Nuclear Reactor Theory

Lesson 7: The Critical Reactor II

The Bare Critical Finite Cylindrical Reactor

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Lesson 7 Objectives

Setup and solve the 1-group diffusion equation for the 2-D bare critical finite cylindrical reactor.

Explain the basic idea behind the Separation of Variables method for solving relatively simple PDEs.

Recognize the standard form of Bessel's equations and write the general solution to ODEs of this form.

Perform integration and differentiation involving Bessel functions (to compute the normalization constant and to find the neutron current or leakage).

Describe the basic functional behavior of the flux and current profiles for the bare finite cylindrical reactor.

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Consider the bare finite cylindrical reactor with extrapolated dimensions R and H, where $R = R_o + d$ and $H = H_o + 2d$.

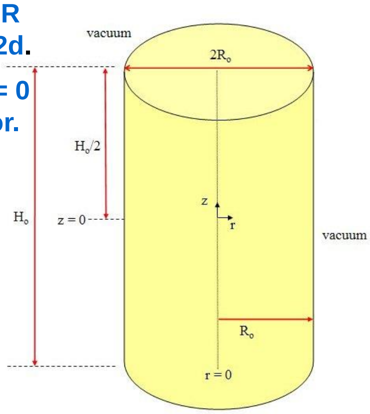
The coordinate system is such that r = 0 and z = 0 is in the center of the reactor.

The 1-group critical reactor model for this two dimensional (2-D) system is

$$\nabla^2\phi(r,z) + B^2\phi(r,z) = 0$$

and the Laplacian in cylindrical coordinates is given as

$$\nabla^2\phi(r,\theta,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$



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Thus, for azimuthal symmetry, the balance equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2} + B^2\phi = 0$$

this is a PDE with two independent variables, r and z

We will use the Method of Separation of Variables to develop a solution to this 2-D PDE problem.

The method assumes that the flux is separable in the two spatial dimensions, r and z.

Although this separability assumption is not valid in most situations, for the simple bare homogeneous reactor, it is indeed valid (we would run into problems during the derivation if our assumption was incorrect).

Bare Critical Finite Cylindrical Reactor

Therefore, we write $\phi(r,z)$ as a product of two functions: one that is only a function of r and the other only dependent on z.

This gives

$$\phi(r,z) = X(r)Y(z)$$

Substituting this assumed form of the solution into the original PDE and dividing by $\phi = XY$, gives

$$\frac{1}{X}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dX}{dr}\right)\right] + \frac{1}{Y}\frac{d^2Y}{dz^2} + B^2 = 0$$

Analysis of this last expression indicates that the first term is only a function of r, the second term is only a function of z, and the third term is a constant.

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In order to satisfy this expression for all values of r and z, the first two terms must separately equal some constant (usually called the separation constant).

Doing this gives the following three expressions,

$$\frac{1}{Y}\frac{d^2Y}{dz^2} = -\alpha^2 \qquad \frac{1}{X}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dX}{dr}\right)\right] = -\beta^2$$

and

$$B^2 = \alpha^2 + \beta^2$$

The negative signs are needed to satisfy the BCs for the problem.

The above procedure gives a separate second-order differential equation in each direction -- it converts the original partial differential equation (PDE) into two ordinary differential equations (ODEs).

This is the basic idea behind the Separation of Variables method.

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Axial Direction

Addressing the axial direction first, we require the solution to

$$\frac{d^2Y}{dz^2} + \alpha^2 Y = 0$$

Noting that this is just the 1-D bare slab reactor problem from previous work, we have

$$Y(z) = A_1 \cos\alpha z + A_2 \sin\alpha z$$

Because of the separability assumption, we can evaluate the boundary conditions in each direction without the interaction of the other direction.

In the z-direction, the appropriate boundary conditions are symmetry at $z = 0$ and the flux goes to zero at $z = H/2$.

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Axial Direction (cont.)

Imposing these conditions gives

$$\left.\frac{dY(z)}{dz}\right|_{z=0} = 0 \qquad \text{implies that } A_2 = 0$$

$$Y(z)\big|_{H/2} = 0 \qquad \text{implies that } \cos(\alpha H/2) = 0$$

or

$$Y_n(z) = \cos(\alpha_n z) \qquad \text{where} \quad \alpha_n = \frac{(2n-1)\pi}{H} \qquad \text{for } n = 1, 2, 3, \ldots$$

This is the same result that we have already seen for the 1-D Cartesian geometry bare reactor problem.

