Nuclear Reactor Theory

Lesson 5: Neutron Diffusion in Moderating Media II

The 2-Group Diffusion Theory Approximation

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Lesson 5 Objectives

Setup and solve the 2-group diffusion equation for a variety of moderating media configurations.

Explain the sequential nature of the solution process for problems with no fission and no upscatter.

Explain the behavior of the fast-to-thermal flux ratio in water versus graphite regions.

Discuss the concern about using diffusion theory for large regions of purely moderating media (with no distributed isotropic sources).

2-Group Theory for Non-Multiplying Media

The 2-group diffusion equation for a homogeneous non-multiplying medium with no upscatter can be written as

$$-D_1\nabla^2\phi_1 + (\Sigma_{a1} + \Sigma_{1\to2})\phi_1 = Q_1$$

$$-D_2\nabla^2\phi_2 + \Sigma_{a2}\phi_2 - \Sigma_{1\to2}\phi_1 = Q_2$$

Since the coefficients are constant, we can divide each equation by the diffusion coefficient, giving

$$\nabla^2\phi_1 - \frac{1}{L_1^2}\phi_1 = -\frac{Q_1}{D_1}$$

$$\nabla^2\phi_2 - \frac{1}{L_2^2}\phi_2 + \frac{\Sigma_{1\to2}}{D_2}\phi_1 = -\frac{Q_2}{D_2}$$

where L_1^2 and L_2^2 are the fast and thermal diffusion areas.

2-Group Theory for Non-Multiplying Media

Now, if it is valid to assume that $\Sigma_{1\to2} >> \Sigma_{a1}$ (which allows us to replace the fast diffusion area with the neutron age), then the fast group equation becomes

$$\nabla^2\phi_1 - \frac{1}{\tau_T}\phi_1 = -\frac{Q_1}{D_1}$$

Thus, our starting point for any 2-group moderating media problem (i.e. no fission) will be

$$\nabla^2\phi_1 - \frac{1}{\tau_T}\phi_1 = -\frac{Q_1}{D_1}$$

$$\nabla^2\phi_2 - \frac{1}{L_2^2}\phi_2 + \frac{\Sigma_{1\to2}}{D_2}\phi_1 = -\frac{Q_2}{D_2}$$

Remember that the $\nabla^2\phi$ term is geometry dependent.

Point Source of Fast Neutrons in an Infinite Medium

To illustrate the application of the 2-group diffusion equation in a diffusing medium, let's assume that we have an isotropic point source of fast neutrons (Q_1 n's/sec) in an infinite homogeneous non-multiplying medium, or

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi_1}{dr}\right)-\frac{1}{\tau_T}\phi_1=0 \qquad r>0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi_T}{dr}\right)-\frac{1}{L_T^2}\phi_T=-\frac{\Sigma_{1\to 2}}{D_2}\phi_1$$

Assumes 1-D spherical geometry with the point isotropic fast neutron source at r = 0.

Here we have decided to use a subscript T to denote the cross sections and flux within the thermal group (except for D_2).

Also, since the fast neutron source is only non-zero at r = 0, we will treat this as a source condition and exclude the point r = 0 when solving the fast group equation.

Point Source of Fast Neutrons (cont.)

One final observation is that these equations can be decoupled. This means that the fast group equation is independent of ϕ_T, and it can be solved without consideration of the thermal equation.

Once ϕ_1 is known, ϕ_T can be obtained from the solution of the thermal balance equation.

In performing this procedure, note that the ϕ_1 equation is identical in form to the 1-group point source problem that was solved previously.

Therefore, by analogy, we can immediately write the solution for the fast flux as

$$\phi_1(r)=\frac{Q_1}{4\pi D_1}\frac{1}{r}e^{-r/\sqrt{\tau_T}}$$

fast flux

recall for 1-g case

$$\phi(r)=\frac{Q}{4\pi D}\frac{1}{r}e^{-r/L}$$

Point Source of Fast Neutrons (cont.)

