Nuclear Reactor Theory

Lesson 3: Neutron Diffusion in Moderating Media I

The 1-Group Diffusion Theory Approximation

Prof. John R. White
Chemical and Nuclear Engineering
UMass-Lowell, Lowell MA

Lesson 3 Objectives

Explain the basic procedure for solving 2nd order, linear, constant coefficient, source-driven ODEs.

Setup and solve the 1-group diffusion equation for a variety of moderating media problems (for various geometry and source configurations).

Formulate and evaluate expressions for the absorption and leakage rates and validate the neutron balance equation in a variety of simple geometries using the 1-group diffusion theory approximation.

Discuss the fundamental difference between Cartesian geometry and curvilinear geometries (spherical and cylindrical configurations) when interpreting the spatial flux profiles in source-driven diffusing media problems.

Lesson 3 Objectives (cont.)

Define the term diffusion length and describe its importance in moderating media problems.

Explain the procedure for treating multiregion problems and configurations containing multiple discrete sources and continuously distributed sources.

2nd Order Linear Constant Coefficient ODEs

One goal of this Lesson is to obtain and interpret the solution to the 1-group diffusion equation for a variety of simple geometries.

However, the 1-group steady state diffusion equation is a 2nd order (linear) differential equation.

And, within a single homogeneous region, the Cartesian geometry problem reduces to a simple 2nd order linear constant coefficient source-driven ODE -- a problem that is rather easy to solve analytically.

As a review, given the following linear, 2nd order, inhomogeneous (i.e. source-driven) system,

$$y''(x) + a\,y'(x) + b\,y(x) = f(x)$$

the general solution is given as the linear combination of the solutions to the homogeneous and particular equations,

$$y(x) = y_h(x) + y_p(x)$$

2^{nd} Order Linear Constant Coeff. ODEs (cont.)

$y_h(x)$ is the solution to the homogeneous (or complementary) equation

$$y_h''(x) + a y_h'(x) + b y_h(x) = 0$$

and $y_p(x)$ is a particular solution to the original source-driven ODE.

The general solution will contain two arbitrary coefficients and the unique solution for a specific source-driven problem is obtained by satisfying two boundary conditions (which uniquely determine the two arbitrary coefficients in the general solution).

2^{nd} Order Linear Constant Coeff. ODEs (cont.)

Homogeneous Solution:

The solution to a linear, 2^{nd} order, constant-coeff., homogeneous ODE can be written in the form of a simple exponential, $y_h \sim e^{rx}$, where r is an unknown constant.

Putting this assumed solution into the complementary equation leads to the characteristic equation for the values of r that satisfy the assumed solution:

$$r^2 + ar + b = 0$$

Referring to r_1 and r_2 as the two distinct solutions to the characteristic equation (the case of repeated roots is a relatively uncommon occurrence), the homogeneous solution can be written as a linear combination of the individual solutions, or

$$y_h(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

2nd Order Linear Constant Coeff. ODEs (cont.)

UMASS LOWELL Learning with Purpose

A Special Case:

Note that for the special case of $a = 0$ and $b = \pm\alpha^2$ (which is consistent with the two different forms of the 1-group diffusion equation that are commonly encountered), the homogeneous equation becomes

$$y_h''(x) \pm \alpha^2 y_h(x) = 0$$

with characteristic roots

$$r^2 \pm \alpha^2 = 0 \qquad \text{or} \qquad r_{1,2} = \pm\sqrt{\mp\alpha^2}$$

Forms of Interest

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi = -\frac{Q}{D}$$

$$\frac{d^2\phi}{dx^2} + B^2\phi = 0$$

Thus, the sign of $b = \pm\alpha^2$ becomes very important, since two completely different forms for the solution can result.

In particular, when $b = -\alpha^2$, the roots are real and distinct and yield a solution written in the form of real exponentials

$$y_h(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

2nd Order Linear Constant Coeff. ODEs (cont.)

UMASS LOWELL Learning with Purpose

However, when $b = +\alpha^2$, the roots are still distinct, but now they are pure imaginary complex conjugates -- and these lead to solutions in the form of complex exponentials -- that is, the roots to the characteristic equation are $r_{1,2} = \pm j\alpha$, where i or j is the imaginary number $i = j = \sqrt{-1}$.