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Radial Direction

In the radial direction, we have a little more work to do since the defining ODE is not a simple constant coefficient equation.

The ODE of interest here is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dX}{dr}\right)+\beta^2 X=0$$

Expanding the first term of this equation gives

$$\frac{d^2X}{dr^2}+\frac{1}{r}\frac{dX}{dr}+\beta^2 X=0$$

and multiplication by r^2 gives

$$r^2X''+rX'+(\beta^2r^2-0)X=0$$

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Radial Direction (cont.)

As shown in the special set of Lecture Notes on Bessel Functions, this form of the diffusion equation for 1-D cylindrical geometry is of the form of an ordinary Bessel equation with order $\nu = 0$.

Since the subject of Bessel functions may be new to many students, it makes sense to give a brief overview of this subject before continuing with the given problem…

See the next few slides!!!

Brief Overview of Bessel Functions

In particular, the standard form of Bessel's equation is usually written as

$$x^2y''+xy'+(\alpha^2x^2-\nu^2)y=0$$

with the general solution given as

$$y(x)=C_1J_\nu(\alpha x)+C_2Y_\nu(\alpha x)$$

where the functions $J_\nu(\alpha x)$ and $Y_\nu(\alpha x)$ are called ordinary Bessel functions of the first and second kind, respectively, of order ν.

Note also that the sign before the α^2x^2y term in the defining equation is positive.

This form is consistent with the critical reactor problem.

Brief Overview of Bessel Functions

For subcritical regions, the sign of this term is negative, and we have a form of the modified Bessel's equation which is generally written as

$$x^2y''+xy'-(\alpha^2x^2+\nu^2)y=0$$

with the general solution given as

$$y(x)=C_1I_\nu(\alpha x)+C_2K_\nu(\alpha x)$$

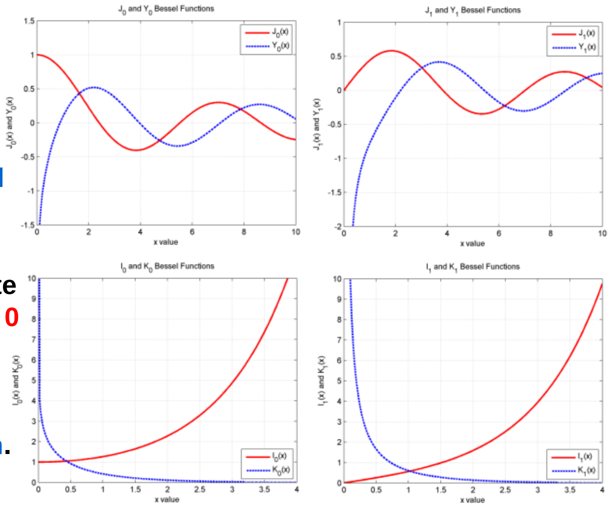
where $I_\nu(\alpha x)$ and $K_\nu(\alpha x)$ are modified Bessel functions of order ν of the first and second kind.

Comparing to previous work, we note that the ordinary Bessel functions behave similar to the oscillatory trigonometric functions and the modified Bessel functions have exponential-like behavior similar to the hyperbolic functions.

Brief Overview of Bessel Functions

The similarities and differences can be seen in the selected plots of the first two integer-order Bessel functions.

One should also note the behavior as $x \rightarrow 0$ and as $x \rightarrow \infty$, since these are useful when applying BCs to Bessel's equation.



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Radial Direction (cont.)

Now, comparing the diffusion equation in the radial direction to Bessel's equation, we see that the general solution can be written in terms of zero-order ordinary Bessel functions,

$$X(r) = C_1 J_0(\beta r) + C_2 Y_0(\beta r)$$

For the radial direction, the appropriate boundary conditions are that the flux must remain finite at r = 0 and that the flux at r = R is zero.

The first condition forces C_2 to be zero, since the $Y_0(\beta r)$ function goes to $-\infty$ as $r \rightarrow 0$.

Thus, the general solution reduces to

$$X(r) = C_1 J_0(\beta r)$$

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Radial Direction (cont.)

