Overview of Bessel Functions

Introduction

If a particular differential equation and its solution occur frequently in applications, one gives them a name and introduces special symbols that define them. The properties of the functions are studied and tabulated and this information becomes a resource that can be exploited by the practicing engineer.

We have seen that linear constant coefficient systems have solutions that can be written in terms of elementary functions (sinusoids, exponentials, etc.). These functions are called elementary because they are treated in detail in introductory algebra, trigonometry, and calculus courses and they are used routinely in a variety of engineering applications. In short, since we are very familiar with these functions, they are easy to work with and we refer to them as elementary functions.

In contrast, functions that we are not as familiar with are initially more difficult to use in applications and sometimes these are referred to as non-elementary functions, special functions, or transcendental functions. In particular, there is a whole range of functions in common use which are solutions to a general class of linear variable coefficient homogeneous 2^{nd} order ordinary differential equations. A technique called the Power Series Method is often used to solve these linear variable coefficient ODEs. The solutions are written in the form of power series, $y(x) = \sum a_n x^n$ where, in most cases, the a_n expansion coefficients are complicated functions of the coefficients within the original ODE. Since the power series solution is cumbersome to work with on a regular basis, the so-called special functions are given unique names and all their important properties (integrals, derivatives, recurrence relationships, etc.) are tabulated in the literature for use by the practicing engineer and scientist.

Also, it should be emphasized that, once you gain a little experience with these special functions, you will no longer be intimidated with their use, and the non-elementary connotation will no longer be applicable -- for example, using Bessel functions is as easy as using sinusoids and exponential functions, once you become comfortable with their use!

Well, as you might have guessed, Bessel functions fall into the class of special functions discussed above. Bessel functions are the linearly independent solutions to a special set of 2^{nd} order linear variable coefficient ODEs (see the details below) that occur frequently in a variety of applications, such as those involving variable area problems. The purpose of this set of Lecture Notes is to define the ordinary and modified Bessel functions and to tabulate some of their more important properties (i.e. differential and integral relationships, limiting conditions, etc.). In addition, we also briefly illustrate the use of the Matlab functions that allow numerical evaluation and plotting of the Bessel functions of interest. This overview is by no means complete -- it only contains the essential relationships needed to work with the Bessel functions within the context of solving several cylindrical geometry problems from nuclear reactor theory. Thus, the student is certainly encouraged to consul other more comprehensive reference material, as needed -- do a Google search and you will find lots of additional information!!!

Ordinary and Modified Bessel Equations and their Solutions

Ordinary Bessel Functions

Ordinary Bessel Equation	$x^2y'' + xy' + \left(\lambda^2x^2 - \nu^2\right)y = 0$
General Solution (ordinary)	$y(x) = A_1J_\nu(\lambda x) + A_2Y_\nu(\lambda x)$
Definition of Y_ν	$Y_\nu(\lambda x) = \dfrac{\cos\nu\pi\, J_\nu(\lambda x) - J_{-\nu}(\lambda x)}{\sin\nu\pi}$
Definition of Y_n	$Y_n(\lambda x) = \lim\limits_{\nu\to n} Y_\nu(\lambda x)$

Modified Bessel Functions

Modified Bessel Equation	$x^2y'' + xy' - \left(\lambda^2x^2 + \nu^2\right)y = 0$
General Solution (modified)	$y(x) = A_1I_\nu(\lambda x) + A_2K_\nu(\lambda x)$
Definition of I_ν	$I_\nu(\lambda x) = i^{-\nu}J_\nu(i\lambda x)$
Definition of K_ν	$K_\nu(\lambda x) = \dfrac{\pi}{2}\dfrac{I_{-\nu}(\lambda x) - I_\nu(\lambda x)}{\sin\nu\pi}$
Definition of K_n	$K_n(\lambda x) = \lim\limits_{\nu\to n} K_\nu(\lambda x)$

In the above tables, $J_\nu(\lambda x)$ is referred to as an ***ordinary*** Bessel function of the ***first kind*** and $Y_\nu(\lambda x)$ is known as an ***ordinary*** Bessel function of the ***second kind***. J_ν and Y_ν are linearly independent for any $\nu > 0$. Similarly, $I_\nu(\lambda x)$ is referred to as a ***modified*** Bessel function of the ***first kind*** and $K_\nu(\lambda x)$ is known as a ***modified*** Bessel function of the ***second kind***. I_ν and K_ν are linearly independent for any $\nu > 0$.

