1-D Bare and Reflected Critical Systems
Using 1-Group Diffusion Theory

Overview

Much of the emphasis in our study of reactor physics until now has targeted applications involving non-multiplying media (i.e. regions with no fissionable material). These examples focused on how neutrons diffuse/attenuate in moderating media, and they were useful for establishing a good understanding of the few-group neutron balance equation, the utility of Fick's law, and the analytical techniques needed for solving the source-driven diffusion equation for a variety of simple one-dimensional (1-D) source and geometry configurations. In addition, a series of Matlab GUIs were used to help visualize the flux and current profiles in some of these situations, and to highlight the importance of the diffusion length in the overall neutron diffusion process (the reader is referred to Refs. 1-9 for a more detailed treatment of these subjects).

With this background, we are now ready to illustrate the critical reactor problem under a set of similar constraints, with focus in this document on a variety of 1-group 1-D bare and reflected core geometries. In particular, this set of Lecture Notes develops the formal theory and documents a Matlab graphical user interface called **core_refl1g_gui** that solves the 1-group critical reactor problem for bare and reflected 1-D slab, spherical, and cylindrical geometries. The main user interface for the **core_refl1g_gui** code is shown in Fig. 1.

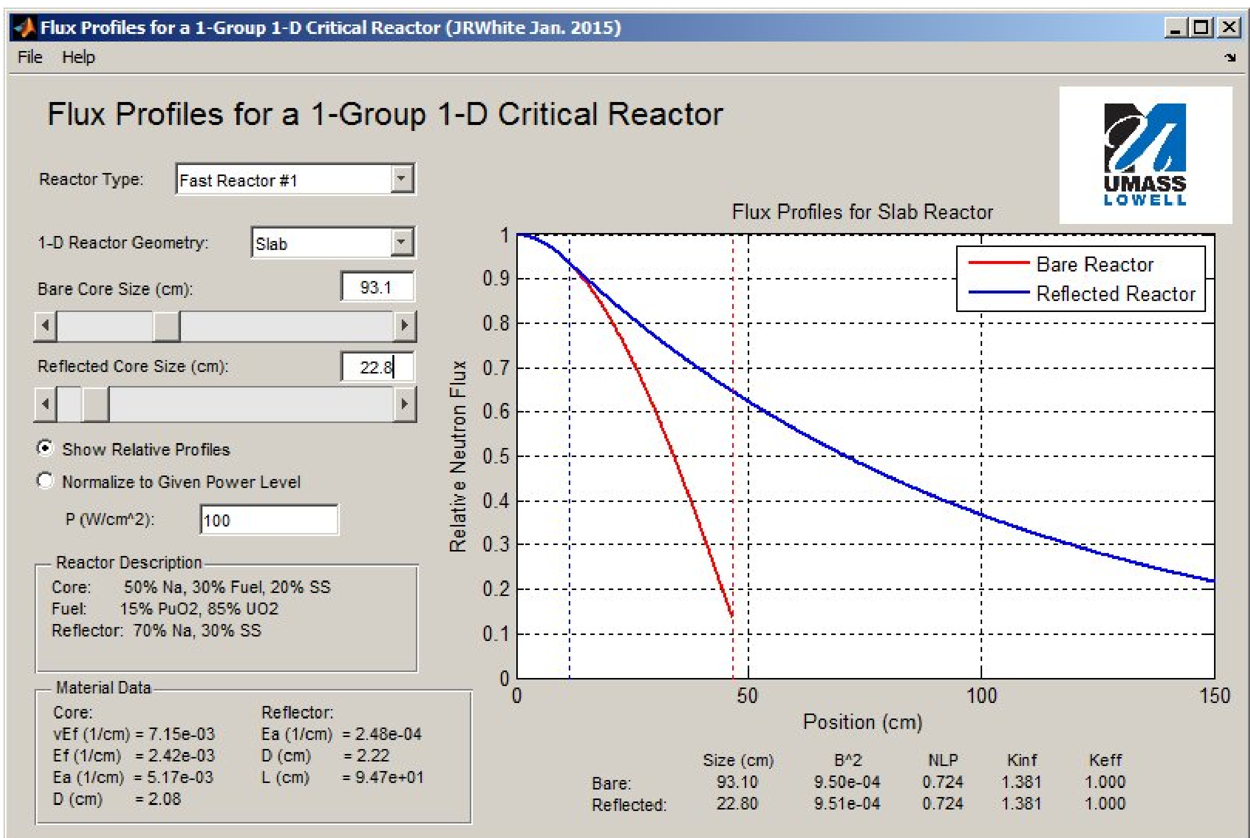


Fig. 1 User interface for the core_refl1g_gui GUI.

The GUI allows the student to easily explore the three different core-reflector geometry options, compare bare and reflected systems, and address how size affects k_{eff} and the flux magnitude for a particular core power level. For a given set of reactor parameters for a specific reactor model (selected from the ***Reactor Type*** menu), the user can adjust the core size for both the bare and reflected cases where, for the core-reflector system, the reflector material is assumed to be very large (infinite) compared to the diffusion length of neutrons within the reflector. The resultant 1-group flux profiles are displayed in the plot window on the right side of the GUI and numerical values for the core size, multiplication factor, core non-leakage probability, and core buckling are tabulated just below the plot area. The goal here is to obtain a good understanding of the basic principles associated with critical systems, and the wealth of information available from the GUI should help one achieve this objective.

The user is encouraged to observe the flux profiles and tabulated information as the core size and 1-D geometry are changed, for both the bare and reflected core configurations. This should help you to better 'visualize' the physical processes that are at work here and to really understand the gains associated with a reflected vs. bare system, the similarity of the flux profiles for the three 1-D geometries, the relationship between core size and multiplication factor, and how the flux magnitude varies with power, core size, and the core configuration (bare vs. reflected geometry).

The remainder of this report documents the underlying theory and the specific equations programmed into the **core_refl1g_gui** code. The derivations here are quite formal, since the development and solution of the diffusion equation for critical systems is quite different from the moderating media problems discussed previously in Refs. 3-6 and 9. We will see that the core physics problem is indeed somewhat unique. For example, we know that, for steady state power production, the reactor has to be just critical. This means that there has to be a precise balance between the neutron production and loss rates. Any arbitrary mixture of fuel, moderator, structure, and control will not satisfy this constraint. We will find that a new restriction, usually called the ***criticality condition***, has to be satisfied. The criticality condition will relate the material composition and geometric configuration such that a critical system can be achieved.

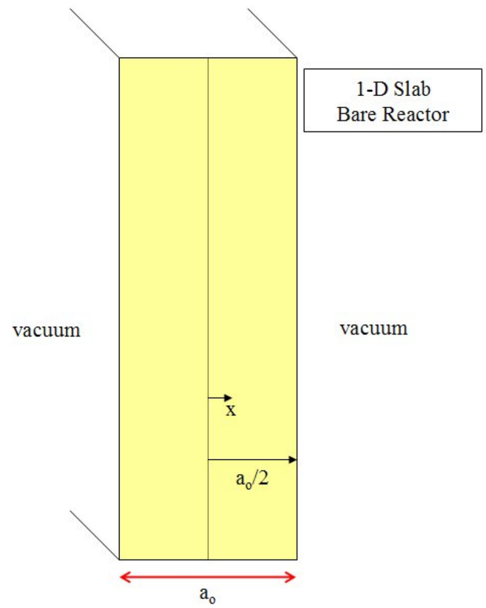
However, for design considerations, it is also important to know if a particular material and geometry combination is subcritical or supercritical and by how much. Thus, during the analysis and design phase, we allow the production and loss terms to be artificially balanced by a mathematical factor, λ, placed in front of the fission source term. For a real operating steady state critical system, $k_{eff} = 1/\lambda$ must be exactly unity. However, in a computational analysis, we can compute k_{eff} for a particular configuration and address whether or not we need to adjust the core size, increase the fuel to coolant ratio, add control to the system, etc., etc. -- allowing the engineer to ask and answer a number of "What if ..." questions during the computational phase of the design process.

All these concepts will become clear as we develop the mathematics that describes the core physics problem. In this set of Lecture Notes, we rely exclusively on the 1-group diffusion approximation to neutron transport and we use many of the same techniques utilized in our previous documentation for the analysis of non-multiplying media. The primary difference in the core physics (versus shielding) problem is that now the fission source (rather than a fixed source) dominates the neutron production term. This gives rise to only a subtle change in the defining equation, but it leads to a substantial change in the character of the solutions. Thus, the solution to the critical reactor problem will exhibit its own unique character…

In the next six subsections we develop the full theory for the bare core and the core-reflector two-region problem for each of the three standard 1-D geometries (e.g. using Cartesian, spherical, and cylindrical coordinates). The bare reactor is done first, followed by the two-region core-reflector problem for the same geometry -- this approach highlights the key differences between the bare core and reflected core. This same type of analysis is done for each of the three 1-D geometries. Although the basic techniques are similar, there are some unique features associated with solution of the diffusion equation in slab, spherical, and cylindrical geometry, so we treat each case separately, and explicitly identify the similarities and differences among the three geometry cases. When complete, you should have a good overview of the terminology and techniques needed for solving 1-group, 1-D critical systems. Of course, we still need to extend our understanding to the 2-group problem (or general multigroup problem) and to multi-dimensional geometries, but these can wait for another day…

1-Group Bare Critical Slab Reactor

To get started, let's first consider the solution of the 1-group diffusion equation for a 1-D critical bare slab reactor model. The reactor has finite thickness a_o in the x direction, but it is infinite in the transverse directions (y and z directions). A rough sketch of the model is shown in the diagram to the right. The "bare" adjective here means that the system has vacuum boundaries and we will apply the typical vacuum boundary condition at the external boundaries of the system. Also, the coordinate system is such that x = 0 is in the center of the reactor, and the system is symmetric about this point. Because of symmetry, we will only consider the region $0 \le x \le a_o/2$.



