Nuclear Reactor Theory

Lesson 6: The Critical Reactor I

Overview and 1-Group 1-D Applications

Prof. John R. White
Chemical and Nuclear Engineering
UMass-Lowell, Lowell MA

Lesson 6 Objectives

Explain the fundamental differences associated with application of the diffusion equation to non-multiplying media problems, subcritical systems, and the critical reactor problem.

Explain the use of the mathematical eigenvalue, λ, within the critical reactor problem.

Write the 1-group fission source and the expression for the 1-group multiplication factor, k, in a variety of ways.

Setup and solve the 1-group 1-D bare critical reactor problem for slab, spherical, and cylindrical geometries.

Discuss the meaning of the criticality condition and the concept of material and geometric bucklings.

Setup and solve the 1-group 1-D 2-region core-reflector critical reactor problem for slab, spherical, and cylindrical geometries.

Lesson 6 Objectives (cont.)

Discuss the meaning of the phrase **"critical determinant"** as applied to the core-reflector critical reactor problem.

Apply a **power constraint** to determine the flux normalization in various geometries.

Perform **critical size and composition calculations** for a variety of 1-D geometries using 1-group theory.

Explain how the **addition of a reflector affects the overall multiplication factor**.

Sketch the **expected shape of the 1-group flux** within a variety of 1-D configurations.

Use the core_refl1g_gui program to **perform a variety of analyses** for various 1-D geometries.

Typical Applications

The **steady state multigroup diffusion equation is quite general** and, for any specific application, only the applicable terms are used.

In most cases of interest, one of the following **three situations** arise:

1. **Subcritical non-multiplying system** (no fission source):

$$L\phi = Q$$

2. **Subcritical multiplying system** (fission & external sources):

$$(L - F)\phi = Q$$

3. **Critical system** (no external sources):

$$(L - \lambda F)\phi = 0$$

Typical Applications (cont.)

Subcritical non-multiplying system:

This first case is applicable primarily in shield design applications and for non-multiplying fusion blanket design.

This situation represents a geometry with no fission source (and this was the situation treated in several previous lessons).

Using 1-group theory for a homogeneous medium, the general equation reduces to

$$-D\nabla^2\phi(\vec{r}) + \Sigma_a\phi(\vec{r}) = Q(\vec{r})$$

$$\nabla^2\phi(\vec{r}) - \frac{1}{L^2}\phi(\vec{r}) = -\frac{Q(\vec{r})}{D} \quad \text{with} \quad L^2 = \frac{D}{\Sigma_a}$$

Typical Applications (cont.)

Subcritical multiplying system:

This second case must be considered in situations where both the fixed source and fission source are important.

The most common situation where this case arises is during reactor startup and shutdown periods.

Using 1-group theory for a homogeneous medium, the general balance equation reduces to

$$-D\nabla^2\phi(\vec{r}) + \Sigma_a\phi(\vec{r}) - \nu\Sigma_f\phi(\vec{r}) = Q(\vec{r})$$

$$\nabla^2\phi(\vec{r}) - \frac{\Sigma_a - \nu\Sigma_f}{D}\phi(\vec{r}) = -\frac{Q(\vec{r})}{D}$$

Typical Applications (cont.)

Now, recall that we previously defined the neutron multiplication factor as

$$k = \frac{\text{neutron production rate}}{\text{neutron loss rate}} = \frac{<\nu\Sigma_f\phi>}{<\Sigma_a\phi> + <-D\nabla^2\phi>}$$

and
$$k_\infty = \frac{<\nu\Sigma_f\phi>}{<\Sigma_a\phi>} = \frac{\nu\Sigma_f}{\Sigma_a}$$

Thus, for a subcritical steady-state system with $k_\infty < 1$, we have

$$\nabla^2\phi(\vec{r}) - \kappa^2\phi(\vec{r}) = -\frac{Q(\vec{r})}{D} \quad \text{with} \quad \kappa^2 = \frac{\Sigma_a - \nu\Sigma_f}{D} > 0$$

solution scheme is identical to the non-multiplying media problem

Typical Applications (cont.)

Critical system:

This last problem classification describes the steady state operation of a nuclear reactor.

In this situation, the total leakage and absorption rates exactly balance the neutron production from fission, and any inherent neutron source that may be present in the fuel is totally dominated by the fission source.

Since the fixed source is negligible, it is simply dropped from the defining equations.

For criticality, there is a very precise balance between the neutron production rate from fission and the total loss rate -- that is, any arbitrary mixture of materials will not satisfy this constraint.

Typical Applications (cont.)

To emphasize this precise relationship, one usually includes a mathematical eigenvalue, λ, before the fission source term.

But, always remember, in an operating critical reactor, λ is unity!!!

In design analysis, however, we often want to know if a particular combination of materials will give a critical reactor.

Thus, for any given material distribution, λ is computed as part of the solution procedure.

It is allowed to vary from unity so that , for steady state operation, the equation can be balanced mathematically (that is, loss = λ * production).

