# A Note Concerning the Flux Spectrum in Thermal Systems

Introduction

This short note is simply to elaborate somewhat on the shape of the expected flux distribution in a thermal system. The additional treatment here was prompted by the different views that are often given and the fact that these can indeed be quite confusing if not labeled properly. Thus, the goal here is simply to try to minimize (or eliminate) this potential confusion.

To illustrate the dilemma that often occurs, consider the typical fast and thermal reactor spectra illustrated in the figure below. This figure is reproduced from page 34 of Module 2 of the DOE Handbook on Nuclear Physics and Reactor Theory (DOE-HDBK-1019). The two curves contrast the typical spectrum for a thermal system and the energy dependence of the flux for a fast breeder reactor. This comparison is exactly as one might expect, with the fast reactor having essentially no thermal neutrons -- since it has no moderator to slow down the original fission neutrons. Therefore, as expected, the spectrum in a fast reactor is indeed much harder than that observed in a thermal reactor. In addition, for the composite thermal spectrum, we observe something that resembles a rough fission spectrum at high energies and a profile that looks Maxwellian in nature at the low end of the energy spectrum. Thus, these profiles seem quite reasonable!!!
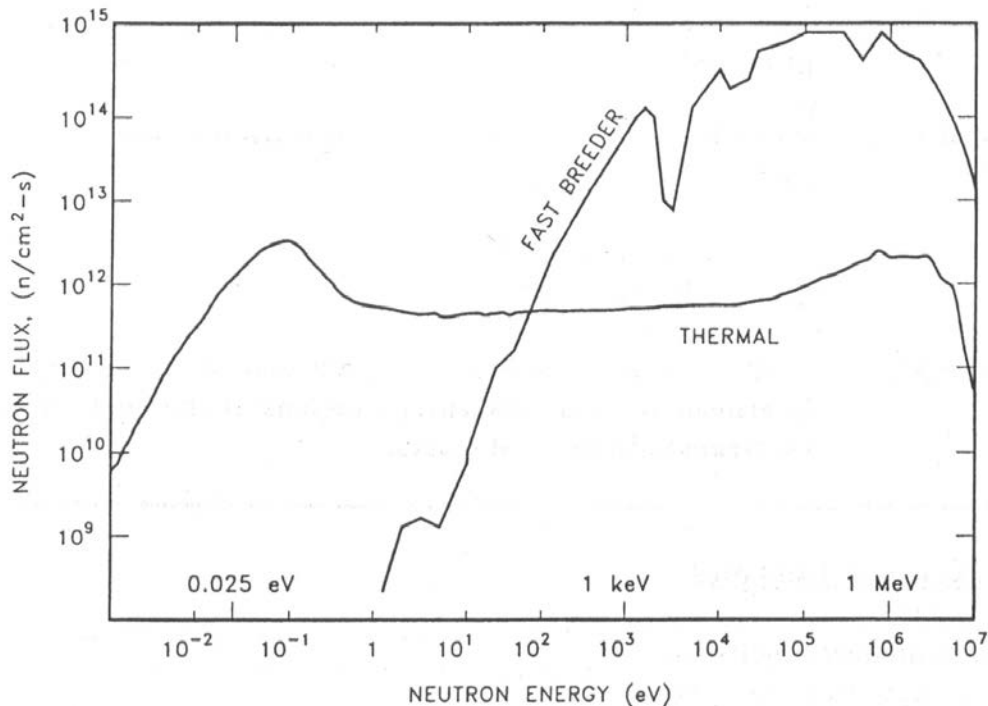
However, the text following the figure in the DOE Handbook goes on to explain the thermal flux profile as follows: "In the thermal reactor, the flux in the intermediate energy region (1 eV to 0.1 MeV) has approximately a 1/E dependence. That is, if the energy (E) is halved, the flux doubles. This 1/E dependence is caused by the slowing down process …".

