The Bare Critical Finite Cylindrical Reactor

Introduction

Up to this point we have found the analytical solution to several 1-D problems involving both critical reactors and systems containing only moderating materials (i.e. the source driven problem).[1-6] However, extension to 2-D geometries is not straightforward -- since analytical solutions to general multidimensional problems are not available for most systems. Thus, in most cases, we need to use numerical methods to solve the 2-D and/or 3-D diffusion equation for the geometries of interest in realistic reactor configurations. One exception, however, is the bare homogeneous 1-region reactor problem, and our goal here is to illustrate the basic solution technique and the resultant flux and current profiles for the case of the bare critical 2-D finite cylindrical reactor, where the quantities of interest here are $\phi(r,z)$ and $\vec{J}(r,z)$.

It should be emphasized that a bare reactor system is clearly not a realistic reactor configuration – all real systems have nearly infinite reflectors on all sides of the reactor to improve the neutron economy within the core and to reduce the radiation environment outside the reactor. However, as we have seen before, solution of the bare reactor problem can give us a lot of insight and, for multidimensional problems, even the simple core-reflector problem cannot be solved analytically (when reflected on all sides). Thus, we will use the 2-D bare reactor model as an illustrative example of multidimensional systems, and gain as much understanding as possible -- we will defer discussion of more complex reactor geometries for a later lesson…

Model Development

Consider a bare finite cylindrical reactor with extrapolated dimensions R and H, where $R = R_o + d$ and $H = H_o + 2d$, as shown in the sketch. The 1-group critical reactor model for this two dimensional (2-D) system is

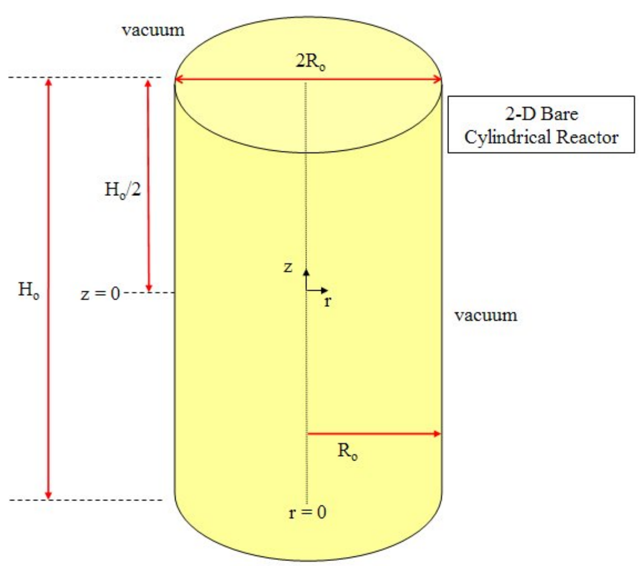
$$\nabla^2\phi(r,z) + B^2\phi(r,z) = 0 \qquad (1)$$

The Laplacian in cylindrical coordinates[7] is given as

$$\nabla^2\phi(r,\theta,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

Thus, for azimuthal symmetry, eqn. (1) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2} + B^2\phi = 0 \qquad (2)$$



We will use the Method of Separation of Variables to develop a solution to this 2-D problem. The method assumes that the flux is separable in the two spatial dimensions, r and z. Although this separability assumption is not valid in most situations, for the simple bare homogeneous reactor, it is indeed valid (we would run into problems during the derivation if our assumption was incorrect). Therefore, we start by letting $\phi(r,z)$ be written as a product of two functions; one that is only a function of r and the other only dependent on z. This gives

$$\phi(r,z) = X(r)Y(z) \tag{3}$$

Substituting this assumed form of the solution into eqn. (2) and dividing by $\phi = XY$, gives

$$\frac{1}{X}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dX}{dr}\right)\right] + \frac{1}{Y}\frac{d^2Y}{dz^2} + B^2 = 0 \tag{4}$$

Analysis of eqn. (4) indicates that the first term is only a function of r, the second term is only a function of z, and the third term is a constant. In order to satisfy this expression for all values of r and z, the first two terms in eqn. (4) must separately equal some constant (usually called the separation constant). Doing this gives the following three expressions,

$$\frac{1}{Y}\frac{d^2Y}{dz^2} = -\alpha^2 \tag{5}$$

$$\frac{1}{X}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dX}{dr}\right)\right] = -\beta^2 \tag{6}$$

and

$$B^2 = \alpha^2 + \beta^2 \tag{7}$$

The above procedure gives rise to a separate second-order differential equation in each direction -- it converts the original partial differential equation (PDE) into two ordinary differential equations (ODEs).

