a specific application (see Refs. 2-4, for example)…

References

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3. A. F. Henry, **Nuclear Reactor Analysis**, MIT Press (1975).
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