At the outer boundary, we have

$$J_0(\beta R) = 0 = J_0(\eta_m)$$

where η_m = m^{th} zero of $J_0(x)$ for m = 1, 2, 3, …

Thus,

$$X_m(r) = J_0(\beta_m r) \quad \text{where} \quad \beta_m = \eta_m / R \quad \text{for } m = 1, 2, 3, \ldots$$

Note that :
$\eta_1 \approx 2.4048$, $\eta_2 \approx 5.5201$, …

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General Solution

Finally, combining the radial solution with the axial solution gives the desired general solution for a bare finite cylindrical reactor,

$$\phi_{mn}(r,z) = AJ_0(\beta_m r)\cos(\alpha_n z)$$

With the geometric buckling given by

$$B_{mn}^2 = \alpha_n^2 + \beta_m^2 = \left[\frac{(2n-1)\pi}{H}\right]^2 + \left[\frac{\eta_m}{R}\right]^2$$

As we have seen before, there are an infinite number of eigenvalues and eigenfunctions that satisfy the defining ODE (n refers to the axial mode shape and m represents the radial mode profile).

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General Solution (cont.)

However, for the same reasons as before (i.e. higher modes vanish at SS), we immediately set m = n = 1 to obtain the fundamental mode solution,

$$\phi(r,z) = AJ_0\left(\frac{2.4048}{R}r\right)\cos\left(\frac{\pi}{H}z\right) \quad \text{and} \quad B^2 = \left(\frac{\pi}{H}\right)^2 + \left(\frac{2.4048}{R}\right)^2$$

where the first zero of the $J_0(x)$ Bessel function occurs at $\eta_1 = 2.4048$.

Power Normalization

To complete the solution for this problem we must find the normalization constant A. Using the total reactor power, P, we have

$$P = \kappa\int \Sigma_f \phi d\vec{r} = \kappa\int \Sigma_f \phi(r,z) 2\pi r dr dz$$

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Power Normalization (cont.)

or

$$P = \kappa\Sigma_f 2\pi A\int_0^{R_0} r J_0\left(\frac{2.4048r}{R}\right)dr \int_{-H_o/2}^{H_o/2}\cos\frac{\pi z}{H}dz$$

The integral over the axial direction gives

$$\int_{-H_o/2}^{H_o/2}\cos\frac{\pi z}{H}dz = \frac{H}{\pi}\sin\frac{\pi z}{H}\Bigg|_{-H_o/2}^{H_o/2} = \frac{H}{\pi}\left[\sin\frac{\pi H_o}{2H} - \sin\frac{-\pi H_o}{2H}\right] = \frac{2H}{\pi}\sin\frac{\pi H_o}{2H}$$

The integral over the radial direction is obtained using the following steps:

First note the following integral relationship involving the J_0 Bessel function,

$$\int xJ_0(x)dx = xJ_1(x)$$

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Power Normalization (cont.)

Now, to do the integral, we let x = 2.4048r/R, which gives

$$r = \frac{Rx}{2.4048} \quad \text{and} \quad dr = \frac{Rdx}{2.4048}$$

and, upon substitution, the radial component of the integral power becomes

$$\int_0^{R_o} rJ_0\left(\frac{2.4048r}{R}\right)dr = \int_0^{2.4048R_o/R}\left(\frac{R}{2.4048}\right)^2 xJ_0(x)dx$$

$$= \left(\frac{R}{2.4048}\right)^2 \left[xJ_1(x)\right]_0^{2.4048R_o/R}$$

$$= \left(\frac{R}{2.4048}\right)^2 \frac{2.4048R_o}{R} J_1\left(\frac{2.4048R_o}{R}\right)$$

$$= \frac{R_oR}{2.4048} J_1\left(\frac{2.4048R_o}{R}\right)$$

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Power Normalization (cont.)

Putting these results (axial and radial integrals) into the power equation gives

$$A = \frac{2.4048\,\pi P}{4\kappa\Sigma_f \pi R_o RH\left[\sin(\pi H_o/2H)\right]\left[J_1(2.4048R_o/R)\right]}$$

Now, for the case where the extrapolation distance, d, is small relative to the reactor dimensions, we have $R \approx R_o$ and $H \approx H_o$.

Noting that the reactor volume is $V = \pi R_o^2 H_o$, for this situation the normalization constant reduces to

$$A = \frac{2.4048\pi}{4J_1(2.4048)}\frac{P}{\kappa\Sigma_f V} = \frac{3.638P}{\kappa\Sigma_f V}$$

where the last equality simply evaluates the first coefficient to be 3.638 (using Matlab to evaluate the J_1 Bessel function).

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Criticality Condition

To complete this problem, we write the formal criticality condition here as

$$k_{eff} = \frac{\nu\Sigma_f}{DB^2 + \Sigma_a} = \frac{k_\infty}{1 + L^2B^2} = k_\infty P_{NL}$$

where, as noted before, this relationship is valid for any 1-group 1-region system.