Axial Direction

Addressing the axial direction first, we require the solution to

$$\frac{d^2Y}{dz^2} + \alpha^2 Y = 0 \tag{8}$$

Noting that this is just the 1-D bare slab reactor problem from previous work, we have

$$Y(z) = A_1 \cos\alpha z + A_2 \sin\alpha z \tag{9}$$

Because of the separability assumption, we can evaluate the boundary conditions in each direction without the interaction (or knowledge) of the other direction. In the z-direction, the appropriate boundary conditions are symmetry at z = 0 (center of the reactor in the axial direction) and the flux goes to zero at z = H/2. Imposing these conditions gives

1. $\left.\frac{dY(z)}{dz}\right|_{z=0} = 0$ implies that $A_2 = 0$

2. $Y(z)\big|_{H/2} = 0$ implies that $\cos(\alpha H/2) = 0$

or

$$\alpha_n = \frac{(2n-1)\pi}{H} \qquad \text{for} \quad n = 1, 2, 3, \ldots \tag{10}$$

and $$Y_n(z) = \cos(\alpha_n z) \tag{11}$$

This solution is identical to that derived in Ref. 6 for the 1-D bare critical slab reactor.

Radial Direction

In the radial direction, we have a little more work to do since the defining ODE is not a simple constant coefficient equation (and no simple substitution technique will simplify things as for the spherical reactor case). The expression of interest here is derived from eqn. (6) as

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dX}{dr}\right) + \beta^2 X = 0 \tag{12}$$

Expanding the first term of this equation gives

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \beta^2 X = 0$$

and multiplication by r^2 gives

$$r^2 X'' + rX' + (\beta^2 r^2 - 0)X = 0 \tag{13}$$

As shown as part of the solution method for the 1-group 1-D critical cylindrical reactor (see Ref. 6) and also in Lamarsh and in the Lecture Notes on Bessel Function (see Refs. 7 and 8, respectively), this form of the diffusion equation for the radial direction is of the form of an ordinary Bessel equation with order $\nu = 0$. In particular, from our previous discussion concerning the 1-D cylindrical reactor in Ref. 6, we already know that the general solution to eqn. (13) can be written in terms of zero-order ordinary Bessel functions,

$$X(r) = C_1 J_0(\beta r) + C_2 Y_0(\beta r) \tag{14}$$

where $J_0(\beta r)$ and $Y_0(\beta r)$ are ordinary Bessel functions of the first and second kind, respectively, and they represent two linearly independent solutions to the given 2nd order ODE. A linear combination of these two independent functions gives the general solution to the radial problem described via eqn. (12) or (13).

For the radial direction, the appropriate boundary conditions are that the flux must remain finite at $r = 0$ and that the flux at $r = R$ is zero. The first condition forces C_2 to be zero in eqn. (14), since the $Y_0(\beta r)$ function goes to $-\infty$ as $r \rightarrow 0$. At the outer boundary, we have

$$J_0(\beta R) = 0 = J_0(\eta_m)$$

or

$$\beta_m = \eta_m / R \quad \text{where} \quad \eta_m = m^{th} \text{ zero of } J_0(x) \text{ for } m = 1, 2, 3, \ldots \tag{15}$$

and

$$X_m(r) = J_0(\beta_m r) \tag{16}$$

And, as expected, this solution is identical to that derived in Ref. 6 for the 1-D bare critical cylindrical reactor.

General Solution

Finally, combining the radial solution [eqns. (15) and (16)] with the axial solution [eqns. (10) and (11)] gives the desired general solution for a bare finite cylindrical reactor,

$$\phi_{mn}(r,z) = AJ_0(\beta_m r)\cos(\alpha_n z) \tag{17}$$

where

$$B_{mn}^2 = \alpha_n^2 + \beta_m^2 \tag{18}$$

As we have seen before (see Ref. 6), there are an infinite number of eigenvalues and eigen-functions that satisfy the defining ODE. However, for the same reasons as before (i.e. the higher modes decay away for the steady state case), we immediately set m = n = 1 to obtain the fundamental mode steady state solution for this problem.

Noting that the first zero of the $J_0(x)$ Bessel function occurs at $\eta = 2.4048$, we can write the fundamental mode solution as

$$\phi(r,z) = AJ_0\left(\frac{2.4048}{R}r\right)\cos\left(\frac{\pi}{H}z\right) \tag{19}$$

where the fundamental mode geometric buckling for this 1-group 2-D problem is

$$B^2 = \alpha^2 + \beta^2 = \left(\frac{2.4048}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2 \tag{20}$$

Power Normalization

As done in Ref. 6, to complete the solution for this problem we must find the normalization constant A. Using the total reactor power, P, as the desired normalization gives

$$P = \kappa\int\Sigma_f\phi d\vec{r} = \kappa\int\Sigma_f\phi(r,z)\,2\pi r dr dz \tag{21}$$

$$P = \kappa\Sigma_f\,2\pi A\left[\int_0^{R_o} r\,J_0\left(\frac{2.4048r}{R}\right)dr\right]\left[\int_{-H_o/2}^{H_o/2}\cos\frac{\pi z}{H}dz\right] \tag{22}$$

The integral over the axial direction gives

$$\int_{-H_o/2}^{H_o/2}\cos\frac{\pi z}{H}dz = \frac{H}{\pi}\sin\frac{\pi z}{H}\bigg|_{-H_o/2}^{H_o/2} = \frac{H}{\pi}\left[\sin\frac{\pi H_o}{2H} - \sin\frac{-\pi H_o}{2H}\right] = \frac{2H}{\pi}\sin\frac{\pi H_o}{2H}$$