For a 1-group 1-region homogeneous critical slab reactor, the general multigroup diffusion equation reduces to

$$\frac{d^2}{dx^2}\phi(x) + B^2\phi(x) = 0 \tag{1}$$

with $$B^2 = \frac{\lambda\nu\Sigma_f - \Sigma_a}{D} \tag{2}$$

where it is assumed that the buckling, B^2, is positive since the production term, $\lambda\nu\Sigma_f\phi$, must be greater than the absorption term, $\Sigma_a\phi$, in a finite critical system ($k_\infty > 1$ for a finite system).

The general solution for this 2[nd] order linear constant-coefficient homogeneous ODE can be written in the form of a simple exponential, $\phi(x) \approx e^{\alpha x}$ and, upon substitution into the defining ODE, we obtain the characteristic equation $\alpha^2 + B^2 = 0$ with solution $\alpha_{1,2} = \pm jB$. These complex conjugate roots lead to a general solution that can be written in terms of simple sinusoids, or

$$\phi(x) = A_1 \cos Bx + A_2 \sin Bx \tag{3}$$

The proper boundary conditions here are symmetry at x = 0 (the center of the slab) and the fact that the flux goes to zero at the extrapolated boundary [at $x = (a_o+2d)/2$, where d is the extrapolation distance]. For convenience, let's define "a" as the extrapolated thickness of the core, where a = (a_o+2d).

Now, applying the symmetry condition gives

$$\left.\frac{d\phi}{dx}\right|_{x=0} = \left[-A_1 B \sin Bx + A_2 B \cos Bx\right]\Big|_{x=0} = A_2 B = 0 \qquad (4)$$

or $\quad A_2 = 0$

With this information, eqn. (3) reduces to $\phi(x) = A_1 \cos Bx$.

At the extrapolated boundary (x = a/2), we have

$$\phi(x)\Big|_{x=a/2} = A_1 \cos\frac{Ba}{2} = 0 \qquad (5)$$

For a nontrivial solution, A_1 must be nonzero. Thus, we have the condition that cos(Ba/2) = 0.

This is somewhat of a peculiar situation -- it is certainly different from the fixed-source problem that was solved previously. In this case, there are multiple solutions -- in fact, an infinite number of possibilities exist. Since the cosine function is zero when evaluated at odd integer multiples of $\pi/2$, we have

$$\cos\left[\frac{(2n-1)\pi}{2}\right] = 0 \quad \text{for } n = 1, 2, \ldots \qquad (6)$$

Comparing this expression to the BC given in eqn. (5) gives

$$\frac{B_n a}{2} = \frac{(2n-1)\pi}{2}$$

or $$B_n = \frac{(2n-1)\pi}{a} \quad \text{for } n = 1, 2, \ldots \qquad (7)$$

where we have included the n subscript to indicate that there are an infinite number of values of buckling (B_1, B_2, … B_n, B_{n+1}, …) that satisfy this expression. And, with an infinite number of different B_n values, we get an infinite number of profiles that satisfy the original ODE and its BCs, or

$$\phi_n(x) = \cos B_n x \qquad (8)$$

The essential result from the above development is that the diffusion equation for the critical reactor problem gives rise to an ***eigenvalue problem***. The B_n's are the eigenvalues and the ϕ_n's are the eigenfunctions. Eigenvalue problems have the characteristic form Ay = λBy, where A and B are operators (or matrices), y is a function (vector), and the eigenvalue λ is a constant. For discrete systems, where A, B, and y are finite matrices and vectors of order N, there are a total of N eigenvalues and N eigenvectors. For the case of a continuous system, there are an infinite number of eigenvalues and eigenfunctions.

The higher eigenmodes of the diffusion equation are of interest in many areas of reactor theory, especially in space-time kinetics work and other more advanced topics. For now, however, we will only work with the fundamental mode eigenfunction and eigenvalue, since all the higher modes decay away leaving only the fundamental mode as the final steady state solution to the critical reactor problem. In this case, n = 1 and the 1-group fundamental mode critical flux distribution in a 1-D bare slab reactor is

$$\phi(x) = A_1 \cos Bx \qquad \text{where} \quad B = \frac{\pi}{a} \quad \text{or} \quad B^2 = \left(\frac{\pi}{a}\right)^2 \tag{9}$$

where the 1 subscript has been omitted for convenience from the definition of $B = B_1$, since at this point we are only interested in the fundamental mode solution.

Notice that there is still an arbitrary constant, A_1, that remains as part of the general solution. Again, this is characteristic of eigenvalue problems (i.e. the solution to any homogeneous equation is only known to within an arbitrary normalization). We need an additional constraint in order to uniquely define this constant (or normalization). Notice that this coefficient only causes the flux level to be higher or lower -- it doesn't affect the distribution. Thus, the appropriate condition here is to simply normalize the flux to the reactor power, P, where, for the 1-D slab problem, this is the power per unit area in the yz plane (since the reactor is infinite in this plane). For the present case, this can be written as

$$\begin{aligned} P &= \kappa \int_{-a_o/2}^{a_o/2} \Sigma_f \phi(x) dx = \kappa \Sigma_f A_1 \int_{-a_o/2}^{a_o/2} \cos\frac{\pi x}{a} dx \\ &= \kappa \Sigma_f A_1 \left[\frac{a}{\pi} \sin\frac{\pi x}{a} \right]_{-a_o/2}^{a_o/2} = \kappa \Sigma_f A_1 \frac{a}{\pi}\left[\sin\frac{\pi a_o}{2a} - \sin\frac{-\pi a_o}{2a} \right] \\ &= A_1 \frac{2\kappa\Sigma_f a}{\pi} \sin\frac{\pi a_o}{2a} \end{aligned} \tag{10a}$$

or

$$A_1 = \frac{P\pi}{2\kappa\Sigma_f a \sin\frac{\pi a_o}{2a}} \tag{10b}$$

Thus, the normalized flux in a 1-D slab reactor can be written as

$$\phi(x) = \frac{P\pi}{2\kappa\Sigma_f a \sin\frac{\pi a_o}{2a}} \cos\frac{\pi x}{a} \tag{11}$$

where, in the above expressions, κ represents the average recoverable energy per fission (recall that $\kappa \approx 200$ MeV per fission = 3.204×10^{-11} J per fission).

As a final note, when the extrapolation distance d is small compared to the reactor size, then a_o/a approaches unity, and then $\sin(\pi a_o/2a) \to 1$. For this case, eqns. (10b) and (11) reduce to

$$A_1 = \frac{P\pi}{2\kappa\Sigma_f a_o} \qquad \text{and} \qquad \phi(x) = \frac{P\pi}{2\kappa\Sigma_f a_o}\cos\frac{\pi x}{a_o} \qquad (\text{for } d << a_o) \tag{12}$$

and this is the result for the infinite slab reactor that is usually tabulated in the standard reactor physics texts (see Ref. 2, for example).

We have essentially completed our discussion of the 1-D bare slab reactor problem except for the fact that two bucklings have been defined; a material buckling B^2_m and a geometric buckling B^2_g.

Recall that B^2_m is a simple function of the material properties as given in eqn. (2) and that B^2_g is the result of forcing the flux distribution to satisfy the appropriate boundary conditions (which are a function of the geometry) as shown in eqn. (9) for the fundamental mode solution. Clearly, $B^2_g = B^2_m$ must be true for a consistent description of a critical reactor. As noted previously, a precise relationship is required between the geometry and material makeup for a just critical system. This relationship is known as the ***critical condition***, and for a 1-group bare homogeneous critical reactor, the critical condition is

$$B^2_g = B^2_m \qquad \text{(1-group critical condition)} \qquad (13)$$

In the development here, we have relaxed this physical constraint slightly by including the mathematical eigenvalue λ in our original expression for the material buckling. Thus the critical condition here simply becomes a relationship for $\lambda = 1/k_{eff}$ in terms of the given material and geometric configuration. Combining eqn. (2) with eqn. (13) gives the desired design relationships

$$\lambda = \frac{DB^2 + \Sigma_a}{\nu\Sigma_f} \qquad \text{or} \qquad k_{eff} = \frac{\nu\Sigma_f}{DB^2 + \Sigma_a} \qquad (14)$$

where, in these expressions, B^2 is the geometric buckling (the g subscript is usually omitted).

Equation (14) is the real ***critical condition*** for all 1-group bare homogeneous systems. It says that the multiplication factor is simply the ratio of the production rate to the loss rate (leakage plus absorption). B^2 is a function of the geometry and the cross sections are a function of the material composition. When these parameters have just the right combination, then the production and loss terms are equal and $k_{eff} = 1.000$ -- giving a critical system. However, for any combination of material composition and geometry, eqn. (14) allows us to compute a value of k_{eff} to determine the criticality level of the given configuration -- and this gives the designer lots of information about the particular system under study. Always remember, however, that a real operating critical reactor has $k_{eff} = 1.000$.