Note also that the ordinary and modified Bessel equations only differ by the sign of the λ^2x^2y term in the defining ODE -- this sign is extremely important in determining the character of the solution. Also, in these equations, ν is referred to as the ***order*** of the Bessel function, and λ is said to be a ***parameter***. Thus, for example, $J_\nu(\lambda x)$ is an ordinary Bessel function of the first kind of order ν and parameter λ.

Also we should note that the $J_\nu(\lambda x)$ and $I_\nu(\lambda x)$ Bessel functions are defined for negative order. However, when the order ν is an integer, then $J_{-\nu}(\lambda x)$ and $I_{-\nu}(\lambda x)$ are not linearly independent from $J_\nu(\lambda x)$ and $I_\nu(\lambda x)$. Thus, as shown in the above tables, one usually defines the Bessel function of the second kind (i.e. the 2^{nd} linearly independent solution to the ODE) as $Y_\nu(\lambda x)$ and $K_\nu(\lambda x)$ for the ordinary and modified Bessel equations, respectively. These functions are linearly independent from the corresponding Bessel functions of the first kind for both non-integer and integer order.

Properties and Relationships among the Bessel Functions

Several important **Recurrence Formulas** are:

$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x)$	$Y_{\nu+1}(x) = \frac{2\nu}{x} Y_\nu(x) - Y_{\nu-1}(x)$
$I_{\nu+1}(x) = I_{\nu-1}(x) - \frac{2\nu}{x} I_\nu(x)$	$K_{\nu+1}(x) = K_{\nu-1}(x) + \frac{2\nu}{x} K_\nu(x)$

Some important **Derivative Formulas** are:

$\frac{d}{dx} J_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x)$ $= -J_{\nu+1}(x) + \frac{\nu}{x} J_\nu(x)$	$\frac{d}{dx} Y_\nu(x) = Y_{\nu-1}(x) - \frac{\nu}{x} Y_\nu(x)$ $= -Y_{\nu+1}(x) + \frac{\nu}{x} Y_\nu(x)$
$\frac{d}{dx} I_\nu(x) = I_{\nu-1}(x) - \frac{\nu}{x} I_\nu(x)$ $= I_{\nu+1}(x) + \frac{\nu}{x} I_\nu(x)$	$\frac{d}{dx} K_\nu(x) = -K_{\nu-1}(x) - \frac{\nu}{x} K_\nu(x)$ $= -K_{\nu+1}(x) + \frac{\nu}{x} K_\nu(x)$

Some important **Integral Formulas** are:

One can use the derivative formulas to derive various integral relations. For example, the above expression for $J_\nu'(x)$, for $\nu = 0$, gives

$$J_0'(x) = -J_1(x)$$

Thus, from this relationship, we have

$$\int J_1(x)dx = -J_0(x)$$

Similarly, the expression for $J_\nu'(x)$, for $\nu = 1$, gives

$$J_1'(x) = J_0(x) - \frac{1}{x} J_1(x) \qquad \text{or} \qquad xJ_1'(x) + J_1(x) = xJ_0(x)$$

The left hand side of the last expression can be written as the derivative of the $xJ_1(x)$ product, or

$$\frac{d}{dx}\left[xJ_1(x)\right] = xJ_0(x)$$

Therefore, integrating this expression gives

$$\int xJ_0(x)dx = xJ_1(x)$$

Plots and Limiting Values for the Low-Order Bessel Functions

It is important to have a feeling for the functional behavior of the Bessel functions for various values of the argument x. This is particularly true for the low, integer-order Bessel functions since they occur so frequently in practical applications. To show this behavior, a short Matlab file, **bessplt.m**, has been written to plot some low-order Bessel functions and the resultant plots are given in Fig. 1. From here it is obvious that the ordinary Bessel functions are oscillatory in nature and that the modified Bessel functions tend to look more like decaying and growing exponentials (this is only a rough description). A listing of **bessplt.m** is given in Table 1.

Also of interest here is the limiting values of the low-order integer Bessel functions on the interval $0 \le x \le \infty$. In particular, the limiting values can be summarized as follows:

	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$	$I_0(x)$	$I_1(x)$	$K_0(x)$	$K_1(x)$
as $x \to 0$	1	0	$-\infty$	$-\infty$	1	0	∞	∞
as $x \to \infty$	oscillates	oscillates	oscillates	oscillates	∞	∞	0	0

These quantities are particularly useful in evaluating boundary conditions for boundary value problems (BVPs) which can be solved in terms of integer-order Bessel functions.

Note: There are many additional useful relationships for the Bessel functions that have not been tabulated in this brief overview, and the student is encouraged to browse the literature for a more comprehensive treatise on this subject -- have fun...