The integral over the radial direction is obtained as follows. First note that from Refs. 7 and 8, we have the integral relationship

$$\int xJ_0(x)dx = xJ_1(x) \tag{23}$$

Therefore, letting x = 2.4048r/R, we have r = Rx/2.4048 and dr = Rdx/2.4048, and the radial component of eqn. (22) becomes

$$\int_0^{R_o} rJ_0\left(\frac{2.4048r}{R}\right)dr = \int_0^{2.4048R_o/R}\left(\frac{R}{2.4048}\right)^2 xJ_0(x)dx = \left(\frac{R}{2.4048}\right)^2\left[xJ_1(x)\right]\Big|_0^{2.4048R_o/R}$$

$$= \left(\frac{R}{2.4048}\right)^2 \frac{2.4048R_o}{R} J_1\left(\frac{2.4048R_o}{R}\right) = \frac{R_oR}{2.4048} J_1\left(\frac{2.4048R_o}{R}\right)$$

Putting these results into eqn. (22) gives

$$A = \frac{2.4048\,\pi P}{4\kappa\Sigma_f\,\pi R_o RH\left[\sin \pi H_o/2H\right]\left[J_1(2.4048R_o / R)\right]} \tag{24}$$

Now, for the case where the extrapolation distance, d, is small relative to the reactor dimensions, we have $R \approx R_o$ and $H \approx H_o$. Noting that the reactor volume is $V = \pi R_o^2 H_o$, for this situation eqn. (24) reduces to

$$A = \frac{2.4048\pi}{4J_1(2.4048)}\frac{P}{\kappa\Sigma_f V} = \frac{3.638P}{\kappa\Sigma_f V} \tag{25}$$

where the last equality simply evaluates the first coefficient to be 3.638 (this was evaluated using Matlab and it agrees approximately with the result given in Table 6.2 in Ref. 7).

Criticality Condition

To complete this problem, we write the formal ***criticality condition*** here as

$$k_{eff} = \frac{\nu\Sigma_f}{DB^2 + \Sigma_a} = \frac{k_\infty}{1 + L^2B^2} = k_\infty P_{NL} \tag{26}$$

where we noted before that this relationship is valid for any 1-group 1-region system. The only unique aspect here is that, for the finite bare cylinder, the buckling used in this expression is given by eqn. (20).

Visualization of the Flux and Current Distributions

The above development formally solves the 1-group finite cylindrical bare critical reactor problem. All the pertinent equations are given and, except for the introduction of the Separation of Variables Method to convert the original PDE into two ODEs, the overall solution procedure was identical to the methods discussed previously for various 1-D reactor geometries (see Ref. 6). However, in two dimensions it is a little more difficult to visualize the resultant flux and current distributions. To address this concern, a short Matlab program was written to display these profiles in various formats to help in the visualization process. The Matlab code, **bare1g_rz.m**, is given in Table 1 (at the end of this set of Lecture Notes).

In the simulations, the extrapolation distance is assumed to be small and the flux magnitude is set to unity. Also, for convenience, the diffusion coefficient, D, is set to 1 cm. The geometry is assumed to be a right circular cylinder where the height, H, is twice the diameter -- thus, H = 2(2R) = 4R. For specificity in the plots, we set R = 1 m.

Figures 1 - 6 detail a number of different views for the flux and current in the critical bare finite cylindrical reactor. The profiles plotted are

Flux Distribution: $\phi(r,z) = AJ_0\left(\frac{2.4048}{R}r\right)\cos\left(\frac{\pi}{H}z\right)$

Current Distribution: $\vec{J}(r,z) = J_r(r,z)\hat{a}_r + J_z(r,z)\hat{a}_z$

where $J_r(r,z) = -DA\frac{\partial}{\partial r}\left[J_0\left(\frac{2.4048}{R}r\right)\cos\left(\frac{\pi}{H}z\right)\right] = \frac{2.4048}{R}DA\,J_1\left(\frac{2.4048}{R}r\right)\cos\left(\frac{\pi}{H}z\right)$

and $J_z(r,z) = -DA\frac{\partial}{\partial z}\left[J_0\left(\frac{2.4048}{R}r\right)\cos\left(\frac{\pi}{H}z\right)\right] = \frac{\pi}{H}DA\,J_0\left(\frac{2.4048}{R}r\right)\sin\left(\frac{\pi}{H}z\right)$

where we used the expression $\frac{d}{dx}J_0(x) = -J_1(x)$ from Ref. 8 and the chain rule [for example, $\frac{d}{dr}J_0(x) = \frac{d}{dx}J_0(x)\frac{dx}{dr}$] to do the above derivative calculations.