Typical Applications (cont.)

Using 1-group theory for a homogeneous medium, the general balance equation for a critical system (with $k_\infty > 1$) reduces to

$$-D\nabla^2\phi(\vec{r}) + \Sigma_a\phi(\vec{r}) - \lambda\nu\Sigma_f\phi(\vec{r}) = 0$$

$$\nabla^2\phi(\vec{r}) + \frac{\lambda\nu\Sigma_f - \Sigma_a}{D}\phi(\vec{r}) = 0$$

for an operating critical system, λ = 1.000

$$\nabla^2\phi(\vec{r}) + B^2\phi(\vec{r}) = 0 \quad \text{where} \quad B^2 = \frac{\lambda\nu\Sigma_f - \Sigma_a}{D} > 0$$

where we note that this equation is homogeneous and that the sign of the 2nd term on the LHS is positive (instead of negative as in the other two application classes) -- and these two differences lead to some rather interesting physical and mathematical behavior…

Typical Applications (cont.)

Also, since the 2nd derivative of a function is related to its curvature (i.e. the slope of the slope), then B^2 is referred to as the buckling -- which is a measure of the curvature of the flux profile.

That is,

$$\nabla^2\phi = -B^2\phi$$

Finally, B^2 in the above expression is called the material buckling since it is only a function of the material properties of the core (for λ = 1.000), where

$$B^2 = \frac{\lambda\nu\Sigma_f - \Sigma_a}{D}$$

Significance of the Eigenvalue, λ

To see the significance of λ, let's integrate the 1-group balance equation over all space, giving

$$< -D\nabla^2\phi > + < \Sigma_a\phi > - \lambda < \nu\Sigma_f\phi > = 0$$

or

$$\lambda = \frac{< -D\nabla^2\phi > + < \Sigma_a\phi >}{< \nu\Sigma_f\phi >} = \frac{\text{loss rate}}{\text{production rate}}$$

for a 1-region homogeneous system

$$k = \frac{< \nu\Sigma_f\phi >}{< \Sigma_a\phi > + < -D\nabla^2\phi >} = \frac{< \nu\Sigma_f\phi >}{< \Sigma_a\phi > + < DB^2\phi >} = \frac{\nu\Sigma_f}{\Sigma_a + DB^2}$$

From the definition of the multiplication factor, k,

$$k_{eff} = \frac{\text{production rate}}{\text{loss rate}} \quad \text{or} \quad \lambda = \frac{1}{k_{eff}}$$

Thus, the addition of the eigenvalue within the defining equation is quite justifiable, since at SS operating conditions, $k_{eff} = 1/\lambda$ = 1.000.

Significance of the Eigenvalue, λ

However, for any given material configuration, the calculated k_{eff} may not be unity, but this just tells the designer that a material and/or geometry modification is required for criticality -- and this can be a great design tool !!!

Nevertheless, a SS system with $k_{eff} \neq 1.000$, is not self-consistent, and can never be achieved in practice -- that is, for Q = 0 and $k < 1$, the steady state flux is zero and, if $k > 1$, the flux magnitude continually increases without bound…

1-Group Expressions for S_{fis} & k_{eff}

Depending on what data may be available for a given problem, it may be convenient to express the 1-group fission source and k_{eff} equation in various forms.

To illustrate some of the alternatives, we first recall the difference between the definitions of η and ν, where

$$\eta = \frac{\text{average number of neutrons emitted}}{\text{absorption in the fuel}}$$

and

$$\nu = \frac{\text{average number of neutrons emitted}}{\text{fission}}$$

1-Group Expressions for S_{fis} & k_{eff}

With these definitions, one can express the **1-group fission source, S_{fis}**, as

$$S_{fis} = \left(\frac{\text{neutrons}}{\text{fission}}\right)\left(\frac{\text{fissions}}{\text{cm}^3 - \text{sec}}\right) = \nu\Sigma_f\phi$$

$$S_{fis} = \left(\frac{\text{neutrons emitted}}{\text{absorption in fuel}}\right)\left(\frac{\text{absorptions in fuel}}{\text{cm}^3\text{-sec}}\right) = \eta\Sigma_a^F\phi$$

Now, we define a new term called the **fuel utilization**, as

$$f = \text{fuel utilization} = \frac{\text{neutrons absorbed in fuel}}{\text{neutrons absorbed in complete system}}$$

important term

1-Group Expressions for S_{fis} & k_{eff}

And, for a **homogeneous 1-region reactor**, **f can be written as**

$$f = \frac{\int \Sigma_a^F(\vec{r})\phi(\vec{r})d\vec{r}}{\int \Sigma_a(\vec{r})\phi(\vec{r})d\vec{r}} = \frac{<\Sigma_a^F\phi>}{<\Sigma_a\phi>} \Rightarrow \frac{\Sigma_a^F}{\Sigma_a}$$