The complex exponentials are almost always written in terms of sinusoids using Euler's formula,

$$e^{\pm j\alpha x} = \cos\alpha x \pm j\sin\alpha x$$

where $\sin\alpha x = \frac{e^{j\alpha x} - e^{-j\alpha x}}{2j}$ & $\cos\alpha x = \frac{e^{j\alpha x} + e^{-j\alpha x}}{2}$

Thus, for the case of pure imaginary roots, the homogeneous solution becomes

$$y_h(x) = c_1 e^{j\alpha x} + c_2 e^{-j\alpha x} = c_1(\cos\alpha x + j\sin\alpha x) + c_2(\cos\alpha x - j\sin\alpha x)$$

$$= (c_1 + c_2)\cos\alpha x + j(c_1 - c_2)\sin\alpha x$$

$$= A_1\cos\alpha x + A_2\sin\alpha x$$

← This is how to write the solution!!!

2^{nd} Order Linear Constant Coeff. ODEs (cont.)

Note that, for the case of real exponential solutions (for $b = -\alpha^2$), one can perform a similar manipulation using hyperbolic sinusoids.

To see this, we first formally define the hyperbolic sine and cosine in terms of real exponential functions,

$$\sinh \alpha x = \frac{e^{\alpha x} - e^{-\alpha x}}{2} \quad \text{and} \quad \cosh \alpha x = \frac{e^{\alpha x} + e^{-\alpha x}}{2}$$

or

$$e^{\pm \alpha x} = \cosh \alpha x \pm \sinh \alpha x$$

For the case of real distinct roots, $r_{1,2} = \pm\alpha$, the homogeneous solution becomes

$$\begin{aligned} y_h(x) &= c_1 e^{\alpha x} + c_2 e^{-\alpha x} = c_1(\cosh \alpha x + \sinh \alpha x) + c_2(\cosh \alpha x - \sinh \alpha x) \\ &= (c_1 + c_2)\cosh \alpha x + (c_1 - c_2)\sinh \alpha x \\ &= A_1 \cosh \alpha x + A_2 \sinh \alpha x \end{aligned}$$

2^{nd} Order Linear Constant Coeff. ODEs (cont.)

Note that, although the first and last forms of $y_h(x)$ are equivalent, it is often more convenient to use the exponential form for infinite systems and the hyperbolic sinusoids for finite systems.

In summary, the following table collects the various homo-geneous solutions to the specialized ODE:

$$y''(x) \pm \alpha^2 y(x) = 0$$

1-group non-multiplying slab problem	1-group critical slab reactor problem
$y'' - \alpha^2 y = 0$	$y'' + \alpha^2 y = 0$
$y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$ (infinite region)	$y(x) = c_1 e^{j\alpha x} + c_2 e^{-j\alpha x}$ (rarely used)
$y(x) = A_1 \cosh \alpha x + A_2 \sinh \alpha x$ (finite region)	$y(x) = A_1 \cos \alpha x + A_2 \sin \alpha x$ (usual form)

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi = -\frac{Q}{D}$$

Actual eqns. of interest

$$\frac{d^2\phi}{dx^2} + B^2\phi = 0$$

2^{nd} Order Linear Constant Coeff. ODEs (cont.)

Particular Solution:

For $y_p(x)$, we will use the Method of Undetermined Coefficients, where one essentially makes an assumption concerning the form of $y_p(x)$, and then, via substitution of the assumed solution into the defining source-driven ODE, determines the unknown coefficients within the assumed solution.

The UC method is easy to apply once a proper form for $y_p(x)$ has been selected, and this can be done by carefully applying the following two rules:

General Rule: Choose $y_p(x)$ to have the same form as the RHS forcing function, f(x), and all its linearly independent derivatives. Then evaluate the unknown coefficients within $y_p(x)$ by substitution into the original inhomogeneous ODE, and equate the coefficients of the terms with similar forms on both sides of the equation.

2^{nd} Order Linear Constant Coeff. ODEs (cont.)

Special Rule: If $y_p(x)$ via the general rule for a constant coefficient linear system contains one or more terms that are solutions to the homogeneous equation, one then multiplies these terms by x^k where k is the smallest integer value that makes all the terms in $y_p(x)$ independent of the terms in $y_h(x)$.

Note: The case of repeated roots within the homogenous solution and the need for the Special Rule within the UC method do not occur very often in practical applications, and we will not need to apply either of these special cases in our focused treatment of the steady state neutron diffusion equation.

Non-Multiplying Source-Driven Systems

To illustrate the use of the diffusion equation, let's apply it to some simple, but representative, situations.

As a starting point, we restrict our analyses to the 1-group approximation with no fission.