The student is encouraged to study these profiles carefully and to try to really visualize the flux and current distributions obtained for this simple 2-D reactor model -- hopefully the variety of plots displayed here is sufficient to accomplish this goal…

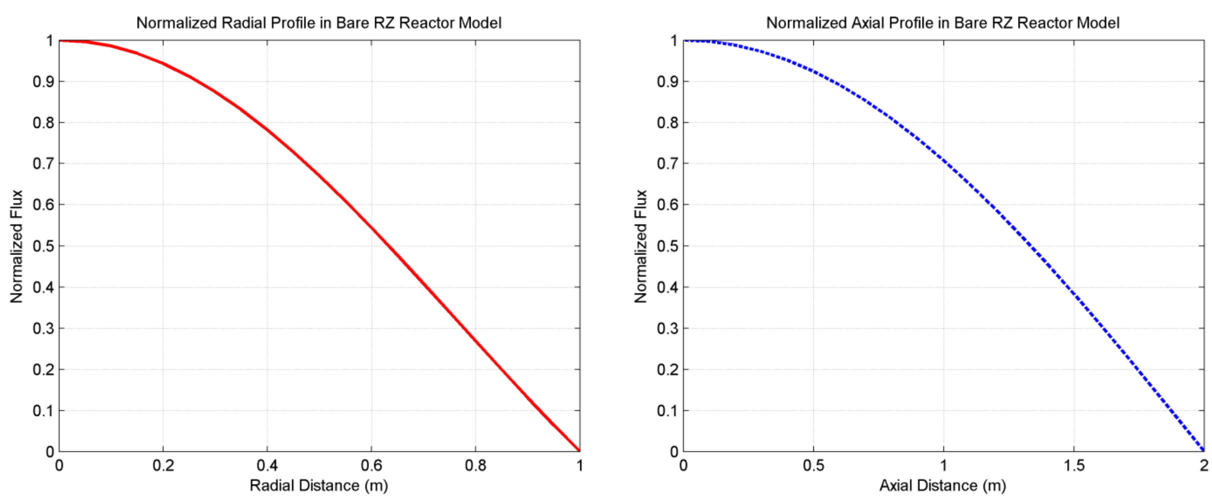


Fig. 1 Separate normalized radial and axial flux profiles.

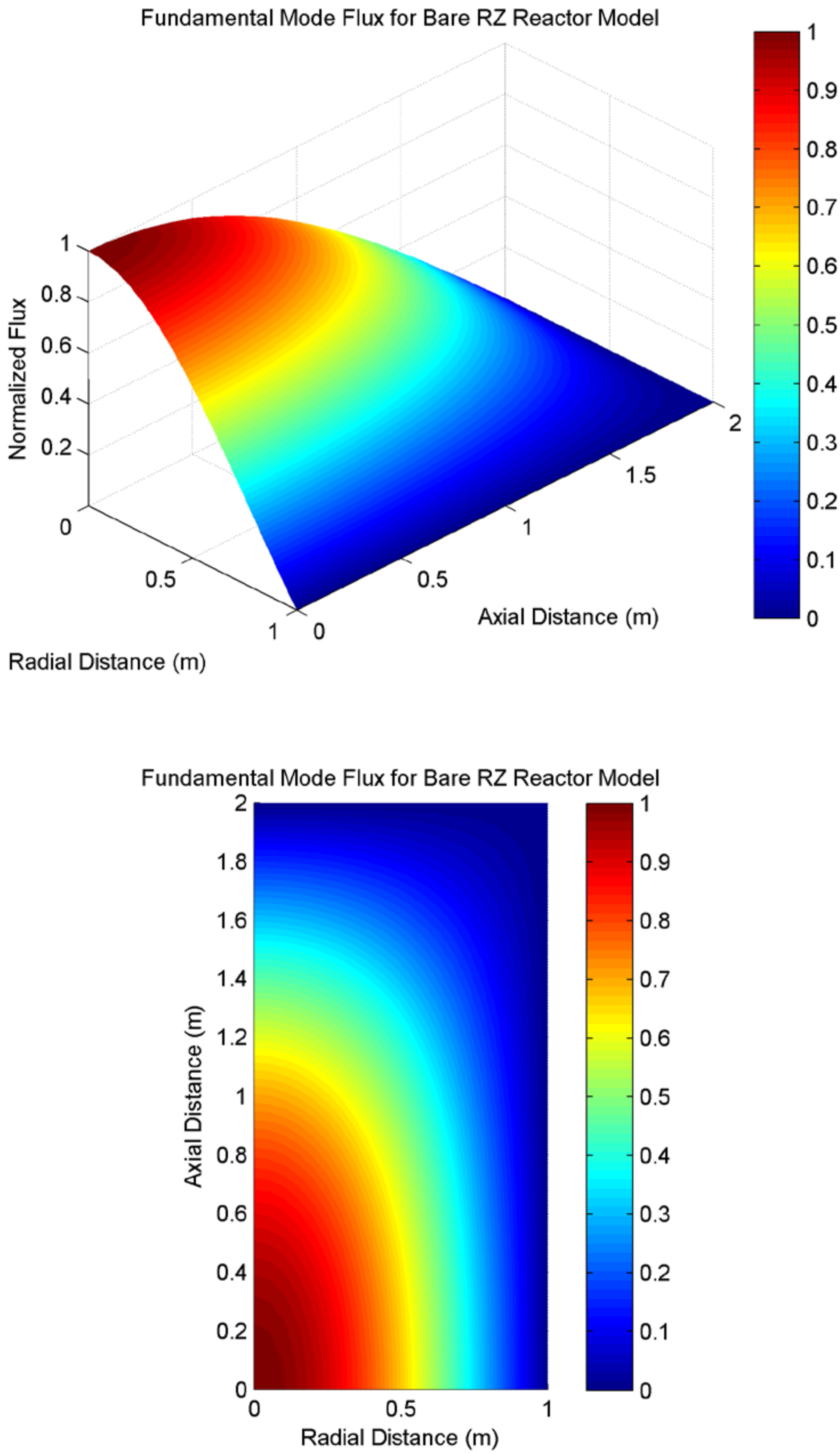
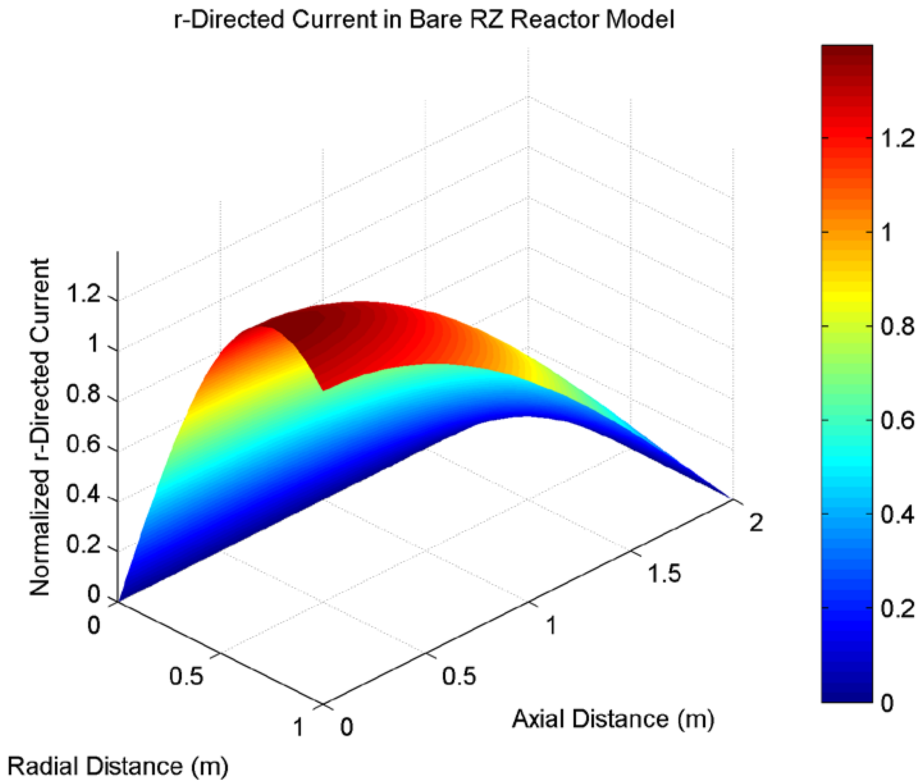