Now, using these expressions, one also has the following **equivalent forms for S_{fis}**,

$$S_{fis} = \nu\Sigma_f\phi = \eta\Sigma_a^F\phi = \eta f\,\Sigma_a\phi = k_\infty\Sigma_a\phi$$

where we have used the 1-group k_∞ formulation, $k_\infty = \nu\Sigma_f/\Sigma_a$

Finally, with these different representations for the fission source, we can also write the multiplication factor in various ways, as follows,

$$k = \frac{\nu\Sigma_f}{\Sigma_a + DB^2} = \frac{\nu\Sigma_f/\Sigma_a}{1+L^2B^2} = \frac{k_\infty}{1+L^2B^2} = \frac{\eta f}{1+L^2B^2}$$

1-Group 1-D Critical Reactor Cases

With the above background, we are now ready to formally analyze a number of 1-group 1-D critical reactor configurations.

In particular, the 1-group 1-D bare critical reactor model can be solved analytically in slab, spherical, and cylindrical geometries -- and these solutions can give significant insight into the behavior of general critical systems.

In addition, a 1-group 2-region core-reflector model can also be solved analytically for these same 1-D geometries -- and this problem is clearly more realistic of a real system (i.e. there are no bare reactors in existence -- except for some very early experimental facilities such as the Godiva and Jezebel bare cores).

We will address both the single-region bare core and the 2-region core-reflector cases…

1-Grp 1-D Critical Reactor Cases (cont.)

The bare and reflected 1-group 1-D models allow an introduction to the solution methodology as well as several new terms and concepts needed in analyzing general critical systems -- and they allow some preliminary core size and composition design calculations.

The formal analytical solutions for all six configurations (bare and reflected cores in slab, spherical, and cylindrical geometries) are treated in detail in the formal Lecture Notes.

Here, we will discuss only the bare and reflected critical slab reactor to highlight the key terminology and procedures associated with problems of this type.

The student can then review the developments for the other geometries as independent study and via related HW problems.

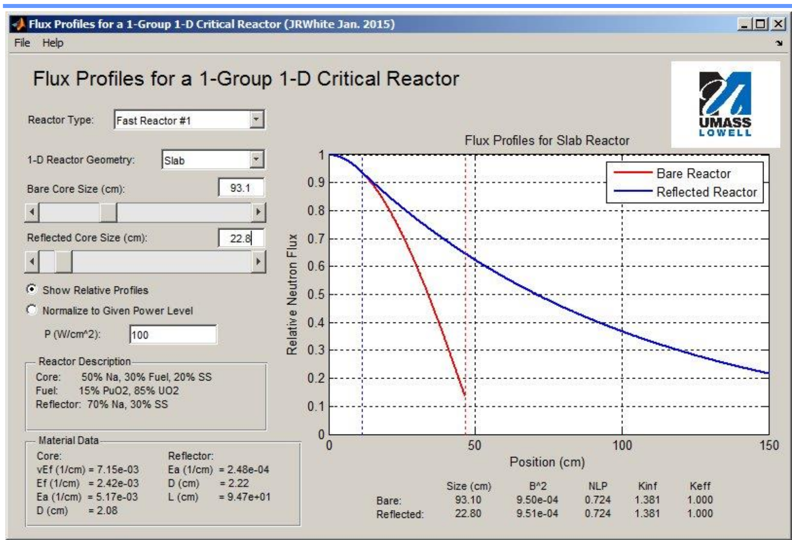
1-Grp 1-D Critical Reactor Cases (cont.)

In addition, a Matlab GUI, core_refl1g_gui, was developed to implement the solutions for all six 1-group 1-D configurations.

This tool should help in the visualization of the resultant flux profiles and in performing various comparisons and analytical studies (bare vs. reflected systems, variation of k_{eff} with core size, how the flux level changes with power, core size, and core configuration, etc.).

The combination of the theory and solution capability within the Matlab GUI should give you a good understanding of 1-group 1-D critical systems -- and we will build upon this background in subsequent lessons to expand your overall knowledge of general steady state critical systems…

The core_refl1g_gui Interface



1-Group Bare Critical Slab Reactor

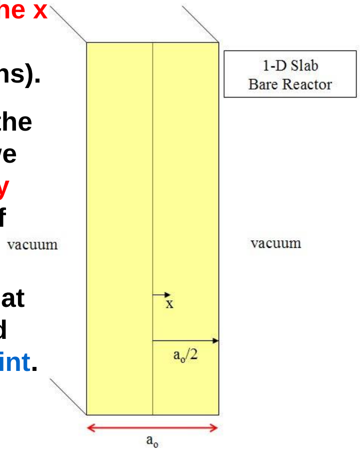
Consider the bare slab reactor shown.

The reactor has finite thickness a_o in the x direction, but it is infinite in the transverse directions (y and z directions).

The "bare" adjective here means that the system has vacuum boundaries and we will apply the typical vacuum boundary condition at the external boundaries of the system.

Also, the coordinate system is such that x = 0 is in the center of the reactor, and the system is symmetric about this point.