Equation (14) is sometimes written in terms of k_∞ and the non-leakage probability. Note that as the reactor size gets large, B^2 and the leakage component approach zero. Thus, for a 1-group homogeneous system, the infinite multiplication factor is simply

$$k_\infty = \frac{\nu\Sigma_f}{\Sigma_a} \qquad (15)$$

Since there are only two loss components in a finite system, the non-leakage probability, P_{NL}, is simply the ratio of the absorption rate to the total loss rate, or

$$P_{NL} = \frac{\langle \Sigma_a \phi \rangle}{\langle DB^2\phi \rangle + \langle \Sigma_a \phi \rangle} = \frac{\Sigma_a}{DB^2 + \Sigma_a} = \frac{1}{DB^2/\Sigma_a + 1} = \frac{1}{1 + L^2B^2} \qquad (16)$$

where $L^2 = D/\Sigma_a$ is the diffusion area of the core material.

If the numerator and denominator of eqn. (14) are divided by Σ_a, one gets

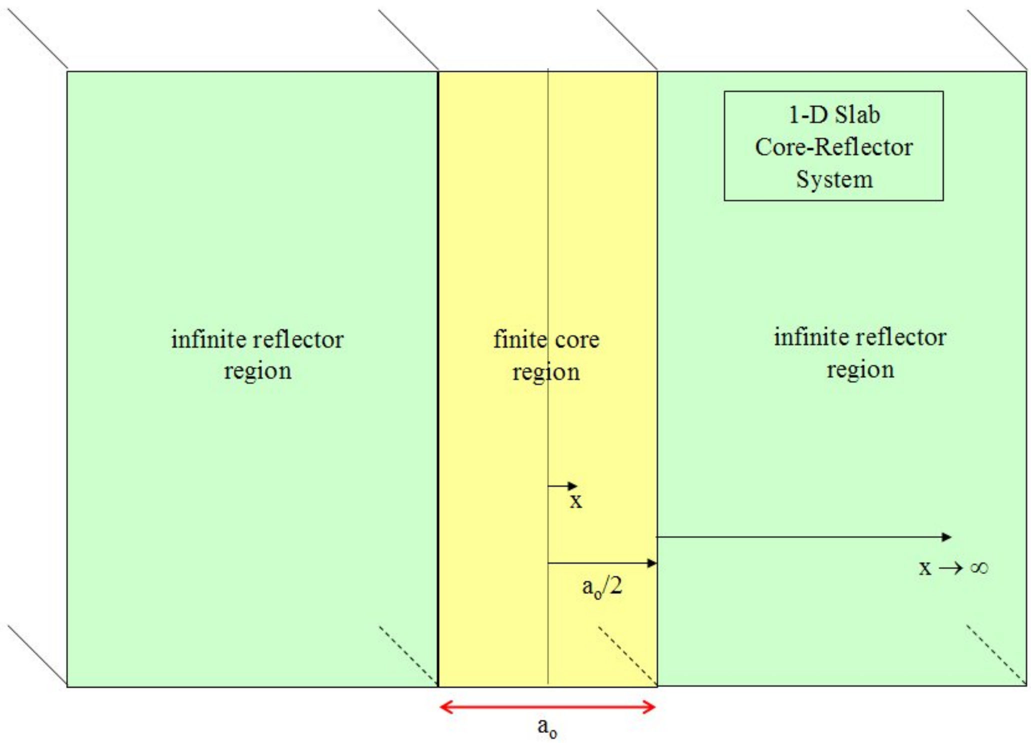
$$k_{eff} = \frac{\nu\Sigma_f}{DB^2 + \Sigma_a} = \frac{\nu\Sigma_f/\Sigma_a}{DB^2/\Sigma_a + 1} = \frac{\nu\Sigma_f/\Sigma_a}{1 + L^2B^2} = k_\infty P_{NL} \qquad (17)$$

Thus, $k_{eff} = k_\infty P_{NL}$ is a common way to write the core multiplication factor for 1-group systems.

Well, there is a lot more that could be said concerning the critical reactor model given here and certainly we need to do some numerical examples to highlight some of the relationships discussed here (this is where the **core_refl1g_gui** code can come in handy). However, for now, we have essentially completed the theoretical development for this example, and it is time to move on to the core-reflector model for slab geometry and to the other 1-D reactor geometries that are discussed in subsequent sections of this set of Lecture Notes.

1-Group Critical Core-Reflector System in 1-D Slab Geometry

We study bare reactor problems (as in the above section) because the mathematics involved is relatively straightforward and they give considerable insight into the general critical reactor problem. However, it should be clear that a bare reactor is not a practical option, and all operating reactor systems have essentially infinite reflectors around the core region (to improve upon the neutron economy from the perspective of core physics and to minimize neutron and gamma radiation problems outside the core). In this context, the simplest two-region reflected system we can address is the critical core-reflector configuration in 1-D Cartesian (slab) geometry using the 1-group diffusion theory approximation. This section addresses this problem in detail, where we have assumed that the reflector region has infinite thickness, as sketched in the diagram below.



In the core region of the model, k_∞ is greater than unity and, in the reflector region, there is no fissionable material. Thus, this simple two-region system combines the critical reactor problem and non-multiplying medium problem into a single system. As shown in the diagram, the composite two-region system is symmetric about x = 0 and as x becomes large the flux must remain finite (in fact, it will decrease towards zero as $x \to \infty$). At the core-reflector interface (i.e. at $x = a_o/2$), the standard continuity of flux and current conditions will apply.

The basic procedure for solving this problem is to write the 1-group 1-D homogenous form of the diffusion equation for each region of the model. Solving this balance equation within each homogeneous region gives a general solution for the flux profile in each zone. Since the diffusion equation is a 2nd order ODE, the general solution within each region will contain two arbitrary coefficients and, for the two-region core-reflector problem, this leads to a total of four unknown coefficients. Thus, we need four independent BCs to help specify these four coefficients -- and, as noted above, the four conditions involve symmetry at $x = 0$, a finite solution as $x \to \infty$, and the continuity of flux and current at $x = a_o/2$.

In the core region, the defining balance equation is identical to the previous critical bare core example,

$$\frac{d^2}{dx^2}\phi_c + B^2\phi_c = 0 \quad \text{with} \quad B^2 = \left.\frac{\lambda\nu\Sigma_f - \Sigma_a}{D}\right|_c \qquad \text{for } 0 \le x \le a_o/2 \tag{18}$$

where we use a 'c' subscript to denote that the properties and flux profile are only valid for the core region. The general solution is also identical as before and, taking into account the symmetry condition at $x = 0$, the general solution for the core flux profile becomes

$$\phi_c(x) = A_1 \cos Bx \tag{19}$$

where, at present, A_1 is an arbitrary constant.

In the reflector region, the balance equation for a non-multiplying medium region applies (i.e. no fissionable material is present), or

$$\frac{d^2}{dx^2}\phi_r - \frac{1}{L_r^2}\phi_r = 0 \quad \text{with} \quad L_r^2 = \left.\frac{D}{\Sigma_a}\right|_r \qquad \text{for } x \ge a_o/2 \tag{20}$$

where we have set $Q = 0$ since there is no external source present in this region and we have used the subscript 'r' to denote that the flux and material properties are associated with the reflector region.

Since the reflector has infinite thickness, we write the general solution to eqn. (20) as

$$\phi_r(x) = A_3 e^{-x/L_r} + A_4 e^{x/L_r}$$

and immediately set A_4 to zero to force the flux solution to be finite as $x \to \infty$. Doing this gives

$$\phi_r(x) = A_3 e^{-x/L_r} \tag{21}$$

At this point, we have used two BCs to reduce the general solutions for the core and reflector regions to the expressions given in eqns. (19) and (21), with A_1 and A_3 yet to be determined. To help find these, we next apply the continuity of flux and current conditions, as follows:

$$\left.\phi_c(x)\right|_{x=a_o/2} = \left.\phi_r(x)\right|_{x=a_o/2} \qquad \text{(continuity of flux)}$$

or

$$A_1 \cos\frac{Ba_o}{2} - A_3 e^{-a_o/(2L_r)} = 0 \tag{22}$$

$$J_c(x)\big|_{x=a_o/2} = J_r(x)\big|_{x=a_o/2} \qquad \text{(continuity of current)}$$

or $$-D_c\left(-A_1 B\sin\frac{Ba_o}{2}\right) + D_r\left(-\frac{A_3}{L_r}e^{-a_o/(2L_r)}\right) = 0 \tag{23}$$

Writing these two homogeneous equations in matrix form gives

$$\begin{bmatrix} \cos\frac{Ba_o}{2} & -e^{-a_o/(2L_r)} \\ D_c B\sin\frac{Ba_o}{2} & -\frac{D_r}{L_r}e^{-a_o/(2L_r)} \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{24}$$

Now, before continuing, we should reflect a little upon the current state of the development. Thus far we have developed the proper general solutions and have applied the four BCs in a formal mathematical sense. Two of the BCs for this particular case lead to zero coefficients (A_2 and A_4 have already been set to zero), which reduced, for this 1-group 2-region case, a 4-equation model to a 2-equation model, as given in eqn. (24) -- with the solution to eqn. (24) giving the remaining coefficients needed to precisely define $\phi_c(x)$ and $\phi_r(x)$.

However, eqn. (24) is a homogeneous equation which, for a non-trivial solution, requires a singular coefficient matrix -- which means that the determinant of the coefficient matrix must be zero. In addition to this constraint, we also know that the solution to a homogeneous equation is only known to within an arbitrary constant. This means that, even with a singular coefficient matrix, we can't solve eqn. (24) explicitly for both A_1 and A_3 -- the best we can do is to write A_3 in terms of A_1, and let A_1 be the arbitrary normalization factor.