Fig. 2 Various surface plots of the normalized flux distribution.



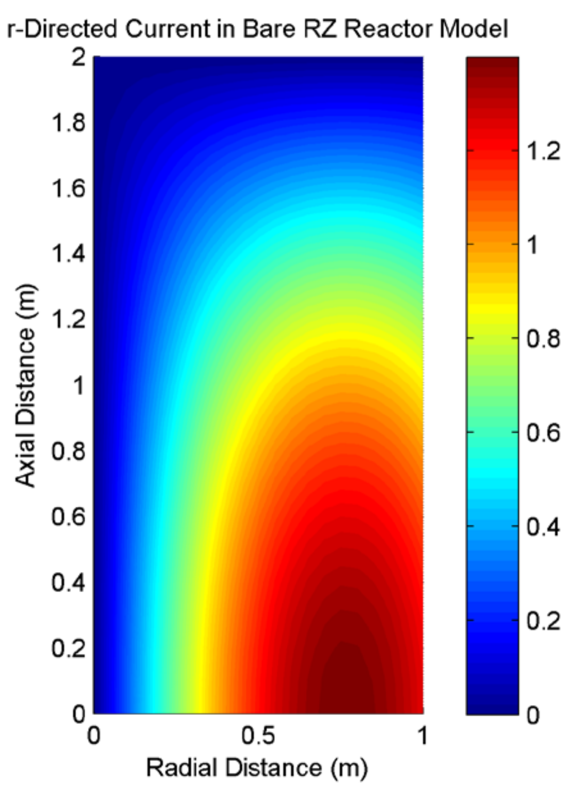
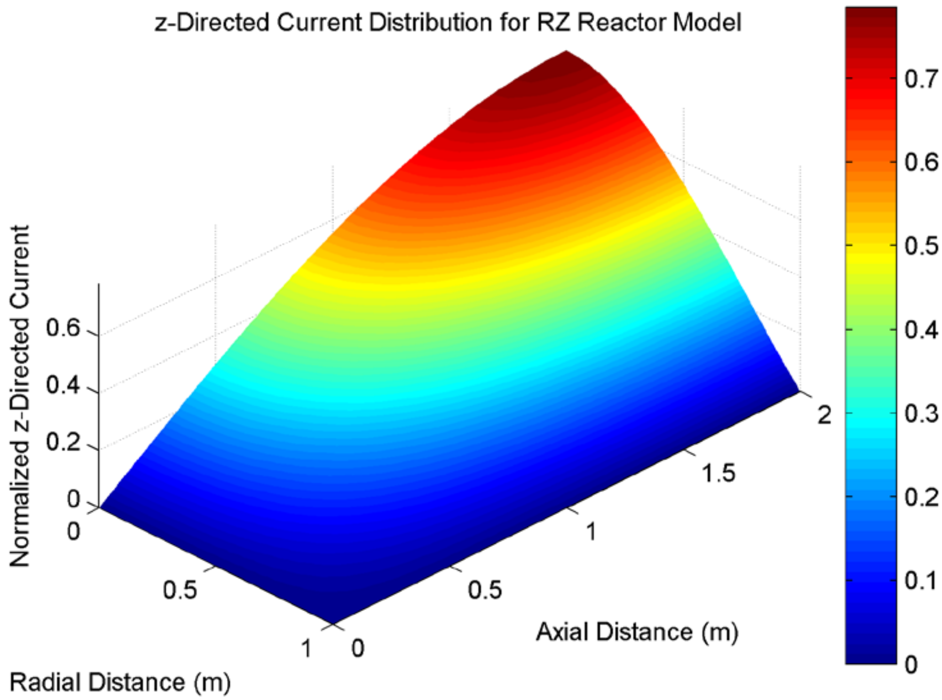