Because of symmetry, we will only consider the region $0 \le x \le a_o/2$.



1-Grp Bare Critical Slab Reactor (cont.)

For a 1-group 1-region homogeneous critical slab reactor, the general multigroup diffusion equation becomes

$$\frac{d^2}{dx^2}\phi(x) + B^2\phi(x) = 0 \quad \text{with} \quad B^2 = \frac{\lambda\nu\Sigma_f - \Sigma_a}{D}$$

where it is assumed that the buckling, B^2, is positive since the production term, $\lambda\nu\Sigma_f\phi$, must be greater than the absorption term, $\Sigma_a\phi$, in a finite critical system ($k_\infty > 1$ for a finite system).

The general solution for this 2nd order linear constant-coefficient homogeneous ODE can be written in the form of a simple exponential,

$$\phi(x) = e^{\alpha x}$$

1-Grp Bare Critical Slab Reactor (cont.)

Upon substitution into the defining ODE, we obtain the characteristic equation

$$\alpha^2 + B^2 = 0 \quad \text{with solution} \quad \alpha_{1,2} = \pm jB$$

These complex conjugate roots lead to a general solution written in terms of simple sinusoids,

$$\phi(x) = A_1 \cos Bx + A_2 \sin Bx$$

General Solution to the Bare Critical Slab Problem

The proper boundary conditions here are symmetry at x = 0 (the center of the slab) and the fact that the flux goes to zero at the extrapolated boundary [at x = $(a_o+2d)/2$, where d is the extrapolation distance].

For convenience, define "a" as the extrapolated thickness of the core, where $a = (a_o+2d)$.

1-Grp Bare Critical Slab Reactor (cont.)

Now, applying the symmetry condition gives

$$\left.\frac{d\phi}{dx}\right|_{x=0} = \left[-A_1 B \sin Bx + A_2 B \cos Bx\right]_{x=0} = A_2 B = 0 \quad \text{or} \quad A_2 = 0$$

With this information, the general solution reduces to

$$\phi(x) = A_1 \cos Bx$$

These could have been written directly since cos(x) is a symmetric even function, and sin(x) is an anti-symmetric odd function.

At the extrapolated boundary (x = a/2), we have

$$\phi(x)\Big|_{x=a/2} = A_1 \cos\frac{Ba}{2} = 0$$

For a nontrivial solution, A_1 must be nonzero.

Thus, we have the condition that $\cos(Ba/2) = 0$.

1-Grp Bare Critical Slab Reactor (cont.)

UMASS LOWELL
Learning with Purpose

This last condition -- that is, $\cos(Ba/2) = 0$ -- is somewhat of a peculiar situation (and it is certainly different from the fixed-source problems that were solved previously)!!!

In this case, there are multiple solutions -- in fact, an infinite number of possibilities exist since the cosine function is zero when evaluated at any odd integer multiple of $\pi/2$.

We can write this statement mathematically as

$$\cos\left[\frac{(2n-1)\pi}{2}\right] = 0 \quad \text{for } n = 1, 2, \ldots$$

Now, comparing this expression to the BC given above gives

$$\frac{B_n a}{2} = \frac{(2n-1)\pi}{2} \quad \text{or} \quad B_n = \frac{(2n-1)\pi}{a} \quad \text{for } n = 1, 2, \ldots$$

where the n subscript is included to indicate that there are an infinite number of values of buckling (B_1, B_2, …).

1-Grp Bare Critical Slab Reactor (cont.)

UMASS LOWELL
Learning with Purpose

Now, with the different B_n values, we get an infinite number of profiles that satisfy the original ODE and its BCs, or

$$\phi_n(x) = \cos B_n x$$

The essential result here is that the diffusion equation for the critical reactor problem gives rise to an eigenvalue problem.

The B_n's are the eigenvalues and the ϕ_n's are the eigenfunctions.

- - - - - - - - - - - - - - - - - -

Eigenvalue problems have the characteristic form $Ay = \lambda By$, where A and B are operators (or matrices), y is a function (vector), and the eigenvalue λ is a constant.

For discrete systems, where A, B, and y are finite of order N, there are a total of N eigenvalues and N eigenvectors.

For the case of a continuous system, there are an infinite number of eigenvalues and eigenfunctions.

1-Grp Bare Critical Slab Reactor (cont.)

The higher eigenmodes of the diffusion equation are of interest in many areas of reactor theory, especially in space-time kinetics work and other more advanced topics.

For now, however, we will only work with the fundamental mode eigenfunction and eigenvalue, since all the higher modes decay away leaving only the fundamental mode as the final steady state solution to the critical reactor problem (this is often demonstrated in detail in graduate Reactor Physics courses).

In this case, n = 1 and the 1-group fundamental mode critical flux distribution in a 1-D bare slab reactor is

$$\phi(x) = A_1 \cos Bx \quad \text{where} \quad B = \frac{\pi}{a} \quad \text{or} \quad B^2 = \left(\frac{\pi}{a}\right)^2$$

where the 1 subscript has been omitted for convenience from the definition of $B = B_1$, since, at this point, we are only interested in the fundamental mode solution.