Well, the two issues noted here are really not unexpected, since they refer to exactly the same situations we saw for the bare critical reactor discussed in the previous subsection of these Lecture Notes. In particular, because of the precise balance between the material composition and geometry that is required for a critical system, we needed to impose a ***criticality condition*** on the problem. For the bare reactor problem, we forced the material and geometric bucklings to be identical, which allowed us to compute the multiplication factor in terms of the composition (i.e. macroscopic cross sections) and the geometry (i.e. core size) -- which lead to eqn. (14) as the criticality condition for the 1-group bare slab reactor problem.

Now, for the current problem, we do essentially the same thing -- that is, set $B^2_m = B^2_g$ -- but now the geometric buckling is determined from the statement that ***the determinant of the coefficient matrix must be zero***. Recalling that B^2_m contains the eigenvalue λ (which is just $1/k_{eff}$), we see that this condition gives the expected relationship for k_{eff} in terms of the material properties of both the core and reflector and the core-reflector geometry. This relationship is a little more complicated than before, but we now have a two-region system (instead of a simple 1-region bare critical reactor), so this should be expected. This general form of the ***criticality condition*** is robust enough to apply here and in more general, multigroup, multi-region systems.

Now, for the current problem, the determinant of a 2×2 matrix is simply the product of the main diagonal elements minus the product of the other diagonal terms, or

$$\cos\frac{Ba_o}{2}\left(-\frac{D_r}{L_r}e^{-a_o/(2L_r)}\right)-D_cB\sin\frac{Ba_o}{2}\left(-e^{-a_o/(2L_r)}\right)=0$$

or $$-\frac{D_r}{L_r}\cos\frac{Ba_o}{2}+D_cB\sin\frac{Ba_o}{2}=0$$

or $$f(B)=\cot\frac{Ba_o}{2}-\frac{L_rD_cB}{D_r}=0 \qquad (25)$$

When written in this form, it is easy to see that the criticality condition can be cast as a classical root finding problem (i.e. given the core size, a_o, and the material properties, D_c, L_r, and D_r, what is the value of B such that f(B) = 0?). And, once B (or B^2) is found from eqn. (25), we can use the definition of B^2 in eqn. (18) to get the value of k_{eff}, or

$$B^2=\left.\frac{\lambda\nu\Sigma_f-\Sigma_a}{D}\right|_c \qquad \text{or} \qquad k_{eff}=\frac{1}{\lambda}=\frac{\nu\Sigma_{fc}}{D_cB^2+\Sigma_{ac}} \qquad (26)$$

This is the same relationship as the 1-group bare reactor problem, but now the value of the buckling, B^2, is obtained from the solution of eqn. (25) instead of the simple relationship given in eqn. (9). Note, in particular, that **B^2 for the core-reflector problem is NOT equal to $(\pi/a_o)^2$** -- in fact, as shown below, B^2 for a reflected core is less than that for the bare core (since, with some neutrons being scattered or reflected back into the core region when a reflector is present, there will be less net neutron flow in the outward direction in the reflected system).

To help fully understand some of these statements and to better visualize the relationship given in eqn. (25), let's re-write this expression as

$$\cot\frac{Ba_o}{2}=\frac{L_rD_cB}{D_r}$$

and let $p = Ba_o/2$. Now, upon substitution, we have

$$\cot p=\frac{L_rD_cBa_o/2}{D_ra_o/2}=\frac{2L_rD_c}{D_ra_o}p \qquad (27)$$

The left hand side (LHS) of this relationship is just the familiar cotangent function and the RHS is a simple linear function of p (with a positive slope and zero intercept). The points where these two functions intersect represent the roots of this nonlinear equation. Note that, with p known, one can compute B, and then k_{eff} via the above relationships…

Figure 2 illustrates this relationship in graphical form. The plot shows a simple cot (p) function over the range $0 \le p \le 5\pi$ and a straight line with a small positive slope. All the intersections of these two curves represent the values of p that satisfy the so-called criticality condition for this simple core-reflector problem. Note that, since the cotangent function is periodic, if the plot is extended for larger values of p, there will be an infinite number of roots (or eigenvalues) as $p \to \infty$. This is exactly what we saw for the bare slab problem! However, as we also argued in the previous situation, only the first or fundamental mode eigenvalue contributes to the steady state solution -- thus, for our current emphasis, we will only focus on the first nontrivial root.

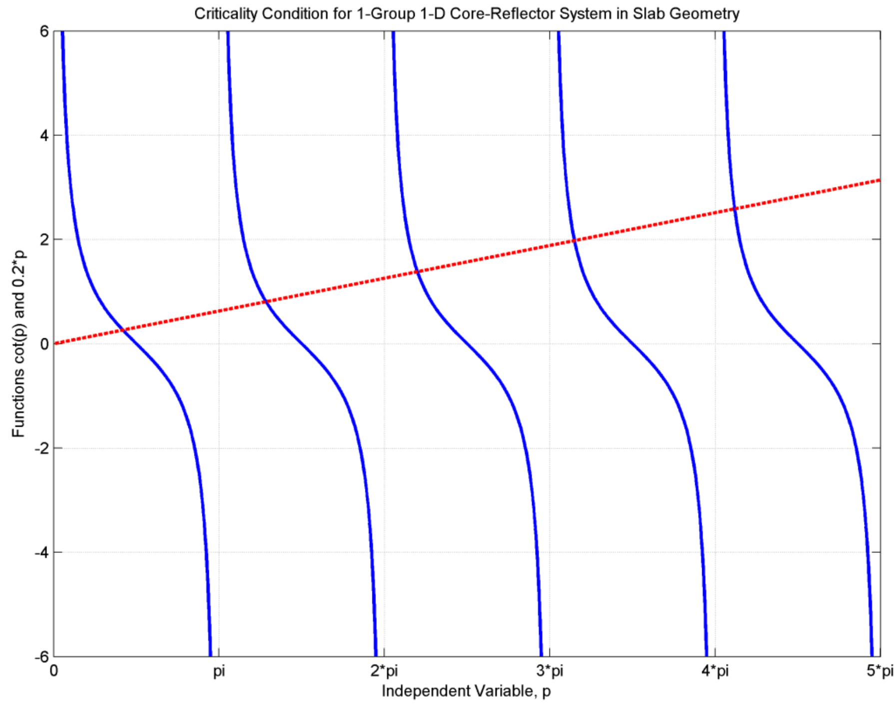


Fig. 2 Illustration showing the criticality condition for the 1-group 1-D core-reflector problem in Cartesian geometry.

As apparent in Fig. 2, the first root occurs for $p \le \pi/2$. Thus, we can write

$$p = \frac{Ba_o}{2} \le \frac{\pi}{2} \qquad \text{or} \qquad B \le \frac{\pi}{a_o} \qquad \text{or} \qquad B^2 \le \left(\frac{\pi}{a_o}\right)^2 \tag{28}$$

and this tells us that $B^2_{reflected} \le B^2_{bare}$ (since the buckling in a bare slab of thickness a_o is about π/a_o). Also, since the DB^2 term in the neutron balance equation and in the expression for the multiplication factor is associated with the leakage term, we see that adding a reflector decreases the net core leakage. And, decreasing the leakage increases the non-leakage probability and the effective core multiplication factor [see eqns. (16) and (17)].

Now, with the fundamental mode value for B from eqn. (25) or eqn. (27), the determinant of the coefficient matrix in eqn. (24) is indeed zero, and we can proceed to actually solve this equation for the unknown values of A_1 and A_3. Expanding the first equation and solving for A_3 gives

$$A_3 = A_1 \cos\frac{Ba_o}{2} e^{a_o/(2L_r)} \tag{29}$$

and putting this result into eqns. (19) and (21) gives

$$\phi_c(x) = A_1 \cos Bx \qquad \text{for} \quad 0 \le x \le a_o/2 \tag{30a}$$

and $$\phi_r(x) = A_1 \cos\frac{Ba_o}{2} e^{-(x-a_o/2)/L_r} \qquad \text{for} \quad x \ge a_o/2 \tag{30b}$$

as the formal solution for the core and reflector flux profiles.