Fig. 3 Various surface plots of the r-directed current distribution.



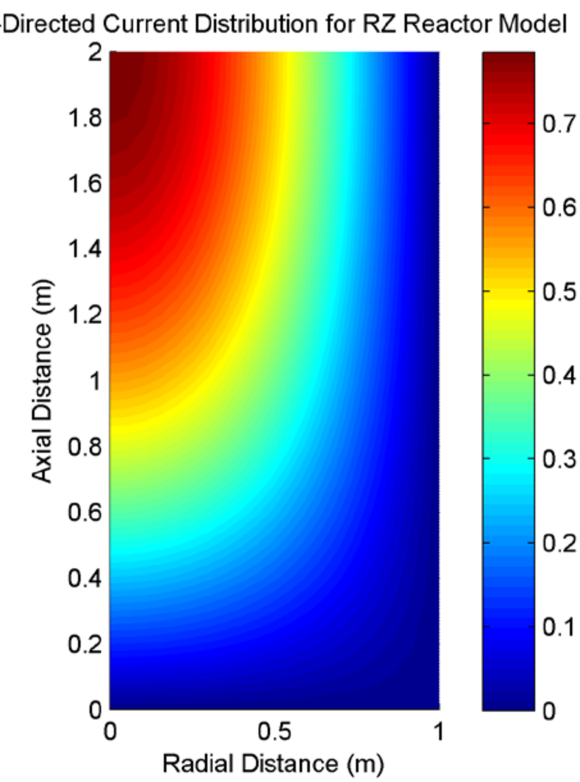


Fig. 4 Various surface plots of the z-directed current distribution.