1-Grp Bare Critical Slab Reactor (cont.)

Notice that there is still an arbitrary constant, A_1, that remains as part of the general solution.

This is characteristic of eigenvalue problems (i.e. the solution to any homogeneous equation is only known to within an arbitrary normalization).

The appropriate condition here is to normalize the flux to the reactor power, P, where, for the semi-infinite 1-D slab problem, P is the power per unit area in the yz plane:

$$P = \kappa \int_{-a_o/2}^{a_o/2} \Sigma_f \phi(x) dx = \kappa \Sigma_f A_1 \int_{-a_o/2}^{a_o/2} \cos \frac{\pi x}{a} dx$$

$$= \kappa \Sigma_f A_1 \left[\frac{a}{\pi} \sin \frac{\pi x}{a} \right]_{-a_o/2}^{a_o/2} = \kappa \Sigma_f A_1 \frac{a}{\pi} \left[\sin \frac{\pi a_o}{2a} - \sin \frac{-\pi a_o}{2a} \right]$$

$$= A_1 \frac{2\kappa \Sigma_f a}{\pi} \sin \frac{\pi a_o}{2a}$$

κ is the recoverable energy per fission

1-Grp Bare Critical Slab Reactor (cont.)

Solving the last expression for A_1 gives

$$A_1 = \frac{P\pi}{2\kappa\Sigma_f a \sin\frac{\pi a_o}{2a}}$$

Thus, the normalized flux in a 1-D slab reactor can be written as

$$\phi(x) = \frac{P\pi}{2\kappa\Sigma_f a \sin\frac{\pi a_o}{2a}} \cos\frac{\pi x}{a}$$

This is the desired normalized flux profile.

When the extrapolation distance d is small compared to the reactor size, then a_o/a approaches unity, and $\sin(\pi a_o/2a) \rightarrow 1$. For this case, the above equations reduce to

$$A_1 = \frac{P\pi}{2\kappa\Sigma_f a_o} \quad \text{and} \quad \phi(x) = \frac{P\pi}{2\kappa\Sigma_f a_o} \cos\frac{\pi x}{a_o} \quad (\text{for } d \ll a_o)$$

and this is the result for the infinite slab reactor that is usually tabulated in the standard reactor physics texts.

1-Grp Bare Critical Slab Reactor (cont.)

We have nearly completed our discussion of the 1-D bare slab reactor problem except for the fact that two bucklings have been defined: a material buckling $B^2{}_m$ and a geometric buckling $B^2{}_g$.

Recall that $B^2{}_m$ is a simple function of the material properties as given from the original balance equation

$$B^2{}_m = \frac{\lambda\nu\Sigma_f - \Sigma_a}{D}$$

and that $B^2{}_g$ is the result of forcing the flux distribution to satisfy the appropriate boundary conditions (which are a function of the geometry) which, for the fundamental mode solution for the current geometry, gives

$$B^2{}_g = \left(\frac{\pi}{a}\right)^2$$

1-Grp Bare Critical Slab Reactor (cont.)

Clearly, $B^2_g = B^2_m$ must be true for a consistent description of a critical reactor, and this is consistent with our previous comment that "a precise relationship is required between the geometry and material makeup for a just critical system".

This general relationship is known as the critical condition for the system of interest.

Equating the two buckling expressions $B^2 = B^2_g = B^2_m$ and solving for λ or k_{eff} gives

$$\lambda = \frac{DB^2 + \Sigma_a}{\nu\Sigma_f} \qquad \text{or} \qquad k_{eff} = \frac{\nu\Sigma_f}{DB^2 + \Sigma_a}$$

where, in these expressions, B^2 is the geometric buckling.

This equation is the real critical condition for all 1-group bare homogeneous systems.

1-Grp Bare Critical Slab Reactor (cont.)

It says that the multiplication factor is simply the ratio of the production rate to the loss rate where B^2 is a function of the geometry and the cross sections are a function of the material composition.

When these parameters have just the right combination, then the production and loss terms are equal and $k_{eff} = 1.000$ (a critical system).

However, for any combination of material composition and geometry, this expression allows us to compute a value of k_{eff} to determine the criticality level of the given configuration -- and this gives the designer lots of information about the particular system under study.

However, remember that a real operating critical reactor has $k_{eff} = 1.000$!!!

1-Grp Bare Critical Slab Reactor (cont.)

As a final note on this example, recall that the expression for k_{eff} can be written in a variety of ways.

In particular, it is often convenient to define a non-leakage probability, P_{NL}, and to write k_{eff} in terms of this quantity.