Now, the only unresolved quantity in these expressions is the normalization factor A_1 and, as before, this quantity is determined by the power constraint on the system. In general, the expression for the power involves integration over all space. However, in regions where there is no fission, there is no power production (except for a small contribution due to gamma energy deposition which we are ignoring in this simplified treatment). Thus, for the current problem, we only need to integrate over the core region, which gives

$$P = \kappa\int_{-a_o/2}^{a_o/2} \Sigma_f \phi_c(x)dx = \kappa\Sigma_f A_1 \int_{-a_o/2}^{a_o/2} \cos Bx\, dx$$

$$= \kappa\Sigma_f A_1 \left[\frac{1}{B}\sin Bx\right]_{-a_o/2}^{a_o/2} = \frac{\kappa\Sigma_f A_1}{B}\left[\sin\frac{Ba_o}{2} - \sin\frac{-Ba_o}{2}\right] \qquad (31a)$$

$$= A_1 \frac{2\kappa\Sigma_f}{B}\sin\frac{Ba_o}{2}$$

or

$$A_1 = \frac{PB}{2\kappa\Sigma_f \sin\frac{Ba_o}{2}} \qquad (31b)$$

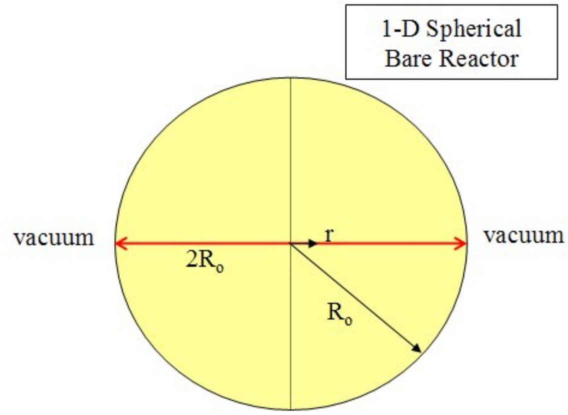
Note that the development of eqns. (31a) and (31b) is nearly identical to the procedure used for finding the normalization constant for the bare critical slab reactor [see eqns. (10a) and (10b)]. There is one subtle, but very important difference, however. Here we do not have a simple analytical expression for B, so we simply carry along the variable B without any formal substitution. For the bare reactor, $B = \pi/a$, and this result was used within the development for the bare system. For the core-reflector problem, however, the numerical value of B that goes into eqn. (31b) is the smallest root of the nonlinear criticality condition given in eqn. (25).

We have finally finished the theoretical development of the 1-group core-reflector problem in 1-D slab geometry. Clearly, this case was somewhat more difficult than the bare critical reactor model from at least an algebraic viewpoint. However, the basic procedures for the two cases were really quite similar, with only subtle differences in the details of working with the criticality condition and the power normalization. Concerning implementation, some added work associated with finding B^2 and k_{eff} for the core-reflector case (i.e. solving the nonlinear criticality condition) is certainly necessary but, again, the overall procedures are very similar -- and we will see that this is true for all the 1-group 1-D geometries treated as part of these notes.

The bare-core and reflected-core models developed in these notes are implemented into the **core_refl1g_gui** code, and you should use this software to compare results for the two cases for a variety of reactor types and bare and reflected core sizes (i.e. different values of a_o). With the theory developed here and the Matlab GUI to easily visualize the resultant flux profiles, you should be able to get a good handle on this subject. For example, keep an eye on the value of k_{eff} as you change the value of a_o for both the bare and reflected systems -- does this behave as expected?

1-Group Bare Critical Spherical Reactor

The complete development for the slab reactor in the previous sections is repeated here for the case of spherical geometry. In this section we focus on the bare critical spherical reactor with the core size characterized by the radius R_o. The core is homogeneous so that the material composition is constant throughout the full sphere. This specification, along with vacuum BCs on the outside of the sphere, leads to a 1-D problem, with the flux distribution only dependent on the radial variable r (i.e. there is no dependence on the θ and ψ angular variables). A rough sketch of the 1-D bare spherical reactor is shown to the right, where r = 0 is in the center of the reactor.



As before, for a 1-group 1-region homogeneous critical reactor, the general multigroup diffusion equation becomes

$$\nabla^2\phi + B^2\phi = 0 \tag{32}$$

where

$$B^2 = \frac{\lambda\nu\Sigma_f - \Sigma_a}{D} \tag{33}$$

However, in 1-D spherical geometry, the Laplacian operator can be written as (see Ref. 2 or 4, for example)

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) \tag{34}$$

and using this expression in eqn. (32) gives

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) + B^2\phi = 0 \tag{35}$$

This 2nd order ODE is linear, but the first term has variable coefficients. However, as discussed for the case of a point source in a moderating media (see Ref. 4), a simple substitution of variables, $\phi(r) = \omega(r)/r$, can convert this expression into a constant coefficient ODE that is much easier to solve. When this substitution is inserted into eqn. (35), the defining equation for $\omega(r)$ becomes

$$\frac{d^2\omega}{dr^2} + B^2\omega = 0 \tag{36}$$

and this has the general solution (note that this is of the same form as the bare slab reactor problem),

$$\omega(r) = A_1\cos Br + A_2\sin Br \tag{37}$$

Now, since $\phi(r) = \omega(r)/r$, we can write the general solution for $\phi(r)$ in the bare spherical system as

$$\phi(r) = A_1 \frac{\cos Br}{r} + A_2 \frac{\sin Br}{r} \tag{38}$$

As before, we now have to address the BCs for the problem. In this case, since the cos (Br)/r term approaches infinity as $r \to 0$, we must set $A_1 = 0$ to guarantee a finite flux everywhere, leaving

$$\phi(r) = A_2 \frac{\sin Br}{r} \tag{39}$$

as the general solution [note that, via L'Hospital's Rule, sin (Br)/r $\to$ B as $r \to 0$].

For the 2nd BC, we first define $R = R_o + d$ as the extrapolated core size, and then force the flux to go to zero at the extrapolated boundary of the system (i.e. we use the standard diffusion theory representation of a vacuum BC), or

$$\phi(r)\big|_{r=R} = 0 \tag{40}$$

With this condition, eqn. (39) becomes

$$\phi(R) = A_2 \frac{\sin BR}{R} = 0 \tag{41}$$

and, with the fact that the sine function goes to zero at integer multiples of π, we have

$$B_n = \frac{n\pi}{R} \qquad \text{for} \quad n = 1, 2, \ldots \tag{42}$$

Focusing on only the fundamental mode solution (since this is the only solution that remains at steady state), the buckling for the 1-D bare sphere is

$$B^2 = \left(\frac{\pi}{R}\right)^2 \tag{43}$$

and the flux becomes

$$\phi(r) = A_2 \frac{1}{r} \sin \frac{\pi r}{R} \tag{44}$$

As before, we define the flux normalization by specifying the power, P, generated in the core. Noting that the differential volume element in 1-D spherical geometry is $d\vec{r} = 4\pi r^2 dr$, we can write the power as

$$P = \kappa\Sigma_f \int \phi(r) d\vec{r} = 4\pi\kappa\Sigma_f \int_0^{R_o} r^2 \phi(r) dr = 4\pi\kappa\Sigma_f A_2 \int_0^{R_o} r \sin \frac{\pi r}{R} dr \tag{45}$$

Now performing the integral (via table lookup) gives

$$P = 4\pi\kappa\Sigma_f A_2 \left[\left(\frac{R}{\pi}\right)^2 \sin \frac{\pi r}{R} - \left(\frac{rR}{\pi}\right) \cos \frac{\pi r}{R} \right]\Bigg|_0^{R_o}$$

$$P = 4\pi\kappa\Sigma_f A_2 \left[\left(\frac{R}{\pi}\right)^2 \sin\frac{\pi R_o}{R} - \left(\frac{R_o R}{\pi}\right)\cos\frac{\pi R_o}{R} \right]$$

and, solving for the normalization factor, gives

$$A_2 = \frac{P}{4\pi\kappa\Sigma_f \left[\left(\frac{R}{\pi}\right)^2 \sin\frac{\pi R_o}{R} - \left(\frac{R_o R}{\pi}\right)\cos\frac{\pi R_o}{R} \right]} \qquad (46)$$

Also, if the extrapolation distance d is small compared to R_o, then $R_o/R \to 1$ and this rather messy expression reduces nicely to

$$A_2 = \frac{P}{4\kappa\Sigma_f R_o^{\,2}} \quad \text{and} \quad \phi(r) = \frac{P}{4\kappa\Sigma_f R_o^{\,2}}\frac{1}{r}\sin\frac{\pi r}{R_o} \quad (\text{for } d \ll R_o) \qquad (47)$$

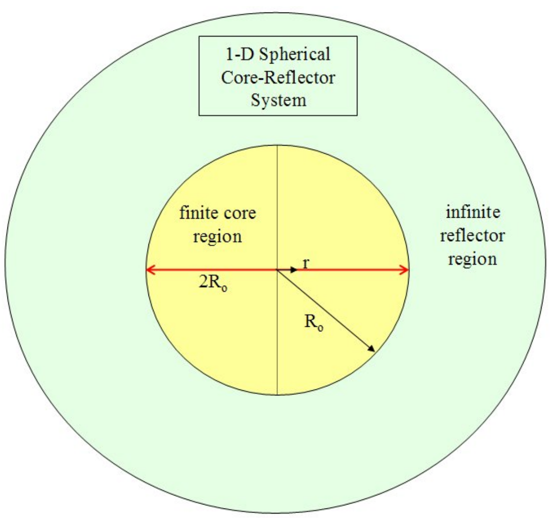
Finally, to complete this development, we note that the criticality condition here is exactly as given for the 1-group bare slab problem as described in eqn. (14) with the use of the buckling relationship for the bare sphere [from eqn. (43)]. Thus, the multiplication factor for the 1-group 1-D bare 'critical' spherical reactor is

$$k_{eff} = \frac{\nu\Sigma_f}{DB^2 + \Sigma_a} \quad \text{with} \quad B^2 = \left(\frac{\pi}{R}\right)^2 \qquad (48)$$

Equations (44) and (46)-(48) completely describe the simple 1-group 1-D bare homogeneous critical spherical reactor problem, and these equations, along with the 1-D core-reflector model in spherical geometry (see below) are implemented for the spherical reactor option within the **core_refl1g_gui** Matlab GUI.

1-Group Critical Core-Reflector System in 1-D Spherical Geometry

To continue our formal development of the various 1-group 1-D systems, we now address the core-reflector problem in spherical geometry. Within the above sections that developed the theory for the bare slab model, the core-reflector slab case, and the bare spherical reactor, we have now addressed all the pieces of the puzzle that are needed to put together the theory for the two-region core-reflector spherical reactor case. Thus, here we will be a little more concise than in the above sections, since it is assumed that the reader has already reviewed the previous material in some detail.