Now, although this description is completely correct, it can indeed be very confusing -- since the flux in the roughly 1 eV to 0.1 MeV range in the figure appears to be nearly constant (or even has a slightly positive slope) instead of a slope of about -1. Since this picture is clearly not consistent with the above discussion, the obvious question is "What gives -- that is, what is wrong with this picture?". Well, in fact, everything is okay with the plot except that the y-axis label in the figure really should indicate that the curves show the flux density with units of neutrons/cm^2-s per unit lethargy. Unfortunately, many authors neglect to include this useful piece of information, and this can indeed be confusing to the student just learning about these concepts.

An Example -- Various Views of the Approximate Flux Profile in a Thermal System

In an attempt to shed some light on the above discussion, we will actually evaluate and plot, in a variety of ways, a rough estimate of the expected flux profile in a thermal system -- to illustrate the different views that are often presented. As we have already discussed, a very rough approximation for the neutron flux spectrum in an infinite homogeneous critical reactor with little resonance absorption is given by a fission spectrum at high energy, a $1/\Sigma_t E$ behavior at intermediate energies, and a thermal Maxwellian at low energies. In particular, these basic profiles, assuming that $\Sigma_t(E)$ is roughly constant in the intermediate region, lead to the following composite spectrum:

$$\phi(E) = \begin{cases} \phi_1(E) = c_1 e^{-1.036E} \sinh\sqrt{2.29E} & \text{for } E_2 \leq E \leq E_1 \\ \phi_2(E) = c_2/E & \text{for } E_3 \leq E \leq E_2 \\ \phi_3(E) = c_3 E e^{-E/kT} & \text{for } 0 \leq E \leq E_3 \end{cases} \qquad (1)$$

where we note that the thermal spectrum function, $\phi_3(E)$, is a product of n(E) and v(E) in the thermal region -- where the product of the $\sqrt{E}$ dependence in each of these separate functions leads to the E dependence given here [the exponential part is from the Maxwellian n(E) component]. In these expressions, E is given in MeV and the Boltzmann constant is 8.617×10^{-11} MeV/K.

To actually plot this functional dependence we need to specify the normalization factors, c_1, c_2, and c_3, and the specific 3-region energy boundaries, E_1, E_2, and E_3. Varying these quantities lead to considerably different spectra, so we need to specify a reasonable set of constraints and conditions if we hope to get something that resembles the flux spectrum in an actual thermal system. In doing this, we first determine the normalization coefficients by forcing continuity in the composite spectrum at E_2 and E_3 and by setting an overall (arbitrary) normalization of 10^{12} neutrons/cm^2-s. Doing this gives

Continuity at E_2: $\qquad\qquad \phi_1(E_2) = \phi_2(E_2)$ $\qquad\qquad\qquad\qquad\qquad$ (2a)

Continuity at E_3: $\qquad\qquad \phi_2(E_3) = \phi_3(E_3)$ $\qquad\qquad\qquad\qquad\qquad$ (2b)

Overall normalization: $\qquad \int_0^{E_1} \phi(E)dE = 1\times10^{12}\dfrac{\text{neutrons}}{\text{cm}^2-\text{s}}$ $\qquad\qquad$ (2c)

These three conditions give three equations with three unknowns and, if written in matrix form, we have

$$\begin{bmatrix} e^{-1.036E_2}\sinh\sqrt{2.29E_2} & -\dfrac{1}{E_2} & 0 \\[2ex] 0 & \dfrac{1}{E_3} & -E_3 e^{-E_3/kT} \\[2ex] \int_{E_2}^{E_1} e^{-1.036E}\sinh\sqrt{2.29E}\,dE & \int_{E_3}^{E_2}\dfrac{dE}{E} & \int_0^{E_3} E e^{-E/kT}\,dE \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10^{12} \end{bmatrix} \tag{3}$$

and, for a given set of energy boundaries and a given temperature, we can evaluate the matrix elements and solve the resultant equation for the desired normalization coefficients (Matlab's *quadl* routine, for example, could be used to perform the required integrals).