Since there are only two loss components, the non-leakage probability for the 1-group model is simply the ratio of the absorption rate to the total loss rate, or

$$P_{NL} = \frac{\langle \Sigma_a \phi \rangle}{\langle DB^2 \phi \rangle + \langle \Sigma_a \phi \rangle} = \frac{\Sigma_a}{DB^2 + \Sigma_a} = \frac{1}{DB^2 / \Sigma_a + 1} = \frac{1}{1 + L^2 B^2}$$

Now, from the definition of k_{eff} for a 1-group 1-region homogeneous model, we have

$$k_{eff} = \frac{\nu \Sigma_f}{\Sigma_a + DB^2} = \frac{\nu \Sigma_f / \Sigma_a}{1 + DB^2 / \Sigma_a} = \frac{\nu \Sigma_f / \Sigma_a}{1 + L^2 B^2} = k_\infty P_{NL}$$

1-Grp Bare Critical Slab Reactor (cont.)

Thus, $k_{eff} = k_\infty P_{NL}$ is a common way to write the core multiplication factor for 1-group 1-region homogeneous systems.

Also recall that $k_\infty = \eta f$ (i.e. a 2-factor formula) -- so the effective multiplication factor for a finite core can be written a

$$k_{eff} = \eta f P_{NL}$$

and this is sometimes referred to as the 3-factor formula.

We will see similar terminology again
-- the 4-factor and 6-factor formulas --
when using 2-group theory later in the course…

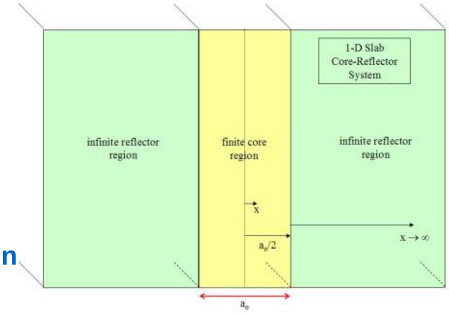
ENGY.4340 Nuclear Reactor Theory
Lesson 6: The Critical Reactor I
(Oct. 2016)

1-Grp 2-Region Critical Slab Reactor

We study bare reactor problems because the mathematics involved is relatively straightforward and they give considerable insight into the general critical reactor problem.

However, a bare reactor is not a practical option, and all operating reactor systems have essentially infinite reflectors around the core region (to improve upon the neutron economy and to minimize neutron and gamma radiation outside the core).

The simplest two-region reflected system is the critical core-reflector configuration in 1-D Cartesian (slab) geometry using the 1-group approximation (as sketched in the diagram).



1-Grp 2-Region Critical Slab (cont.)

In the core region of the model $k_\infty > 1$ and, in the reflector region, there is no fissionable material.

Thus, this simple two-region system combines the critical reactor problem and non-multiplying medium problem into a single system.

The composite two-region system is symmetric about $x = 0$ and, as x becomes large, the flux must remain finite.

At the core-reflector interface (i.e. at $x = a_o/2$), the standard continuity of flux and current conditions will apply.

These last two statements define the problem boundary conditions (BCs).

1-Grp 2-Region Critical Slab (cont.)

The basic solution procedure is as follows (but there will be a few twists…):

1. Write the 1-group 1-D homogenous form of the diffusion equation for each region of the model.

2. Solve these equations to get a general solution for the flux profile in each zone (there will be four arbitrary coefficients for a 2-region model).

3. Apply the four independent BCs indicated above to help specify these four coefficients (the "twists" noted above will occur in this step).

1-Grp 2-Region Critical Slab (cont.)

In the core region, the defining balance equation is identical to the previous critical bare core example,

$$\frac{d^2}{dx^2}\phi_c + B^2\phi_c = 0 \quad \text{with} \quad B^2 = \left.\frac{\lambda\nu\Sigma_f - \Sigma_a}{D}\right|_c \qquad \text{for } 0 \le x \le \frac{a_o}{2}$$

where we use a 'c' subscript to denote that the properties and flux profile are only valid for the core region.

The general solution is also identical as before and, taking into account the symmetry condition at x = 0, the general solution for the core flux profile becomes

$$\phi_c(x) = A_1 \cos Bx$$

1-Grp 2-Region Critical Slab (cont.)

In the reflector region, the balance equation for a non-multiplying medium region applies, or

$$\frac{d^2}{dx^2}\phi_r - \frac{1}{L_r^{\,2}}\phi_r = 0 \quad \text{with} \quad L_r^{\,2} = \left.\frac{D}{\Sigma_a}\right|_r \qquad \text{for } x \geq \frac{a_o}{2}$$

where we have set Q = 0 since there is no external source present in this region, and we have used the subscript 'r' to denote that the flux and material properties are associated with the reflector region.

Since the reflector has infinite thickness, we will write the general solution as

$$\phi_r(x) = A_3 e^{-x/L_r} + A_4 e^{x/L_r}$$

and immediately set A_4 to zero to force the flux solution to be finite as $x \to \infty$.

1-Grp 2-Region Critical Slab (cont.)