The geometry of interest has a homogeneous spherical core of radius R_o surrounded by an infinite region of a non-multiplying medium, as shown in the sketch. The reflector region is used to reflect a portion of the neutrons back into the core, and to act as a near perfect shield to minimize neutron and gamma exposure to the outside environment.

The defining equations and general solutions for the core and reflector zones within this 1-group 1-D two-region system are:

Core Region

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi_c}{dr}\right)+B^2\phi_c=0 \qquad \text{where} \qquad B^2=\left.\frac{\lambda\nu\Sigma_f-\Sigma_a}{D}\right|_c \tag{49}$$

with $\phi_c(r)=A_1\dfrac{\cos Br}{r}+A_2\dfrac{\sin Br}{r}$ (50)

Also, as before, since the cos (Br)/r term approaches infinity as $r \rightarrow 0$, we must set $A_1 = 0$ to guarantee a finite flux everywhere, leaving

$$\phi_c(r)=A_2\frac{\sin Br}{r} \tag{51}$$

Reflector Region

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi_r}{dr}\right)-\frac{1}{L_r^{\,2}}\phi_r=0 \qquad \text{where} \qquad L_r^{\,2}=\left.\frac{D}{\Sigma_a}\right|_r \tag{52}$$

with $\phi_r(r)=A_3\dfrac{e^{-r/L_r}}{r}+A_4\dfrac{e^{r/L_r}}{r}$ (53)

Since the reflector has infinite thickness, r can become very large -- thus, we must set $A_4 = 0$ so that the flux remains finite everywhere. Doing this gives

$$\phi_r(r)=A_3\frac{e^{-r/L_r}}{r} \tag{54}$$

At this point, we now apply the continuity of flux and continuity of current conditions at the core-reflector interface (i.e. at $r = R_o$), giving

$$\left.\phi_c(r)\right|_{r=R_o}=\left.\phi_r(r)\right|_{r=R_o} \qquad \text{(continuity of flux)}$$

or $A_2\dfrac{\sin BR_o}{R_o}-A_3\dfrac{e^{-R_o/L_r}}{R_o}=0$ (55)

$$\left.J_c(r)\right|_{r=R_o}=\left.J_r(r)\right|_{r=R_o} \qquad \text{(continuity of current)}$$

or $-D_cA_2\left(\dfrac{B\cos BR_o}{R_o}-\dfrac{\sin BR_o}{R_o^{\,2}}\right)+D_rA_3\left(-\dfrac{e^{-R_o/L_r}}{R_oL_r}-\dfrac{e^{-R_o/L_r}}{R_o^{\,2}}\right)=0$ (56)

Writing these two homogeneous equations in matrix form gives

$$\begin{bmatrix} \dfrac{\sin BR_o}{R_o} & -\dfrac{e^{-R_o/L_r}}{R_o} \\ D_c\left(\dfrac{B\cos BR_o}{R_o}-\dfrac{\sin BR_o}{R_o^{\,2}}\right) & D_r\left(\dfrac{1}{R_oL_r}+\dfrac{1}{R_o^{\,2}}\right)e^{-R_o/L_r} \end{bmatrix}\begin{bmatrix} A_2 \\ A_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (57)$$

As discussed previously in some detail, to obtain a non-trivial solution to this homogeneous system of equations requires that ***the determinant of the coefficient matrix is zero***. This gives the geometric buckling for this critical reactor problem and, as before, $B^2_m = B^2_g$ is the formal ***criticality condition*** for this case.

Thus, expanding the determinant of the 2×2 coefficient matrix in eqn. (57) and setting this to zero gives

$$\left(\frac{\sin BR_o}{R_o}\right)D_r\left(\frac{1}{R_oL_r}+\frac{1}{R_o^{\,2}}\right)e^{-R_o/L_r}-D_c\left(\frac{B\cos BR_o}{R_o}-\frac{\sin BR_o}{R_o^{\,2}}\right)\left(-\frac{e^{-R_o/L_r}}{R_o}\right)=0$$

Cancelling the common exponential and $1/R_o^2$ terms, and combining the two global negative signs in the second term give

$$D_r\sin BR_o\left(\frac{1}{L_r}+\frac{1}{R_o}\right)+D_c\left(B\cos BR_o-\frac{\sin BR_o}{R_o}\right)=0$$

Now, dividing by $\sin BR_o$ gives

$$D_r\left(\frac{1}{L_r}+\frac{1}{R_o}\right)+D_c\left(B\cot BR_o-\frac{1}{R_o}\right)=0$$

or $$f(B)=D_c\left(B\cot BR_o-\frac{1}{R_o}\right)+D_r\left(\frac{1}{L_r}+\frac{1}{R_o}\right)=0 \qquad (58)$$

Equation (58) is written in the form of a classical root finding problem (i.e. given the core size, R_o, and the material properties, D_c, L_r, and D_r, what is the value of B such that f(B) = 0?). And, once B (or B^2) is found from eqn. (58), we can use the definition of B^2 in eqn. (49) to get the value of k_{eff}, or

$$B^2=\left.\frac{\lambda\nu\Sigma_f-\Sigma_a}{D}\right|_c \qquad \text{or} \qquad k_{eff}=\frac{1}{\lambda}=\frac{\nu\Sigma_{fc}}{D_cB^2+\Sigma_{ac}} \qquad (59)$$

which is the same general result as all the other 1-group 1-D cases -- where the difference for each geometry, of course, is the unique definition of the (geometric) buckling term. For the 1-group 1-D spherical core-reflector problem, the smallest root of the nonlinear expression given in eqn. (58) gives the desired fundamental mode 'critical' B^2 value. (Note that, because of the cotangent function, there will be an infinite number of B values that satisfy eqn. (58), but the solution associated with the smallest root gives the final steady state flux profile).

Once the proper B^2 value is obtained, the final flux solutions in the core and reflector regions are given by

$$\phi_c(r) = A_2 \frac{\sin Br}{r} \qquad \text{for} \quad 0 \le r \le R_o \qquad (60a)$$

and $$\phi_r(r) = A_2 \sin BR_o \frac{e^{-(r-R_o)/L_r}}{r} \qquad \text{for} \quad r \ge R_o \qquad (60b)$$

where A_3 in the expression for $\phi_r(r)$ is written in terms of A_2 using eqn. (55).

As for the previous cases, the last step in the theoretical development is to define the flux normalization factor, A_2, such that the desired power is produced in the reactor. For this case, only the core region has a non-zero fission cross section; thus, the power constraint can be written as

$$P = \kappa\Sigma_f \int \phi_c(r) d\vec{r} = 4\pi\kappa\Sigma_f \int_0^{R_o} r^2 \phi_c(r) dr = 4\pi\kappa\Sigma_f A_2 \int_0^{R_o} r \sin Br \, dr$$

Performing the integral (via table lookup) gives

$$P = 4\pi\kappa\Sigma_f A_2 \left[\frac{1}{B^2} \sin Br - \left(\frac{r}{B}\right) \cos Br \right]_0^{R_o}$$

$$P = 4\pi\kappa\Sigma_f A_2 \left[\frac{1}{B^2} \sin BR_o - \frac{R_o}{B} \cos BR_o \right]$$

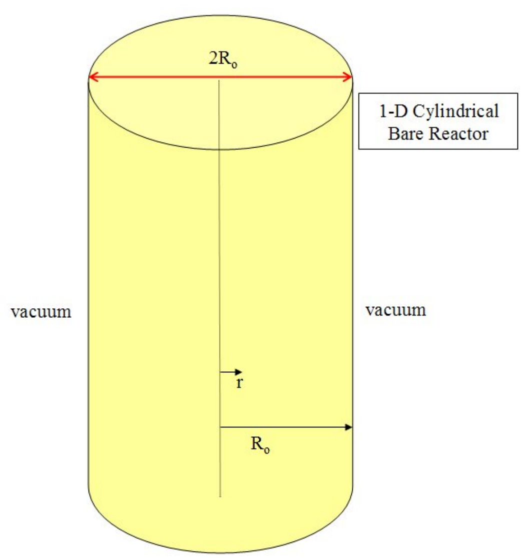
and, solving for the normalization factor, gives

$$A_2 = \frac{PB^2}{4\pi\kappa\Sigma_f \left[\sin BR_o - BR_o \cos BR_o \right]} \qquad (61)$$

This completes the theoretical model development for this case. Equations (58) – (61) completely define the criticality condition (defines B^2 and k_{eff}), the flux shape, and the flux magnitude -- everything needed to do quantitative analyses for a particular 1-group 1-D critical core-reflector system in spherical geometry…

1-Group Bare Critical 1-D Cylindrical Reactor

In this section we look at an infinite 1-D cylindrical critical reactor model. In this case, the core is assumed to be infinitely long and it has a finite radius, R_o. In addition, the core is homogeneous so that the material composition is constant throughout the full cylinder. These specifications, along with vacuum BCs on the outside of the cylinder, lead to a 1-D problem, with the flux distribution only dependent on the radial variable r (i.e. there is no dependence on the z or θ variables). A rough sketch of the 1-D bare cylindrical reactor is shown to the right, where r = 0 is in the center of the reactor.