The only unique aspect is that the buckling used here is specific for the bare finite cylindrical reactor, or

$$B^2 = \left(\frac{\pi}{H}\right)^2 + \left(\frac{2.4048}{R}\right)^2$$

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Visualization of the Flux and Current Distributions

A short Matlab program, bare1g_rz.m, was written to evaluate and display the flux and current profiles in various formats for the bare finite cylindrical reactor.

In the simulations, the extrapolation distance is assumed to be small and the flux magnitude is set to unity.

Also, for convenience, the diffusion coefficient, D, is set to unity.

The geometry is assumed to be a right circular cylinder where the height, H, is twice the diameter -- thus, H = 2(2R) = 4R.

Also, for specificity in the plots, we set R = 1 m.

The goal here is simply to be able to visualize the shape of the flux and current profile in a 2-D system.

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Visualization of the Flux and Current Distributions (cont.)

The profiles plotted are:

Flux Distribution: $\phi(r,z) = AJ_0\left(\frac{2.4048}{R}r\right)\cos\left(\frac{\pi}{H}z\right)$

Current Distribution: $\vec{J}(r,z) = J_r(r,z)\hat{a}_r + J_z(r,z)\hat{a}_z$

where the radial and axial components of the current are given by

$$J_r(r,z) = -DA\frac{\partial}{\partial r}\left[J_0\left(\frac{2.4048}{R}r\right)\cos\left(\frac{\pi}{H}z\right)\right] = \frac{2.4048}{R}DA\,J_1\left(\frac{2.4048}{R}r\right)\cos\left(\frac{\pi}{H}z\right)$$

$$J_z(r,z) = -DA\frac{\partial}{\partial z}\left[J_0\left(\frac{2.4048}{R}r\right)\cos\left(\frac{\pi}{H}z\right)\right] = \frac{\pi}{H}DA\,J_0\left(\frac{2.4048}{R}r\right)\sin\left(\frac{\pi}{H}z\right)$$

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Visualization of the Flux and Current Distributions (cont.)

where we used the following derivative expression for the J_0 Bessel function

$$\frac{d}{dx}J_0(x) = -J_1(x)$$

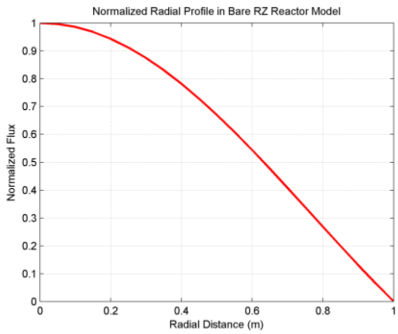
and, via the chain rule, we get

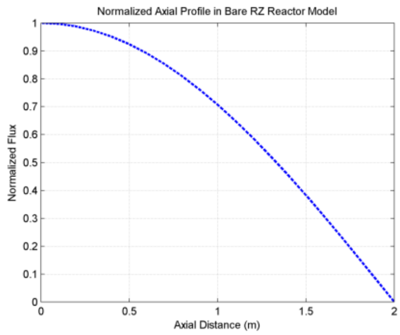
$$\frac{d}{dr}J_0(x) = \frac{d}{dx}J_0(x)\frac{dx}{dr} = -\left(\frac{2.4048}{R}\right)J_1\left(\frac{2.4048}{R}r\right)$$

to formally do the above $J_r(r,z)$ calculation (with $x = 2.4048r/R$).

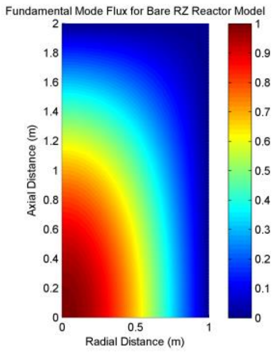
Normalized Flux Profiles

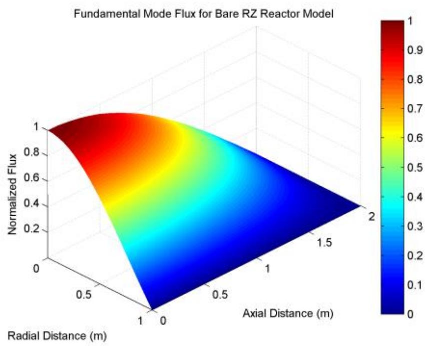
A set of summary plots is displayed here and on the next several slides, and you are encouraged to study these profiles carefully to try to really visualize the flux and current distributions obtained for this simple 2-D reactor model -- hopefully the variety of plots displayed here is sufficient to accomplish this goal…