With $\phi_1(r)$ known, the thermal group equation becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\phi_T\right)-\frac{1}{L_T^2}\phi_T=-\frac{\Sigma_{1\to 2}}{D_2}\frac{Q_1}{4\pi D_1}\frac{1}{r}e^{-r/\sqrt{\tau_T}}=-\frac{Q_1}{4\pi\tau_T D_2}\frac{1}{r}e^{-r/\sqrt{\tau_T}}$$

This expression is a linear non-homogeneous 2nd order variable-coefficient differential equation but, as shown previously, the substitution $\phi_T = \omega/r$ can simplify this expression considerably, giving

$$\frac{d^2}{dr^2}\omega-\frac{1}{L_T^2}\omega=-\frac{Q_1}{4\pi\tau_T D_2}e^{-r/\sqrt{\tau_T}}$$

which is now a linear non-homogeneous 2nd order constant-coefficient ODE.

Point Source of Fast Neutrons (cont.)

The standard solution technique for solving this ODE is to write the general solution as the linear combination of the homogeneous and particular solutions.

From the 1-group point source problem treated earlier, we know that the homogeneous solution is simply

$$\omega_h(r)=A_1e^{-r/L_T}+A_2e^{r/L_T}$$

Now, if we assume that the particular solution has the same functional behavior as the downscatter source, we might try

$$\omega_p(r)=Ce^{-r/\sqrt{\tau_T}}$$

Putting this assumed solution into the defining source-driven ODE gives

$$\frac{C}{\tau_T}e^{-r/\sqrt{\tau_T}}-\frac{C}{L_T^2}e^{-r/\sqrt{\tau_T}}=-\frac{Q_1}{4\pi\tau_T D_2}e^{-r/\sqrt{\tau_T}}$$

Point Source of Fast Neutrons (cont.)

and, solving for C gives

$$C = -\frac{Q_1}{4\pi\tau_T D_2}\left[\frac{1}{1/\tau_T - 1/L_T^2}\right] = -\frac{Q_1 L_T^2}{4\pi D_2 (L_T^2 - \tau_T)}$$

Thus, we can write the general solution as

general solution for thermal flux

$$\phi_T(r) = A_1 \frac{e^{-r/L_T}}{r} + A_2 \frac{e^{r/L_T}}{r} - \frac{Q_1 L_T^2}{4\pi D_2 (L_T^2 - \tau_T)} \frac{e^{-r/\sqrt{\tau_T}}}{r}$$

Now, the next step is to apply appropriate boundary conditions so as to uniquely determine the A_1 and A_2 coefficients within the general solution for $\phi_T(r)$.

Point Source of Fast Neutrons (cont.)

In this case, the fact that the flux must remain finite as $r \to \infty$, immediately forces $A_2 = 0$.

With this constraint, ϕ_T becomes

$$\phi_T(r) = A_1 \frac{e^{-r/L_T}}{r} - \frac{Q_1 L_T^2}{4\pi D_2 (L_T^2 - \tau_T)} \frac{e^{-r/\sqrt{\tau_T}}}{r}$$

Also, since there is no discrete source of thermal neutrons at $r = 0$, the second boundary condition is simply that the thermal leakage out of a sphere of radius r is zero as $r \to 0$.

We can impose this constraint as a mathematical limiting expression, or

$$\underset{\text{as } r\to 0}{\text{leakage}} = \lim_{r\to 0} J_T(r) 4\pi r^2 = 0$$

Point Source of Fast Neutrons (cont.)

To use this BC, we first write the net current in the thermal group in the outward radial direction as

$$J_T(r) = -D_2 \frac{d}{dr}\phi_T(r) = -D_2 \frac{d}{dr}\left[A_1 \frac{e^{-r/L_T}}{r} + C\frac{e^{-r/\sqrt{\tau_T}}}{r}\right]$$

$$= D_2\left[A_1\left(\frac{1}{rL_T} + \frac{1}{r^2}\right)e^{-r/L_T} + C\left(\frac{1}{r\sqrt{\tau_T}} + \frac{1}{r^2}\right)e^{-r/\sqrt{\tau_T}}\right]$$

Now, substituting into the 2nd BC and taking the limit as r → 0 gives

$$\lim_{r\to 0} 4\pi D_2\left[A_1\left(\frac{r}{L_T}+1\right)e^{-r/L_T} + C\left(\frac{r}{\sqrt{\tau_T}}+1\right)e^{-r/\sqrt{\tau_T}}\right] = 4\pi D_2 (A_1 + C) = 0$$

or simply that $A_1 = -C$.

Point Source of Fast Neutrons (cont.)