This situation is appropriate for non-multiplying (diffusing) media such as reflector or shield geometries.

For 1-group theory and no fission the multigroup diffusion equation becomes

$$-\vec{\nabla}\cdot D(\vec{r})\vec{\nabla}\phi(\vec{r}) + \Sigma_a(\vec{r})\phi(\vec{r}) = Q(\vec{r})$$

For a homogeneous region, the macroscopic absorption cross section and diffusion coefficient are constants, giving

$$-D\nabla^2\phi(\vec{r}) + \Sigma_a\phi(\vec{r}) = Q(\vec{r}) \quad \text{or} \quad \nabla^2\phi(\vec{r}) - \frac{\Sigma_a}{D}\phi(\vec{r}) = -\frac{Q(\vec{r})}{D}$$

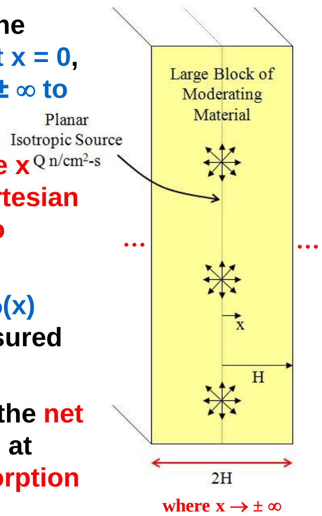
diffusion area $L^2 = D/\Sigma_a$

$$\nabla^2\phi(\vec{r}) - \frac{1}{L^2}\phi(\vec{r}) = -\frac{Q(\vec{r})}{D}$$

This is the starting point for the subsequent examples.

Planar Source in an Infinite Medium

Consider a geometry that is centered on the infinitesimally thin planar source region at x = 0, is infinite in the y-z plane, and where $x \to \pm\infty$ to create a full infinite moderating medium.



In this system, the flux will only vary in the x direction, and this allows a simple 1-D Cartesian geometry (slab) treatment, with a non-zero discrete isotropic source only at x = 0.

Our goal is to formally derive a result for $\phi(x)$ assuming 1-group theory, where x is measured relative to the planar source location.

We also desire analytical expressions for the net neutron leakage out of the right-half plane at some finite location x = H and for the absorption rate within this same volume.

Planar Source in an Infinite Medium (cont.)

The 1-group diffusion equation for a homogeneous medium with no fission is given by

$$\nabla^2\phi - \frac{1}{L^2}\phi = -\frac{Q}{D}$$

For 1-D slab geometry, the Laplacian simply becomes the 2nd derivative of the flux with respect to x, and since the source is only non-zero at the centerline of the block at x = 0, we can write this expression as a homogeneous equation for x > 0, or

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi = 0 \qquad x > 0$$

The general solution to this simple 2nd order constant coefficient homogeneous ODE is

$$\phi(x) = C_1 e^{-x/L} + C_2 e^{x/L}$$

To obtain a unique solution to a 2nd order ODE, we must apply two boundary conditions -- one as $x \to \infty$ and one as $x \to 0$.

Planar Source in an Infinite Medium (cont.)

In the case of the infinite system, where x can become large, we require that the flux must remain finite as $x \to \infty$.

Therefore, the growing exponential term in the general solution immediately forces us to set C_2 to zero.

This condition reduces the flux and net current ($\vec{J} = -D\vec{\nabla}\phi$) for this case to the following expressions,

$$\phi(x) = C_1 e^{-x/L}$$

$$\vec{J}(x) = J_x(x)\hat{i} = -D\frac{d\phi}{dx}\hat{i} = \frac{DC_1}{L}e^{-x/L}\hat{i}$$

where, for convenience, we will refer to the x-directed current, J_x, in all subsequent usage simply as J, since this is the only nonzero component for the 1-D slab problem (i.e. $J_y = J_z = 0$).

Planar Source in an Infinite Medium (cont.)

To find an explicit expression for C_1, we apply a second boundary condition -- the source condition -- at x = 0 (i.e. the discrete discontinuous source at x = 0 requires that a special source condition be applied).

For 1-D Cartesian geometry, this source condition at x = 0 can be written as

$$\lim_{\Delta x \to 0} \begin{Bmatrix} \text{leakage from left} & + & \text{leakage from right} \\ \text{side of a thin box} & & \text{side of a thin box} \end{Bmatrix} = \begin{matrix} \text{source contained} \\ \text{in the thin box} \end{matrix}$$

In mathematical terms, this statement translates to the following equation:

$$\lim_{x \to 0} \left[\vec{J}(x<0) \cdot (-\hat{i}) + \vec{J}(x>0) \cdot (\hat{i}) \right] = \lim_{x \to 0} \left[-J(x<0) + J(x>0) \right] = Q$$

Planar Source in an Infinite Medium (cont.)