Doing this gives

$$\phi_r(x) = A_3 e^{-x/L_r}$$

At this point, we have used two BCs to reduce the general solutions for the core and reflector regions to only contain two arbitrary constants (A_1 & A_3).

To help find these, we apply the continuity of flux and current conditions at the core-reflector interface,

$$\left.\phi_c(x)\right|_{x=a_o/2} = \left.\phi_r(x)\right|_{x=a_o/2} \quad \text{or} \quad A_1 \cos\frac{Ba_o}{2} - A_3 e^{-a_o/(2L_r)} = 0$$

$$\left.J_c(x)\right|_{x=a_o/2} = \left.J_r(x)\right|_{x=a_o/2} \quad \text{or} \quad -D_c\left(-A_1 B \sin\frac{Ba_o}{2}\right) + D_r\left(-\frac{A_3}{L_r} e^{-a_o/(2L_r)}\right) = 0$$

1-Grp 2-Region Critical Slab (cont.)

Writing these two homogeneous equations in matrix form gives

$$\begin{bmatrix} \cos\frac{Ba_o}{2} & -e^{-a_o/(2L_r)} \\ D_c B \sin\frac{Ba_o}{2} & -\frac{D_r}{L_r} e^{-a_o/(2L_r)} \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is a homogeneous matrix equation which, for a non-trivial solution, requires a singular coefficient matrix -- which means that the determinant of the coefficient matrix must be zero.

In addition, we also know that the solution to a homogeneous equation is only known to within an arbitrary constant.

This means that, even with a singular coefficient matrix, we can't solve this matrix equation explicitly for both A_1 and A_3 -- the best we can do is to write A_3 in terms of A_1, and let A_1 be the arbitrary normalization factor.

1-Grp 2-Region Critical Slab (cont.)

These two issues are not really unexpected, since they refer to exactly the same situations we saw for the bare critical reactor.

In particular, concerning the "critical determinant", this comes about because of the precise balance between the material composition and geometry that is required for a critical system -- thus, we expected that a "criticality condition" of some form would be needed.

For the bare reactor problem, we forced the material and geometric bucklings to be identical, which allowed us to compute the multiplication factor in terms of the material composition and the geometry.

Now, for the current problem, we do essentially the same thing -- that is, set $B^2 = B^2_m = B^2_g$ -- but now the geometric buckling is determined from the statement that "the determinant of the coefficient matrix must be zero".

1-Grp 2-Region Critical Slab (cont.)

Recalling that B^2_m contains the eigenvalue $\lambda = 1/k_{eff}$, we see that this condition gives the expected relationship for k_{eff} in terms of the material properties of both the core and reflector and the core-reflector geometry.

This relationship is indeed the desired criticality condition!!!

To be explicit, the determinant of a 2×2 matrix is simply the product of the main diagonal elements minus the product of the other diagonal terms, or

$$\cos\frac{Ba_o}{2}\left(-\frac{D_r}{L_r}e^{-a_o/(2L_r)}\right) - D_c B\sin\frac{Ba_o}{2}\left(-e^{-a_o/(2L_r)}\right) = 0$$

$$-\frac{D_r}{L_r}\cos\frac{Ba_o}{2} + D_c B\sin\frac{Ba_o}{2} = 0$$

1-Grp 2-Region Critical Slab (cont.)

and $$f(B) = \cot\frac{Ba_o}{2} - \frac{L_r D_c B}{D_r} = 0$$

This is the criticality condition for this problem.

From this last form, it is easy to see that the criticality condition can be cast as a classical root finding problem (i.e. given the core size and the material properties, what is the value of B such that f(B) = 0?).

Once B (or B^2) is has been determined, we can use the definition of B^2 in the core balance equation to get the value of k_{eff}, or

$$B^2 = \left.\frac{\lambda\nu\Sigma_f - \Sigma_a}{D}\right|_c \quad \text{or} \quad k_{eff} = \frac{1}{\lambda} = \frac{\nu\Sigma_{fc}}{D_c B^2 + \Sigma_{ac}}$$

This is the same result as before with a different B^2 value.

1-Grp 2-Region Critical Slab (cont.)

To better visualize the criticality condition, let's re-write this expression as

$$\cot\frac{Ba_o}{2} = \frac{L_r D_c B}{D_r}$$

and let $p = Ba_o/2$. Now, upon substitution, we have

$$\cot p = \frac{L_r D_c B}{D_r}\frac{a_o/2}{a_o/2} = \frac{2 L_r D_c}{D_r a_o} p$$

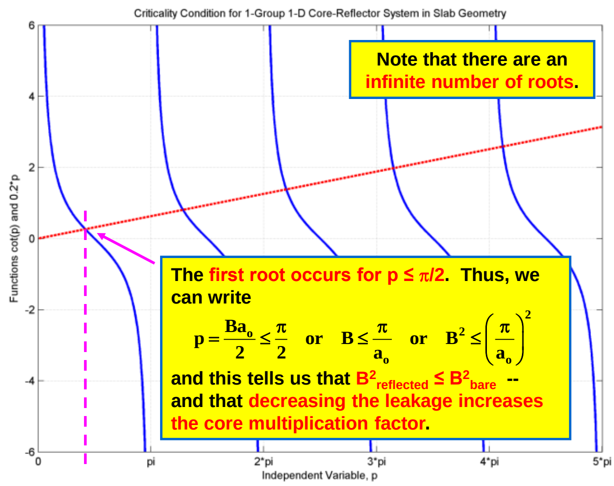
The graphical solution of this nonlinear equation is shown on the next slide.