For the energy boundaries, we set $E_1 = 10$ MeV. This is a convenient choice for generating the desired plots and it represents a sufficiently high energy level such that essentially all the neutrons born in the fission process exist below this energy. For E_2 and E_3, these were changed somewhat arbitrarily to give a fast-to-thermal flux ratio in the range of 2.5 to 3.5 (since this ratio in the UMLRR is in this range). In doing this we defined the fast-to-thermal flux ratio as

$$\frac{\phi_{fast}}{\phi_{thermal}} = \frac{\int_{E_2}^{E_1}\phi_1(E)\,dE + \int_{E_3}^{E_2}\phi_2(E)\,dE}{\int_0^{E_3}\phi_3(E)\,dE} \tag{4}$$

Thus, the procedure followed was to make a reasonable guess for E_2 and E_3, solve eqn. (3) for the normalization coefficients, and then evaluate eqn. (4) to determine $\phi_{fast}/\phi_{thermal}$. This was done several times with various choices of E_2 and E_3 until a suitable value of $\phi_{fast}/\phi_{thermal}$ was obtained. This process eventually lead to the selection of $E_2 = 0.05$ MeV and $E_3 = 2\times10^{-7}$ MeV which gives a fast-to-thermal flux ratio of about 3.35 for T = 20 C (and this is a typical value for a fuel-structure-water mixture in a research reactor operating near room temperature).

With the selection of all the free variables in eqn. (1) for the composite flux spectrum $\phi(E)$, we can now turn our attention to the original goal of plotting this function in a variety of ways. However, it should be emphasized that our resultant $\phi(E)$ is only a very rough approximation to reality, since many simplifying assumptions were made in this development. Nevertheless, although the approach taken here is very simplistic, the resultant profile is quite reasonable and, more importantly for this exercise, it is certainly reasonable enough for illustration of how the various plotting techniques affect our visualization of the flux spectrum.

The most common visualization techniques include plotting $\phi(E)$ vs. E, ϕ_g vs. E, and $\phi(u)$ vs. E, where $\phi(E)$ is the flux per unit energy, ϕ_g is the group flux (energy integrated flux over the particular energy interval associated with group g), and $\phi(u)$ is the flux per unit lethargy, with lethargy defined as

$$u(E) = \ln\frac{E_{ref}}{E} \tag{5}$$

where E_{ref} is an arbitrarily high reference energy level.

For plotting $\phi(E)$ vs. E, we simply evaluate the composite flux spectrum given by eqn. (1) (as described above) and plot it versus energy using logarithmic axes for both $\phi(E)$ and E. The

resultant plot, generated by the **phiE_thermal.m** Matlab code (given at the end of these Notes), is shown below.
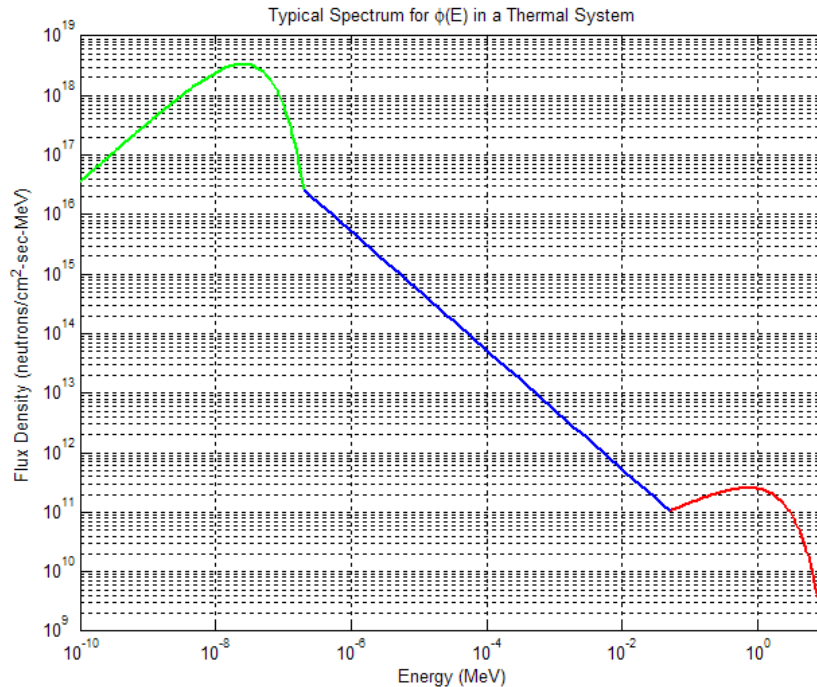
This flux distribution clearly shows the three distinct spectrum functions that make up the composite $\phi(E)$ spectrum, with the fission spectrum at high energies, a clearly observable 1/E behavior at intermediate energies, and a Maxwellian distribution at low energy. This visualization is clearly more consistent with the above quote from the DOE Handbook for the intermediate regions, but it is also clearly different from the thermal spectrum given in the Introduction section.