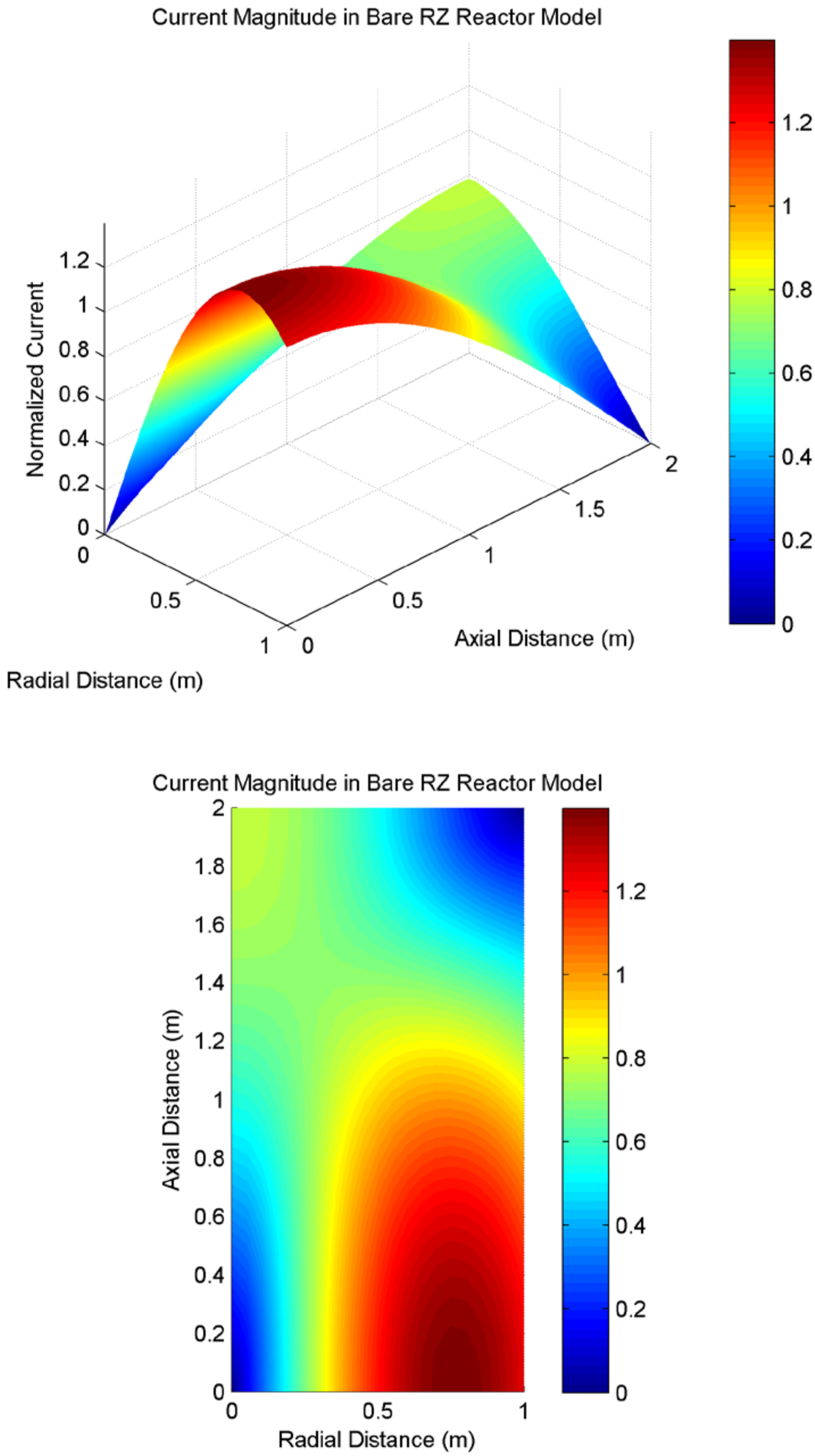


Fig. 5 Various surface plots for the distribution of current magnitude.

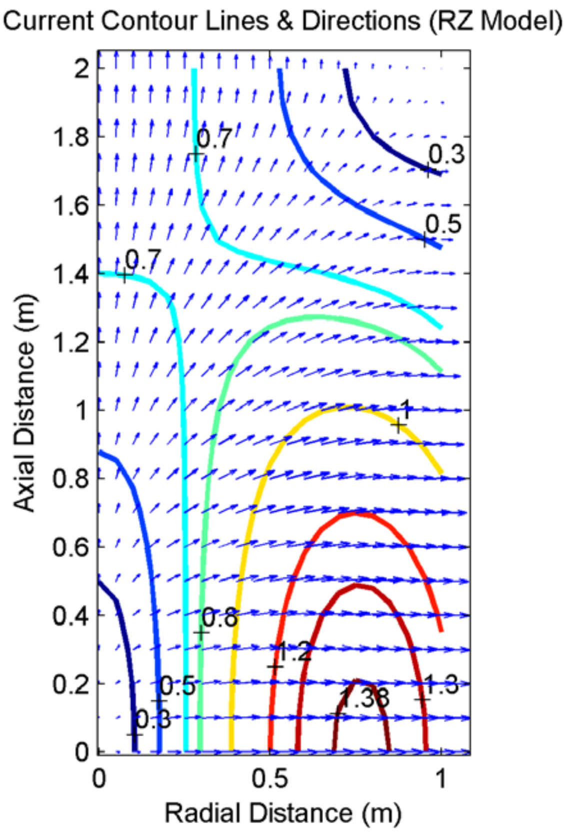


Fig. 6 Contours and directions for the current in the critical bare RZ reactor.