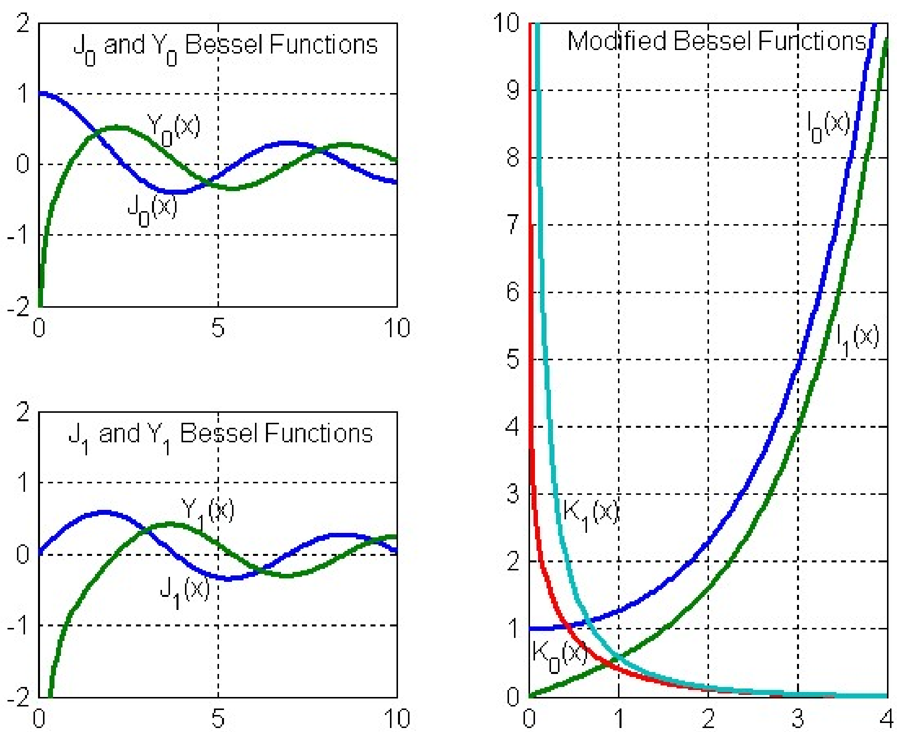

Fig. 1 Some plots for the low-order Bessel functions.

Table 1 Listing of Matlab m-file bessplt.m.

```
%
%   BESSPLT.M    Sample script file to plot some low-order Bessel Functions
%
%   This is a sample file to generate plots of the zero and first order Bessel
%   functions - J0(x), J1(x)   and   Y0(x), Y1(x)
%             - I0(x), I1(x)   and   K0(x), K1(x)
%
%   File written by J. R. White, UMass-Lowell (Jan. 2015)
%

     clear all, close all,  nfig = 0;
%
%   setup independent variable, but don't evaluate at exactly zero since some
%   of the functions have a singular point at zero
     Nx1 = 201;   x1 = linspace(eps,10,Nx1);   % range for ordinary BF plots
     Nx2 = 201;   x2 = linspace(eps,4,Nx2);    % range for modified BF plots
%
%   evaluate ordinary Bessel functions
     J0 = besselj(0,x1);    Y0 = bessely(0,x1);
     J1 = besselj(1,x1);    Y1 = bessely(1,x1);
%
%   evaluate modified Bessel functions
     I0 = besseli(0,x2);    K0 = besselk(0,x2);
     I1 = besseli(1,x2);    K1 = besselk(1,x2);
%
%   now let's plot these curves
     nfig = nfig+1;   figure(nfig)
     subplot(2,2,1),plot(x1,[J0;Y0],'LineWidth',2),grid
     axis([0 10 -2 2]);
     gtext('J_0(x)'),gtext('Y_0(x)')
     gtext('J_0 and Y_0 Bessel Functions')
%
     subplot(2,2,3),plot(x1,[J1;Y1],'LineWidth',2),grid
     axis([0 10 -2 2]);
     gtext('J_1(x)'),gtext('Y_1(x)')
     gtext('J_1 and Y_1 Bessel Functions')
%
     subplot(1,2,2),plot(x2,[I0;I1;K0;K1],'LineWidth',2),grid
     axis([0 4 0 10]);
     gtext('I_0(x)'),gtext('I_1(x)'),gtext('K_0(x)'),gtext('K_1(x)')
     gtext('Modified Bessel Functions')
%
%   end of demo
```