To plot ϕ_g vs. E, we first must create a fine-group energy grid, compute the discrete ϕ_g values, and then plot these values versus energy using a histogram-type plot that shows a constant group flux in each energy bin, with discontinuous changes at the group boundaries. For the energy grid, it is common to have roughly equal lethargy intervals. Thus, for plotting over a range of 10^{-10} MeV to 10 MeV, the total change in lethargy is given by

$$\Delta u_{tot} = \ln\left(\frac{10\,\text{MeV}}{10^{-10}\,\text{MeV}}\right) = \ln 10^{11} = 25.33$$

and, for a $\Delta u_g \approx 0.5$, this would lead to about 50 or 51 total groups. However, since we desire that the discrete boundaries for the three spectrum functions are included directly within the multigroup specification, we must be sure to compute the fine-group boundaries such that they fit evenly within each of the three larger supergroups. To do this, we first define the change in lethargy for group g, Δu_g, and the group boundaries in terms of Δu_g, as

$$\Delta u_g = \ln\left(\frac{E_g}{E_{g+1}}\right) \qquad \text{or} \qquad E_{g+1} = \frac{E_g}{e^{\Delta u_g}} \tag{6}$$

and then, using these expressions within a looping structure, we can easily define the fine-group energy boundaries and corresponding group fluxes separately for each of the three large energy groups, where ϕ_g is given by
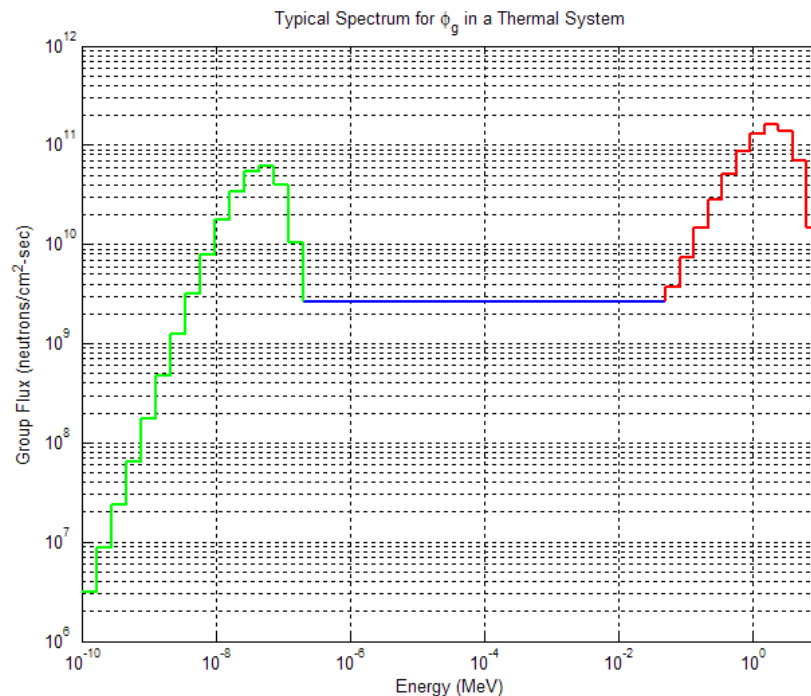
$$\phi_g = \int_{E_{g+1}}^{E_g} \phi(E)\,dE \qquad (7)$$

For example, for the fast region (from E_2 to E_1), we have

$$\Delta u_g = \frac{1}{Ng}\ln\left(\frac{E_1}{E_2}\right) = \frac{1}{11}\ln\left(\frac{10}{.05}\right) = 0.4817 \approx 0.5$$

Thus, Ng = 11 fine groups in the fast region of the full spectrum leads to a change in lethargy per group near 0.5 and the energy boundaries are determined via the 2nd part of eqn. (6) with E_1 known. The logic for creating the energy grid in this fashion and for computing the fine-group fluxes using *quadl* was coded into the **phiE_thermal1.m** Matlab function file and this code was called three times within **phiE_thermal.m** -- once for each of the three supergroups. In addition, this function file also expanded the group flux vector and the vector of energy boundaries, by repeating each entry such that the resultant plot gives the desired histogram structure with use of the usual Matlab *plot* command (see the code listing at the end of these notes for the details).