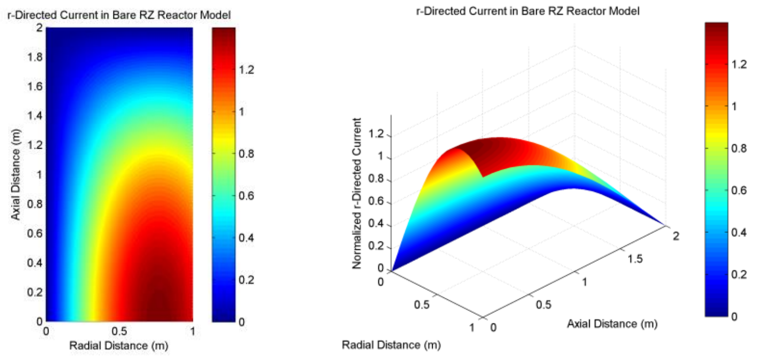


Surface Plots of the Normalized Flux Distribution

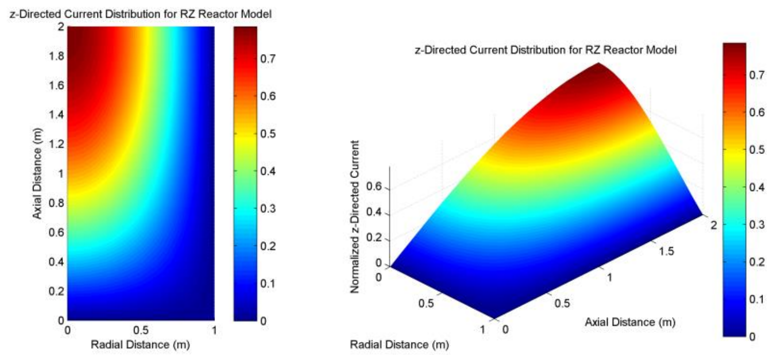




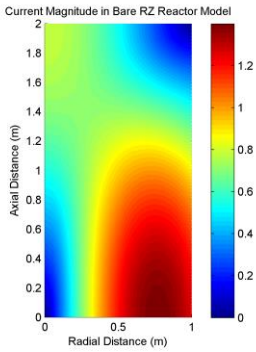
Surface Plots of the r-directed Current Distribution

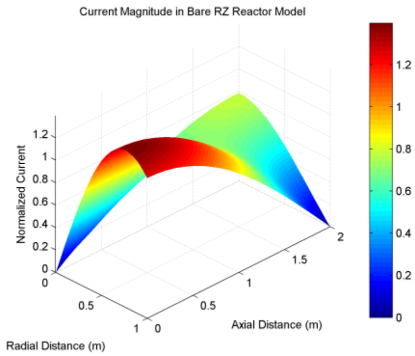


Surface Plots of the z-directed Current Distribution



Surface Plots of the Current Magnitude





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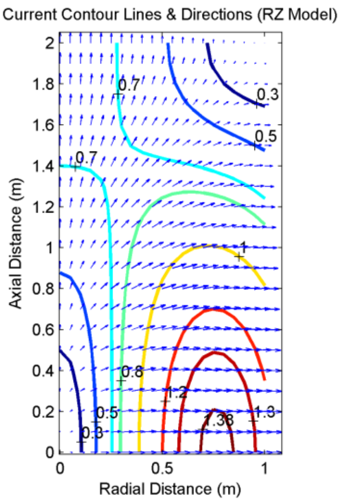
Lesson 7: The Critical Reactor II

(Oct. 2016)

Contours and Directions for the Current in the Bare Critical RZ Reactor



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Lesson 7: The Critical Reactor II

(Oct. 2016)

Lesson 7 Summary

In this Lesson we have briefly discussed the following subjects:

How to setup and solve the 1-group diffusion equation for the 2-D bare critical finite cylindrical reactor.

The basic idea behind the Separation of Variables method for solving relatively simple PDEs.

The standard form of Bessel's equations and how to write the general solution to ODEs of this form.

How to perform integration and differentiation involving Bessel functions (to compute the normalization constant and to find the neutron current or leakage).

The basic functional behavior of the flux and current profiles for the bare finite cylindrical reactor.