Therefore, the final solution for the thermal flux is

$$\phi_T(r) = \frac{Q_1 L_T^2}{4\pi D_2 (L_T^2 - \tau_T)} \frac{1}{r}\left(e^{-r/L_T} - e^{-r/\sqrt{\tau_T}}\right)$$

thermal flux

Thus, we have solved our first 2-group problem!!!

To summarize, we collect the final expressions for the fast and thermal flux profiles and also compute the fast-to-thermal flux ratio on the following slide:

Point Source of Fast Neutrons (cont.)

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$$\phi_1(r) = \frac{Q_1}{4\pi D_1}\frac{1}{r}e^{-r/\sqrt{\tau_T}}$$ fast flux

$$\phi_T(r) = \frac{Q_1 L_T^2}{4\pi D_2 (L_T^2 - \tau_T)}\frac{1}{r}\left(e^{-r/L_T} - e^{-r/\sqrt{\tau_T}}\right)$$ thermal flux

$$\frac{\phi_1(r)}{\phi_T(r)} = \frac{D_2}{D_1}\frac{(L_T^2 - \tau_T)}{L_T^2}\frac{e^{-r/\sqrt{\tau_T}}}{e^{-r/L_T} - e^{-r/\sqrt{\tau_T}}}$$ fast-to-thermal flux ratio

The flux profiles have the expected 1/r geometric attenuation factor associated with spherical geometry and an exponential attenuation term due to neutron diffusion and removal, and the ϕ_1/ϕ_T expression shows how the 2-group spectrum changes with increasing distance from the source.

A Numerical Example

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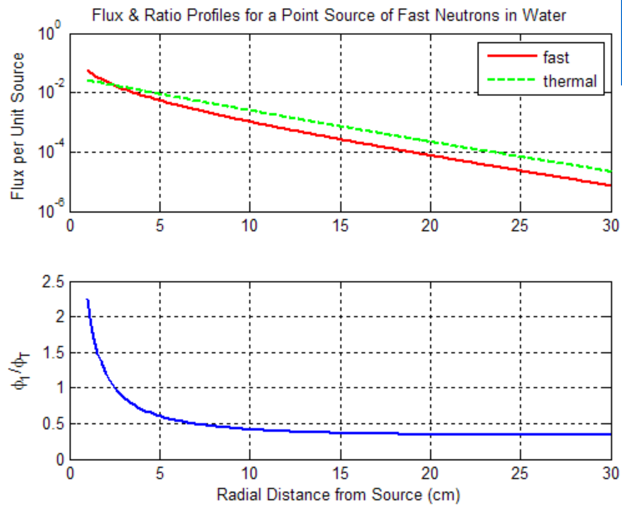
To help visualize the spatial behavior of these profiles, a Matlab program (ptsrc2g.m) was written to evaluate and plot $\phi_1(r)$, $\phi_T(r)$, and ϕ_1/ϕ_T versus r for various moderators with Q_1 = 1 neutron/sec.

The fast and thermal material data were obtained from the Lamarsh data tables, and the resultant profiles are shown in the next few slides for two separate cases: an infinite water region and an infinite medium of graphite.

Concerning the results, first we note that the decrease in the flux level in both groups is faster in water than in graphite since the diffusion lengths in graphite are much larger than for water (for both groups).

Note also that, after a short distance, the fast-to-thermal flux ratio approaches a constant in water but, in graphite, the fast flux continues to attenuate at a faster rate than the thermal flux, with the ϕ_1/ϕ_T ratio eventually approaching zero.

A Numerical Example (cont.)



infinite water medium

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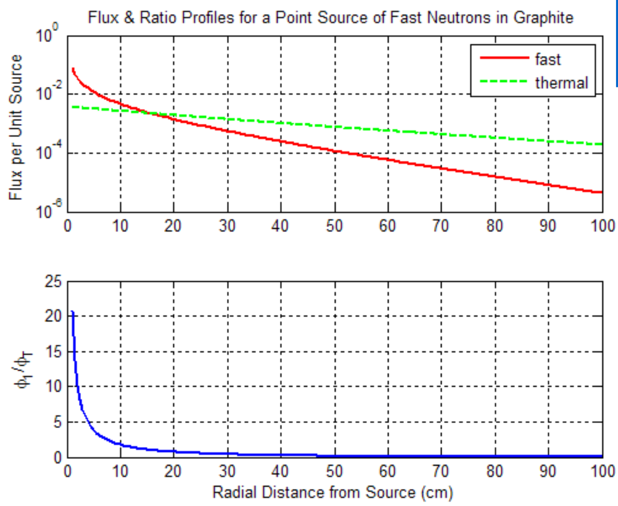
A Numerical Example (cont.)



infinite graphite medium

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A Numerical Example (cont.)