Finally, after the processing steps indicated above, the resultant plot of ϕ_g vs. E is shown below, where we note that this indeed looks quite different from the $\phi(E)$ vs. E plot. In fact, its basic structure actually looks quite similar to the thermal spectrum from the DOE Handbook -- except for the discrete versus continuous nature of the overall profile.

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Dr. John R. White, Chemical and Nuclear Engineering, UMass-Lowell (Sept. 2015)

As a final visualization method, let's look at a continuous plot of the flux per unit lethargy versus energy. To develop this relationship, we note that both $\phi(E)$ and $\phi(u)$ are density functions, and the total number of neutrons/cm^2-s in a given energy interval, dE, or corresponding lethargy interval, du, must be the same, or

$$\phi(u)\,du = -\phi(E)\,dE = \text{ \# neutrons/cm}^2\text{-s in interval} \tag{8}$$

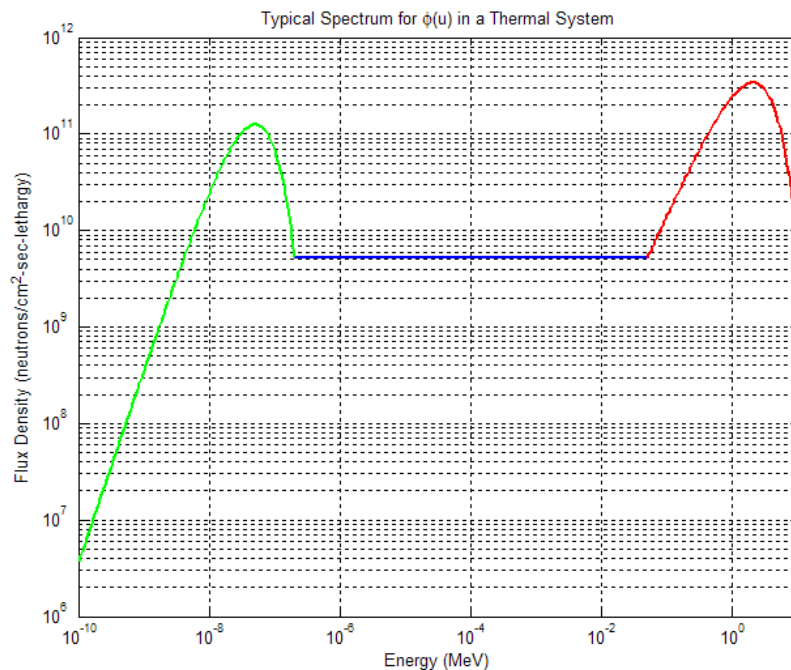where the negative sign accounts for the fact that a decrease in energy leads to an increase in lethargy.

Now with the relationship given in eqn. (8) and the definition of u(E) in eqn. (5), the flux per unit lethargy is simply given as

$$\phi(u) = -\phi(E)\frac{dE}{du} = -\phi(E)(-E) = E\phi(E) \qquad \text{or} \qquad \phi(u) = E\phi(E) \tag{9}$$

and, applying this relationship to the formulae for $\phi(E)$ from eqn. (1) for a thermal system, gives

$$\phi(u) = \begin{cases} c_1 E e^{-1.036E}\sinh\sqrt{2.29E} & \text{for } E_2 \leq E \leq E_1 \\ c_2 & \text{for } E_3 \leq E \leq E_2 \\ c_3 E^2 e^{-E/kT} & \text{for } 0 \leq E \leq E_3 \end{cases} \tag{10}$$

A plot of eqn. (10) versus energy, as shown below, gives the desired $\phi(u)$ vs. E. This visualization is very similar to that for ϕ_g vs. E (for the case where the energy intervals have roughly equal lethargy widths), except for the discontinuous versus discrete nature of the two profiles. Also, as implied above, this profile has the same basic structure as the original thermal profile from the DOE Handbook. Thus, from this comparison, it should be clear that the original y-axis labeling is not quite correct, since it should really say neutron flux per unit lethargy.