As before, for a 1-group 1-region homogeneous critical reactor, the general multigroup diffusion equation becomes

$$\nabla^2\phi + B^2\phi = 0 \tag{62}$$

where

$$B^2 = \frac{\lambda\nu\Sigma_f - \Sigma_a}{D} \tag{63}$$

However, in 1-D cylindrical geometry, the Laplacian operator can be written as (see Ref. 2, for example)

$$\nabla^2\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) \tag{64}$$

and using this expression in eqn. (62) gives

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + B^2\phi = 0 \tag{65}$$

This 2^{nd} order ODE is linear, but the first term has variable coefficients. Expanding the first term of this equation gives

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + B^2\phi = 0$$

and multiplication by r^2 gives

$$r^2\phi'' + r\phi' + (B^2r^2 - 0)\phi = 0 \tag{66}$$

As shown in the special set of Lecture Notes on Bessel Functions (see Ref. 10), this form of the diffusion equation for 1-D cylindrical geometry is of the form of an ordinary Bessel equation with order $\nu = 0$. In fact, since many cylindrical reactor problems can be put into the form of Bessel equations, it makes sense to review this subject briefly here before continuing with the given problem (however, the reader should certainly consult Ref. 10 and possibly other sources, as needed, for further information on this important class of functions).

In particular, the standard form of Bessel's equation is usually written as

$$x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0 \tag{67}$$

with the general solution given as

$$y(x) = C_1J_\nu(\alpha x) + C_2Y_\nu(\alpha x) \tag{68}$$

where the functions $J_\nu(\alpha x)$ and $Y_\nu(\alpha x)$ are called ordinary Bessel functions of the first and second kind, respectively, of order ν. Note also that the sign before the last term in the defining equation, $(\alpha^2x^2 - \nu^2)y$, is positive. This is the form that is consistent with the critical reactor problem.

For subcritical regions, the sign of this last term is negative (since the removal term dominates the fission term). In this case, we have a form of the modified Bessel's equation which is generally written as

$$x^2y''+xy'-(\alpha^2x^2+\nu^2)y=0 \tag{69}$$

with the general solution given as

$$y(x)=C_1I_\nu(\alpha x)+C_2K_\nu(\alpha x) \tag{70}$$

where $I_\nu(\alpha x)$ and $K_\nu(\alpha x)$ are modified Bessel functions of order ν of the first and second kind, respectively.

By way of comparison to previous work, it should be emphasized that the ordinary Bessel functions are somewhat similar to the trigonometric functions [sin (αx) and cos (αx)] and the modified Bessel functions are similar in form to the hyperbolic functions [sinh (αx) and cosh (αx)]. The similarities and differences can be seen in the plots of the first two integer-order Bessel functions given in Fig. 3.

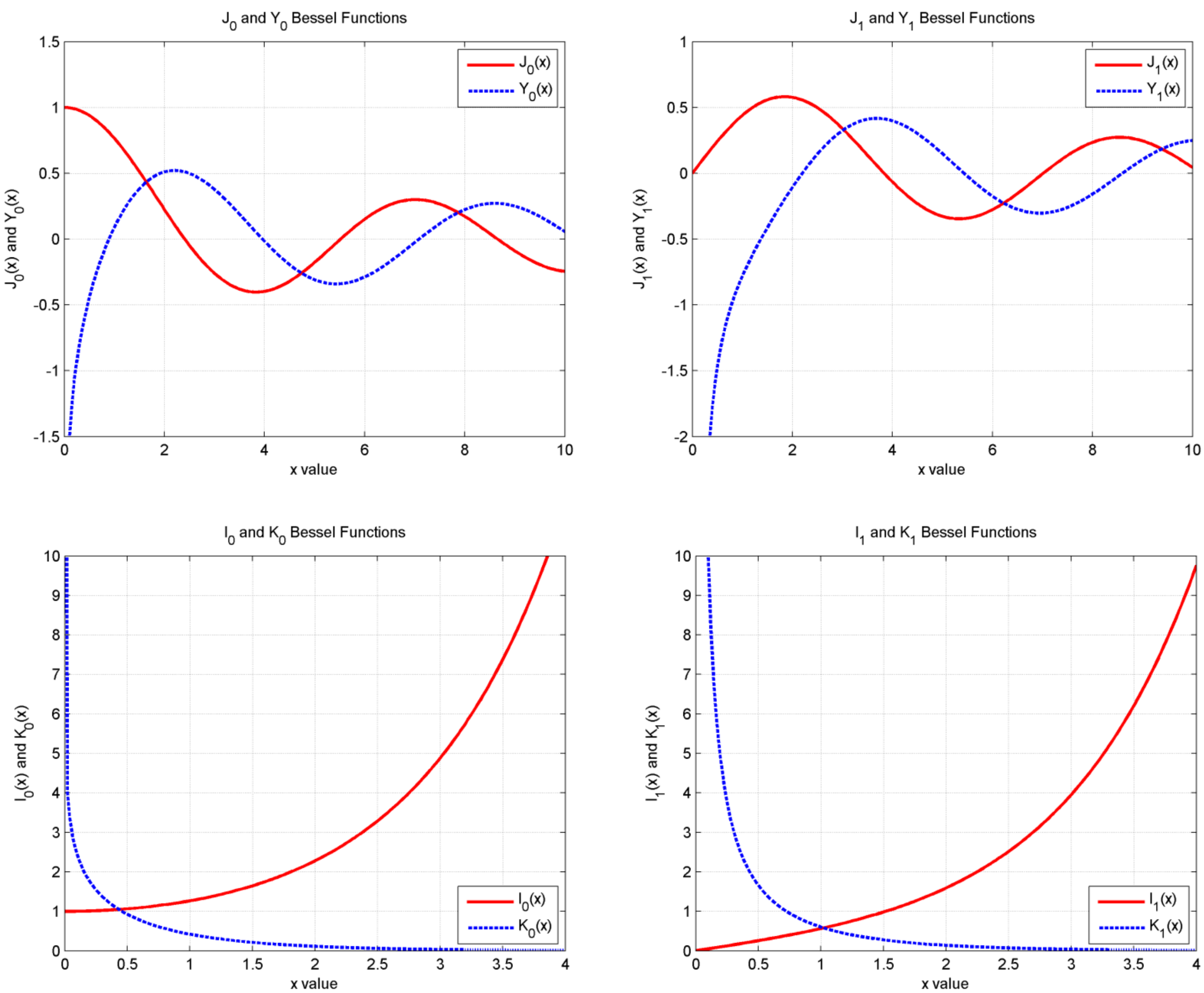


Fig. 3 Some integer-order ordinary and modified Bessel functions.

Before imposing boundary conditions on eqn. (71), we should also look carefully at the general nature of the common low-order Bessel functions displayed in Fig. 3. In addition to the oscillatory and exponential behavior noted above, it should also be apparent that both the ordinary and modified Bessel functions of the second kind [$Y_\nu(x)$ and $K_\nu(x)$, respectively] diverge to positive or negative infinity as the argument approaches zero. Also, as seen in Fig. 3, the $I_\nu(x)$ function approaches infinity as x becomes large. These general trends are important to consider when evaluating boundary conditions for specific applications.

Thus, from the above discussion, we see that the general solution to eqn. (66) can be written in terms of zeroth-order ordinary Bessel functions,

$$\phi(r) = A_1 J_0(Br) + A_2 Y_0(Br) \tag{71}$$

where $J_0(Br)$ and $Y_0(Br)$ are ordinary Bessel functions of the first and second kind, respectively, and they represent two linearly independent solutions to the given 2^{nd} order ODE. A linear combination of these two independent functions gives the general solution to the 1-D cylindrical reactor problem described via eqns. (62) and (63).

With the above discussion, we can now generate a unique solution to our 1-D cylindrical reactor problem. We will impose the conditions that the flux must remain finite at r = 0 and that the flux at r = R is zero (note that $R = R_o + d$ where R_o is the physical dimension and d is the extrapolation distance). The first condition forces A_2 to be zero in eqn. (71), since the $Y_0(Br)$ function goes to $-\infty$ as $r \rightarrow 0$. At the outer boundary, we have

$$J_0(BR) = 0 = J_0(\eta_n) \tag{72a}$$

or

$$B_n = \eta_n / R \quad \text{where} \quad \eta_n = n^{th} \text{ zero of } J_0(x) \text{ for } n = 1, 2, 3, \ldots \tag{72b}$$

where we note, from Fig. 3, that the $J_0(x)$ function is oscillatory in nature and has an infinite number of zero crossings -- which leads to an infinite number of eigenvalues, B_n, and eigenfunctions, $\phi_n(r)$, where

$$\phi_n(r) = J_0(B_n r) \tag{73}$$

However, as we have seen before, only the fundamental mode solution is needed for the steady state flux profile. Thus, the fundamental mode flux profile for the critical reactor problem in 1-D cylindrical geometry is simply

$$\phi(r) = A_1 J_0\left(Br\right) \qquad \text{with} \quad B = \frac{2.4048}{R} \tag{74}$$

where $\eta_1 = 2.4048 \approx 2.405$ is the first zero of the $J_0(x)$ Bessel function and the fundamental mode geometric buckling is $B^2 = (2.4048/R)^2$.