To see why graphite behaves differently from water, let's look at the diffusion properties of these two materials (from Lamarsh):

Material/Property	D_1 (cm)	τ_T (cm^2)	$1/\sqrt{\tau_T}$ (cm^{-1})	D_2 (cm)	L_T^2 (cm^2)	$1/L_T$ (cm^{-1})
water	1.13	27	0.192	0.16	8.1	0.351
graphite	1.02	368	0.052	0.84	3500	0.017

From here we see that, for water, $1/\sqrt{\tau_T} < 1/L_T$, which says that e^{-r/L_T} decreases faster than $e^{-r/\sqrt{\tau_T}}$. Thus, as $r \to \infty$, we have

$$\lim_{r\to\infty} \frac{e^{-r/\sqrt{\tau_T}}}{e^{-r/L_T} - e^{-r/\sqrt{\tau_T}}} \to -1$$

infinite water medium

and

$$\left.\frac{\phi_1}{\phi_T}\right|_\infty = -\frac{D_2}{D_1}\frac{(L_T^2 - \tau_T)}{L_T^2} = -\left(\frac{0.16}{1.13}\right)\left(\frac{8.1-27}{8.1}\right) \approx 0.33$$

A Numerical Example (cont.)

But, for graphite, $1/\sqrt{\tau_T} > 1/L_T$, which says that $e^{-r/\sqrt{\tau_T}}$ decreases faster than e^{-r/L_T} . Thus, as $r \to \infty$, we have

$$\lim_{r\to\infty} \frac{e^{-r/\sqrt{\tau_T}}}{e^{-r/L_T} - e^{-r/\sqrt{\tau_T}}} \to 0 \quad \text{and} \quad \left.\frac{\phi_1}{\phi_T}\right|_\infty \approx 0$$

infinite graphite medium

Material/Property	D_1 (cm)	τ_T (cm^2)	$1/\sqrt{\tau_T}$ (cm^{-1})	D_2 (cm)	L_T^2 (cm^2)	$1/L_T$ (cm^{-1})
water	1.13	27	0.192	0.16	8.1	0.351
graphite	1.02	368	0.052	0.84	3500	0.017

These results suggest, for example, that a large thickness of graphite could be used within an experimental facility to selectively filter out the high energy neutrons, leaving a nearly pure population of thermal neutrons -- such as in the graphite thermal column in the UMass-Lowell research reactor (UMLRR)…

A Final Note Concerning the Use of Diffusion Theory for Non-Multiplying Media Problems

In closing our discussion of non-multiplying media problems, we emphasize once again that the 1-group and 2-group diffusion theory models used here only give a qualitative perspective on the behavior of neutron diffusion in these systems.

Since there is a strong angular dependence of the flux in the direction away from the discrete sources utilized in these problems, the original approximations made to reduce the general neutron balance to the diffusion equation are not very accurate for these situations.

In practice, transport theory is used to get much better quantitative estimates of neutron transport in these systems (but not in this introductory course!!!).

A Final Note Concerning the Use of Diffusion Theory for Non-Multiplying Media Problems

However, our simple 1-group and 2-group diffusion theory examples still give us a good qualitative understanding of these systems, and this was the primary physics-oriented goal of the last few lessons.

Thus, along with the demonstration of how to mathematically solve source-driven diffusing media problems (with no upscatter), we have also gained a little physical insight into the neutronic behavior of general 1-group and 2-group non-multiplying systems -- and both accomplishments should add significantly to your growing inventory of tools and experiences for understanding general steady state problems in reactor theory…

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Lesson 5 Summary

In this Lesson we have briefly discussed the following subjects:

How to setup and solve the 2-group diffusion equation for a variety of moderating media configurations.

The sequential nature of the solution process for problems with no fission and no upscatter.

The behavior of the fast-to-thermal flux ratio in water versus graphite regions.

The concern about using diffusion theory for large regions of purely moderating media (with no distributed isotropic sources).