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Well, we have finally completed the task at hand -- that is, highlighting the differences between $\phi(E)$, ϕ_g, and $\phi(u)$, with a focus on thermal systems. The most common visualizations include plots of $\phi(u)$ or ϕ_g vs. E (i.e. continuous or discrete representations) -- and, more often than not, the labeling simply indicates that the neutron flux is being plotted (even though one should always be very explicit to avoid any possible ambiguities). However, after studying this example, you should be able to distinguish the difference between a plot of $\phi(E)$ or $\phi(u)$, even if the plot is not properly labeled. For example, it should be clear that the thermal spectrum plot from the DOE Handbook is clearly a flux per unit lethargy and, with this knowledge, the quote that accompanies the plot also makes perfect sense…

To conclude this note, we simply list the Matlab code used to generate the plots of $\phi(E)$, ϕ_g, and $\phi(u)$ versus energy, since this might prove to be useful in additional plotting exercises that you may encounter -- and remember to always properly label any plots that you may generate (since what may be obvious to the author is probably not always as clear to the reader…)…

Listing of Matlab codes phiE_thermal.m and phiE_thermal1.m

```
%
%   phiE_thermal.m   Plots typical phi(E), phi(u), and phig versus energy
%                      for a thermal system
%
%   The purpose of this file is to plot the flux distribution (versus energy) in a
%   variety of ways  --  specifically comparing the flux density, phi(E), with the
%   group integrated flux, phig.  A plot of the flux density per unit lethargy, phi(u),
%   is also given.
%
%   Note that the profiles here only give a very rough approximation and the two
%   energy boundaries where the three regions connect are somewhat arbitrary. However,
%   the plots still give a good picture of the differences between phi(E) and phig,
%   and the similarities between phi(u) and phig (especially for the case of nearly
%   equal lethargy widths). Here, E2 and E3 were chosen to give a fast to thermal
%   flux ratio somewhere in the range 2.5 - 3.5 (this is typical of a fuel element
%   in the UMLRR).
%
%   File prepared by J. R. White, UMass-Lowell  (Sept. 2015)
%

    clear all,  close all,  nfig = 0;
%
%   define energy boundaries for the 3 different functional relationships (E in MeV)
    E1 = 10;  E2 = 0.05;  E3 = 2e-7;  E4 = 1e-10;
%
%   define anonymous functions for phi(E) over 3 regions
    kT = (8.617e-11)*293;                      % kT at 20 C for E in MeV
    phiE1 = @(E) exp(-1.036*E).*sinh(sqrt(2.29*E));  % fission spectrum
    phiE2 = @(E) 1./E;                         % slowing down spectrum
    phiE3 = @(E) E.*exp(-E/kT);                % Maxwellian spectrum
%
%   compute coeffs to give continuity and a specified normalization (phi total = 1e12)
    a11 = phiE1(E2);  a12 = -phiE2(E2);
    a22 = phiE2(E3);  a23 = -phiE3(E3);
    a31 = quadl(phiE1,E2,E1); a32 = quadl(phiE2,E3,E2); a33 = quadl(phiE3,0,E3);
    A = [a11 a12 0; 0 a22 a23; a31 a32 a33];
    phitot = 1e12;   b = [0;  0;  phitot];
    c = A\b;
    disp('check on solution of A*c = b: '),  berr = A*c-b
%
%   plot phi(E) vs E (use log-log scale)
    Np1 = 100;   e1 = logspace(log10(E2),log10(E1),Np1);  flxE1 = c(1)*phiE1(e1);
    Np2 = 50;    e2 = logspace(log10(E3),log10(E2),Np2);  flxE2 = c(2)*phiE2(e2);