Table 1 Listing of Matlab code bare1g_rz.m

```
%
%   BARE1G_RZ.M   Plot Spatial Flux and Current Distribution
%                   for a 1-G RZ Bare Cylindrical Reactor
%
%   This demo displays the flux and current profiles for a bare homogeneous critical
%   finite cylindrical reactor.  For convenience, the extrapolation distance is
%   assumed to be small relative to the reactor geometry.  In addition, the
%   fundamental mode flux is normalized to a maximum value of unity and the
%   diffusion coeff in the current equation is also unity.
%
%   Code written by J. R. White, UMass-Lowell (Jan. 2015)
%

      clear all, close all, nfig = 0;

%
%   calc normalized fluxes & currents
%    --> let R = 1 m, then H = 2*Dia = 4R = 4 m
%    --> let A = 1 in the flux equation  and  let DA = 1 in the current equation
      R = 1;  H = 4*R;  a = 2.4048;
      npts = 21;  r = linspace(0,R,npts);   z = linspace(0,H/2,npts);
      [rr,zz] = meshgrid(r,z);
      Xr = besselj(0,a*rr/R);  Yz = cos(pi*zz/H);        % radial & axial components
      flxrz = Xr.*Yz;                                    % flux profile (A = 1)
      currzr = (a/R)*besselj(1,a*rr/R).*cos(pi*zz/H);    % r-directed current
      currzz = (pi/H)*besselj(0,a*rr/R).*sin(pi*zz/H);   % z-directed current
```

```
      mcurrz = sqrt(currzr.*currzr + currzz.*currzz);    % current magnitude (DA = 1)
%
%   plot fluxes
      nfig = nfig+1;     figure(nfig);
      plot(r,Xr(1,:),'r-','LineWidth',2),grid on
      title('Normalized Radial Profile in Bare RZ Reactor Model')
      xlabel('Radial Distance (m)'),ylabel('Normalized Flux')
%
      nfig = nfig+1;     figure(nfig);
      plot(z,Yz(:,1),'b--','LineWidth',2),grid on
      title('Normalized Axial Profile in Bare RZ Reactor Model')
      xlabel('Axial Distance (m)'),ylabel('Normalized Flux')
%
      nfig = nfig+1;     h = figure(nfig);  set(h,'Renderer','Zbuffer')
      surf(rr,zz,flxrz),view(45,30),shading interp, colorbar, axis image
      title('Fundamental Mode Flux for Bare RZ Reactor Model')
      xlabel('Radial Distance (m)'),ylabel('Axial Distance (m)')
      zlabel('Normalized Flux')
%
      nfig = nfig+1;     h = figure(nfig);  set(h,'Renderer','Zbuffer')
      surf(rr,zz,flxrz),view(2),shading interp, colorbar, axis image
      title('Fundamental Mode Flux for Bare RZ Reactor Model')
      xlabel('Radial Distance (m)'),ylabel('Axial Distance (m)')
      zlabel('Normalized Flux')
%
%   plot currents
      nfig = nfig+1;     h = figure(nfig);  set(h,'Renderer','Zbuffer')
      surf(rr,zz,currzr),view(45,30),shading interp, colorbar, axis image
      title('r-Directed Current in Bare RZ Reactor Model')
      xlabel('Radial Distance (m)'),ylabel('Axial Distance (m)')
      zlabel('Normalized r-Directed Current')
%
      nfig = nfig+1;     h = figure(nfig);  set(h,'Renderer','Zbuffer')
      surf(rr,zz,currzr),view(2),shading interp, colorbar, axis image
      title('r-Directed Current in Bare RZ Reactor Model')
      xlabel('Radial Distance (m)'),ylabel('Axial Distance (m)')
      zlabel('Normalized r-Directed Current')
%
      nfig = nfig+1;     h = figure(nfig);  set(h,'Renderer','Zbuffer')
      surf(rr,zz,currzz),view(45,30),shading interp, colorbar, axis image
      title('z-Directed Current Distribution for RZ Reactor Model')
      xlabel('Radial Distance (m)'),ylabel('Axial Distance (m)')
      zlabel('Normalized z-Directed Current')
%
      nfig = nfig+1;     h = figure(nfig);  set(h,'Renderer','Zbuffer')
      surf(rr,zz,currzz),view(2),shading interp, colorbar, axis image
      title('z-Directed Current Distribution for RZ Reactor Model')
      xlabel('Radial Distance (m)'),ylabel('Axial Distance (m)')
      zlabel('Normalized z-Directed Current')
%
      nfig = nfig+1;     h = figure(nfig);  set(h,'Renderer','Zbuffer')
      surf(rr,zz,mcurrz),view(45,30),shading interp, colorbar, axis image
      title('Current Magnitude in Bare RZ Reactor Model')
      xlabel('Radial Distance (m)'),ylabel('Axial Distance (m)')
      zlabel('Normalized Current')
%
      nfig = nfig+1;     h = figure(nfig);  set(h,'Renderer','Zbuffer')
      surf(rr,zz,mcurrz),view(2),shading interp, colorbar, axis image
      title('Current Magnitude in Bare RZ Reactor Model')
      xlabel('Radial Distance (m)'),ylabel('Axial Distance (m)')
      zlabel('Normalized Current')
%
      nfig = nfig+1;     h = figure(nfig);  set(h,'Renderer','Zbuffer')
      vc = [.3 .5 .7 .8 1.0 1.2 1.3 1.38];
      [c,h] = contour(rr,zz,mcurrz,vc); clabel(c); hold on
      set(h,'Linewidth',2)
      quiver(rr,zz,currzr,currzz), hold off, axis image
      title('Current Contour Lines & Directions (RZ Model)')
      xlabel('Radial Distance (m)'),ylabel('Axial Distance (m)')
%
%   end of simulation
```

References

1. J. R. White, "Planar Source in a Moderating Medium," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***slabmm_gui*** Matlab program.

2. J. R. White, "Point Source in a Moderating Medium," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***spheremm_gui*** Matlab program.

3. J. R. White, "Two-Region Slab with a Planar Source at the Interface," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***two-regions_gui*** Matlab program.

4. J. R. White, "Two Planar Sources in an Infinite Moderating Medium," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***two_planar_sources_gui*** Matlab program.

5. J. R. White, "A 2-Group Example: Point Source of Fast Neutrons in an Infinite Moderating Medium," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.

6. J. R. White, "1-D Bare and Reflected Critical Systems Using 1-Group Diffusion Theory," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***core_refl1g_gui*** Matlab program.

7. J. R. Lamarsh and A. J. Baratta, ***Introduction to Nuclear Engineering***, 3rd Edition, Prentice Hall (2001).

8. J. R. White, "Overview of Bessel Functions," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.