As before, we define the flux normalization by specifying the power per unit length, P, generated in the core. Noting that the differential volume element in 1-D cylindrical geometry is given by $d\vec{r} = 2\pi r dr$, we can write the power per unit length as

$$P = \kappa \int \Sigma_f \phi d\vec{r} = \kappa \int \Sigma_f \phi(r)\, 2\pi r dr = \kappa \Sigma_f\, 2\pi A_1 \int_0^{R_0} r\, J_0\left(Br\right) dr \tag{75}$$

The integral over the radial direction is obtained as follows. First note that from Ref. 10, we have the integral relationship

$$\int xJ_0(x)dx = xJ_1(x) \tag{76}$$

Now, letting x = Br and dx = Bdr, we have

$$\int_0^{R_o} rJ_0(Br)dr = \int_0^{BR_o} \frac{1}{B^2} xJ_0(x)dx = \frac{1}{B^2}\left[xJ_1(x)\right]\Big|_0^{BR_o} = \frac{R_o}{B} J_1(BR_o)$$

Putting this result into eqn. (75) gives

$$A_1 = \frac{PB}{2\pi\kappa\Sigma_f R_o J_1(BR_o)} \qquad \text{with} \quad B = \frac{2.4048}{R} \tag{77}$$

Now, for the case where the extrapolation distance, d, is small relative to the reactor dimensions, we have $R \approx R_o$ and eqn. (77) reduces to

$$A_1 = \frac{2.4048P}{2\pi\kappa\Sigma_f R_o^{\;2} J_1(2.4048)} = \frac{0.7372P}{\kappa\Sigma_f R_o^{\;2}} \tag{78}$$

where the last equality simply evaluates the numerical coefficients $2.4048/[2\pi J_1(2.4048)]$ to be 0.7372 (this numerical value was obtained by using Matlab's ***besselj*** function).

Finally, to complete this development, we note that the criticality condition here is exactly as given for the other 1-group bare reactor geometries as long as we use of the buckling relationship specific for the bare cylinder. Thus, the multiplication factor for the 1-group 1-D bare 'critical' cylindrical reactor is

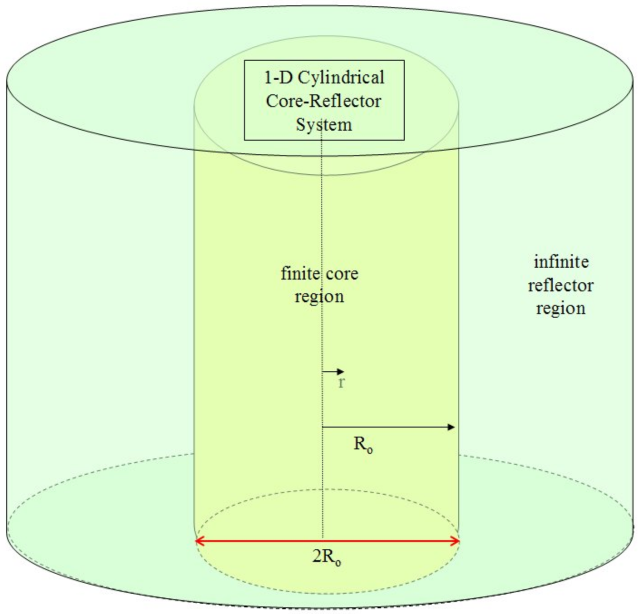
$$k_{eff} = \frac{\nu\Sigma_f}{DB^2 + \Sigma_a} \qquad \text{with} \quad B^2 = \left(\frac{2.4048}{R}\right)^2 \tag{79}$$

Equations (74), (77), and (79) completely describe the simple 1-group 1-D bare homogeneous critical cylindrical reactor problem, and these equations, along with the 1-D core-reflector model in cylindrical geometry (see below), are implemented for the cylindrical reactor option within the **core_refl1g_gui** Matlab GUI.

1-Group Critical Core-Reflector System in 1-D Cylindrical Geometry

Earlier in this set of notes it was indicated that all six 1-D configurations would be worked out in detail (i.e. bare and reflected systems in 1-D slab, spherical, and cylindrical geometry). Well, I have changed my mind, since now I have decided **not to include** the detailed analytical development for the 1-D core-reflector model in cylindrical geometry within these Lecture Notes. In fact, having the student work out the details for this model is an excellent test of your comprehension of the development for the previous five 1-D 1-group reactor models. Thus, we will purposely leave the theoretical development for the current configuration as an exercise for the student -- it should be instructive as a gauge of how much you have learned from the examples given previously…

However, the summary results that you should obtain have been included in Table 1, and these results have been implemented within the **core_refl1g_gui** program. Also, just so the notation makes perfect sense, we have included a simple sketch of the 1-D reflected system in the diagram below.



Please note that, even if you are not completely successful with your development, the data given in Table 1 (on the last page of this document) and the visualization and computational capability within the **core_refl1g_gui** code should be more than sufficient for practical calculations (i.e. core size and material composition analyses) involving 1-D reflected reactors in cylindrical coordinates. Thus, you still have all the same analysis capability as for the other 1-D 1-group bare and reflected cases…

Summary

This document provides a detailed derivation of the 1-group flux profiles in several 1-D bare and core-reflector geometries, with particular focus on comparing the bare critical reactor to its corresponding critical core-reflector configuration. Also, a comparison of the three standard 1-D geometries is made (slab, spherical, and cylindrical geometries), with a note that all three geometries have similar physical behavior, although the mathematical functions that describe the flux profiles are quite different for each geometry.

A summary of the key relationships needed to perform analytical analyses of the three common 1-group 1-D critical reactor geometries is given in Table 1. This can be used as a good summary reference and the information here clearly documents the equations actually implemented within the **core_refl1g_gui** code.

Well, have fun using the **core_refl1g** GUI and the documentation provided in these notes!!! We hope that these educational tools help in the visualization/understanding of the basic processes associated with 1-group 1-D critical systems…

References

1. J. R. White, "The Multigroup Neutron Balance Equation," part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.

2. J. R. Lamarsh and A. J. Baratta, ***Introduction to Nuclear Engineering***, 3rd Edition, Prentice Hall (2001).
3. J. R. White, “Planar Source in a Moderating Medium,” part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***slabmm_gui*** Matlab program.
4. J. R. White, “Point Source in a Moderating Medium,” part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***spheremm_gui*** Matlab program.
5. J. R. White, “Two-Region Slab with a Planar Source at the Interface,” part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***two-regions_gui*** Matlab program.
6. J. R. White, “Two Planar Sources in an Infinite Moderating Medium,” part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***two_planar_sources_gui*** Matlab program.
7. J. R. White, “Interpretation of the Diffusion Length,” part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.
8. J. R. White, “Cross Section Data for Preliminary Calculations,” part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell. This set of Lecture Notes also provides documentation for the ***cross_sections_gui*** Matlab program.
9. J. R. White, “A 2-Group Example: Point Source of Fast Neutrons in an Infinite Moderating Medium,” part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.
10. J. R. White, “Overview of Bessel Functions,” part of a series of Lecture Notes for the Nuclear Engineering Program at UMass-Lowell.

Table 1 Summary equations and various relationships for several 1-group 1-D critical systems.

Geometry	Configuration	Flux Profile	Power Normalization	Geometric Buckling
1-D Slab	**Bare Core**	$\phi(x) = A_1 \cos Bx$	$A_1 = \dfrac{PB}{2\kappa\Sigma_f \sin\dfrac{Ba_o}{2}}$	$B^2 = \left(\dfrac{\pi}{a}\right)^2$
	Core-Reflector	$\phi_c(x) = A_1 \cos Bx$ $\phi_r(x) = A_1 \cos\dfrac{Ba_o}{2} e^{-(x-a_o/2)/L_r}$		$f(B) = \cot\dfrac{Ba_o}{2} - \dfrac{L_r D_c B}{D_r} = 0$
1-D Sphere	**Bare Core**	$\phi(r) = A_2 \dfrac{\sin Br}{r}$	$A_2 = \dfrac{PB^2}{4\pi\kappa\Sigma_f\left[\sin BR_o - BR_o \cos BR_o\right]}$	$B^2 = \left(\dfrac{\pi}{R}\right)^2$
	Core-Reflector	$\phi_c(r) = A_2 \dfrac{\sin Br}{r}$ $\phi_r(r) = A_2 \sin BR_o \dfrac{e^{-(r-R_o)/L_r}}{r}$		$f(B) = D_c\left(B \cot BR_o - \dfrac{1}{R_o}\right)$ $+ D_r\left(\dfrac{1}{L_r} + \dfrac{1}{R_o}\right) = 0$
1-D Cylinder	**Bare Core**	$\phi(r) = A_1 J_0(Br)$	$A_1 = \dfrac{PB}{2\pi\kappa\Sigma_f R_o J_1(BR_o)}$	$B^2 = \left(\dfrac{2.4048}{R}\right)^2$
	Core-Reflector	$\phi_c(r) = A_1 J_0(Br)$ $\phi_r(r) = A_1 \dfrac{J_0(BR_o)}{K_0(R_o/L_r)} K_0(r/L_r)$		$f(B) = D_c B J_1(BR_o) K_0(R_o/L_r)$ $- \dfrac{D_r}{L_r} J_0(BR_o) K_1(R_o/L_r) = 0$
Notes:	For the bare cores, the extrapolated core size is given by: $a = a_o + 2d$ or $R = R_o + d$			
	The multiplication factor for all the 1-group 1-D critical systems is given by: $k_{eff} = \dfrac{\nu\Sigma_{fc}}{D_c B^2 + \Sigma_{ac}}$			
	For the core-reflector systems, the statement $f(B) = 0$ represents a classical root finding problem, where one searches for the smallest value of B to obtain the fundamental mode solution.			