    Np3 = 100;   e3 = logspace(log10(E4),log10(E3),Np3);  flxE3 = c(3)*phiE3(e3);
```

```
                nfig = nfig+1;  figure(nfig)
                loglog(e1,flxE1,'r',e2,flxE2,'b',e3,flxE3,'g','LineWidth',2),grid
                rr = axis;  rr(2:4) = [10 1e9 1e19];  axis(rr);
                title('Typical Spectrum for \phi(E) in a Thermal System')
                xlabel('Energy (MeV)'),ylabel('Flux Density (neutrons/cm^2-sec-MeV)')
%
%   create a discrete energy grid with a lethargy width of roughly 0.5
%   and compute and plot the group fluxes over each discrete interval
                Ng1 = 11;   [eb1,flxg1] = phiE_thermal1(phiE1,c,E2,E1,Ng1);
                Ng2 = 25;   [eb2,flxg2] = phiE_thermal1(phiE2,c(2),E3,E2,Ng2);
                Ng3 = 15;   [eb3,flxg3] = phiE_thermal1(phiE,c(3),E4,E3,Ng3);
                nfig = nfig+1;  figure(nfig)
                loglog(eb1,flxg1,'r',eb2,flxg2,'b',eb3,flxg3,'g','LineWidth',2),grid on
                hold on
                loglog([eb1(1) eb1(1)],[flxg2(end) flxg1(1)],'r','LineWidth',2)
                loglog([eb2(1) eb2(1)],[flxg3(end) flxg2(1)],'g','LineWidth',2)
                rr = axis;  rr(2) = 10;  axis(rr);
                title('Typical Spectrum for \phi_g in a Thermal System')
                xlabel('Energy (MeV)'),ylabel('Group Flux (neutrons/cm^2-sec)')
                hold off
%
%   get a rough measure of the fast to thermal flux ratio
                fast2thm = (sum(flxg1)+sum(flxg2))/sum(flxg3);
                disp('rough estimate of fast to thermal flux ratio: '),fast2thm
%
%   now we want to generate a plot of phi(u) versus energy, where u = lethargy
%
%   define anonymous functions for phi(u) = E*phi(E) over 3 regions
                phiu1 = @(E) c(1)*E.*exp(-1.036*E).*sinh(sqrt(2.29*E)); % fission spectrum
                phiu2 = @(E) c(2)*E./E;                                % slowing down spectrum
                phiu3 = @(E) c(3)*E.^2.*exp(-E/kT);                    % Maxwellian spectrum
%
%   plot phi(u) vs E (log-log scale)
                flxu1 = phiu1(e1);  flxu2 = phiu2(e2);  flxu3 = phiu3(e3);
                nfig = nfig+1;  figure(nfig)
                loglog(e1,flxu1,'r',e2,flxu2,'b',e3,flxu3,'g','LineWidth',2),grid
                rr = axis;  rr(2) = 10;  axis(rr);
                title('Typical Spectrum for \phi(u) in a Thermal System')
                xlabel('Energy (MeV)'),ylabel('Flux Density (neutrons/cm^2-sec-lethargy)')
%
%   end of program




%
%   phiE_thermal1.m   Creates group flux vector for plotting as histogram
%                       (for use with phiE_thermal.m)
%
%   File prepared by J. R. White, UMass-Lowell  (Sept. 2015)
%
                function [x,y] = phiE_thermal1(phi,c,E2,E1,Ng)
%
%   create a discrete energy grid with a constant lethargy width
%   and compute the group fluxes over each discrete interval
                delu = log(E1/E2)/Ng,    rr = exp(delu);
                eb = zeros(Ng+1,1);  flxg = zeros(Ng,1); eb(1) = E1;
                for k = 1:Ng
                   eb(k+1) = eb(k)/rr;
                   flxg(k) = c*quadl(phi,eb(k+1),eb(k));
                end
%
%   expand group flux vector so it can be plotted as a histogram (constant in the grp)
                icnt = 0;   x = zeros(2*Ng,1);  y = zeros(2*Ng,1);
                for k = Ng:-1:1
                   icnt = icnt+1;  x(icnt) = eb(k+1);  y(icnt) = flxg(k);
                   icnt = icnt+1;  x(icnt) = eb(k);     y(icnt) = flxg(k);
                end
```