The left hand side (LHS) of this relationship is just the familiar cotangent function and the RHS is a simple linear function of p (with a positive slope and zero intercept).

The points where these two functions intersect represent the roots of this nonlinear equation -- that is, the values of p that satisfy the criticality condition.

With p known, one can compute B, and then k_{eff}…

1-Grp 2-Region Critical Slab (cont.)



1-Grp 2-Region Critical Slab (cont.)

With B known for the fundamental mode, the determinant of the coefficient matrix is indeed zero, and we can proceed to actually solve this matrix equation for the unknown values of A_1 and A_3,

$$\begin{bmatrix} \cos\frac{Ba_o}{2} & -e^{-a_o/(2L_r)} \\ D_c B \sin\frac{Ba_o}{2} & -\frac{D_r}{L_r} e^{-a_o/(2L_r)} \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Expanding the first equation and solving for A_3 gives

$$A_3 = A_1 \cos\frac{Ba_o}{2} e^{a_o/(2L_r)}$$

Thus, the solutions for the core and reflector fluxes become

These are the desired flux profiles.

$$\phi_c(x) = A_1 \cos Bx \quad \text{for } 0 \le x \le a_o/2$$

$$\phi_r(x) = A_1 \cos\frac{Ba_o}{2} e^{-(x-a_o/2)/L_r} \quad \text{for } x \ge a_o/2$$

1-Grp 2-Region Critical Slab (cont.)

Now, the only unresolved quantity is the normalization factor A_1 which can be determined by the power constraint on the system.

Note that the power constraint involves integration over all space -- but, in the non-fuel regions, there is no power production.

Thus, for the current problem, integrating only over the core region, gives

$$P = \kappa \int_{-a_o/2}^{a_o/2} \Sigma_f \phi_c(x)dx = \kappa \Sigma_f A_1 \int_{-a_o/2}^{a_o/2} \cos Bx\, dx = \kappa \Sigma_f A_1 \left[\frac{1}{B}\sin Bx\right]_{-a_o/2}^{a_o/2}$$

$$= \frac{\kappa \Sigma_f A_1}{B}\left[\sin\frac{Ba_o}{2} - \sin\frac{-Ba_o}{2}\right] = A_1 \frac{2\kappa\Sigma_f}{B} \sin\frac{Ba_o}{2}$$

and

This is the desired flux normalization.

$$A_1 = \frac{PB}{2\kappa\Sigma_f \sin\frac{Ba_o}{2}}$$

Note that, since we do not have an analytical expression for B, we simply carry along the variable B as part of the development.

Summary Comments

We have finally finished the theoretical development of the 1-group critical reactor problem in 1-D slab geometry.

Clearly, solution of the reflected core model was algebraically more difficult than the bare reactor model. However, the basic procedures for the two cases were really quite similar, with only subtle differences associated with the criticality condition.

Concerning implementation, some added work is required to find B^2 and k_{eff} for the core-reflector case (i.e. solving the nonlinear criticality condition) but, again, the overall procedures are very similar -- and this is also true for the 1-group 1-D spherical & cylindrical core geometries.

Both bare-core and reflected-core models for all three 1-D geometries have been implemented into the core_refl1g_gui code, and you should use this software to compare results for a variety of situations…

Lesson 6 Summary

In this Lesson we have briefly discussed the following subjects:

The fundamental differences associated with application of the diffusion equation to non-multiplying media problems, subcritical systems, and the critical reactor problem.

The use of the mathematical eigenvalue, λ, within the critical reactor problem.

Various ways to write the 1-group fission source and the expression for the 1-group multiplication factor, k.

The setup and solution of the 1-group 1-D bare critical reactor problem for slab, spherical, and cylindrical geometries.

The meaning of the criticality condition and the concept of material and geometric bucklings.

Lesson 6 Summary (cont.)

The setup and solution of the 1-group 1-D 2-region core-reflector problem for slab, spherical, and cylindrical geometries.

The meaning of the phrase "critical determinant" as applied to the core-reflector critical reactor problem.

How to apply a power constraint to determine the flux normalization in various geometries.

Critical size and composition calculations for a variety of 1-D geometries using 1-group theory.

How the addition of a reflector affects the overall multiplication factor.

The expected shape of the 1-group flux within a variety of 1-D configurations.

The use of the core_refl1g_gui program to perform a variety of analyses for various 1-D geometries.