Or, so we don't have to treat the x < 0 case, for a symmetric geometry we can simply write this condition as

$$\lim_{x \to 0} J(x>0) = \frac{Q}{2} \quad \text{(for a symmetric block)}$$

This statement makes perfect sense because only half of the original source neutrons enter the right half of the homogeneous block centered at x = 0 (because of the isotropic nature of the source).

Using the equation for J(x), we can apply the above source condition to give an explicit expression for C_1, or

$$\lim_{x \to 0} \left\{ \frac{DC_1}{L} e^{-x/L} \right\} = \frac{DC_1}{L} = \frac{Q}{2} \qquad \text{or} \qquad C_1 = \frac{QL}{2D}$$

Planar Source in an Infinite Medium (cont.)

Finally, substituting this expression for C_1 into the equations for flux and current, gives

$$\phi(x) = \frac{QL}{2D} e^{-x/L} \quad \text{and} \quad J(x) = \frac{Q}{2} e^{-x/L}$$

flux and current due to a planar source in an infinite medium

Thus, the neutron flux and current decrease with distance from the source location in a simple exponential manner.

However, of note, is that the rate of decrease is directly related to the material's diffusion length, L.

Clearly, the neutron flux attenuates at a greater rate for a material with a smaller diffusion length!!!

Planar Source in an Infinite Medium (cont.)

To address the neutron balance within a portion of the infinite slab, we need to compute:

1. the leakage out of the right side of a block defined by x = H,
2. the absorption rate within this portion of the slab, and then
3. add these to show that they sum to the total source within this region.

Treating these terms individually, we have

$$\begin{matrix}\text{leakage} \\ \text{per unit area}\end{matrix} = \int_A \vec{J} \cdot \hat{n}\, dA = J(H)(1) = \frac{Q}{2} e^{-H/L}$$

$$\begin{matrix}\text{absorption rate} \\ \text{per unit area}\end{matrix} = \int_0^H \Sigma_a \phi(x)(1)dx = \frac{QL\Sigma_a}{2D} \int_0^H e^{-x/L} dx$$

$$= \frac{Q}{2L}\left[-Le^{-x/L} \Big|_0^H \right] = \frac{Q}{2}\left(1 - e^{-H/L}\right)$$

Planar Source in an Infinite Medium (cont.)

$$\begin{array}{c}\text{source}\\ \text{per unit area}\end{array} = \int_V Q\,\delta(x)\,d\vec{r} = \int_0^H Q\,\delta(x)\,dx = \frac{Q}{2}$$

$$\left[\frac{Q}{2}e^{-H/L}\right] + \left[\frac{Q}{2}\left(1 - e^{-H/L}\right)\right] = \frac{Q}{2}$$

Overall Neutron Balance
[leakage + absorption = source]

And, for a proper balance, Q/2 = Q/2, as expected!!!

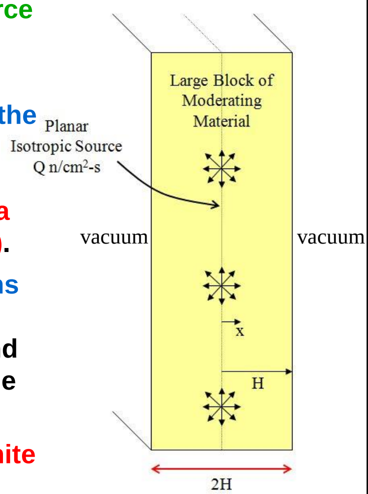
Planar Source in a Bare Finite Slab

Now consider an isotropic planar source placed along the centerline of a finite slab of moderator of thickness 2H.

If the external boundary condition for the bare slab is such that the flux goes to zero at the extrapolated boundary (i.e. at H+d), our goal is to formally derive a 1-group diffusion theory result for $\phi(x)$.

Again, we desire analytical expressions for the net neutron leakage out of the right-half of the finite block at x = H and for the absorption rate within this same volume.

This system is referred to as a bare finite slab of width 2H…



where $-H \le x \le H$

Planar Source in a Bare Slab (cont.)

The solution of the balance equation for this case follows the same procedure as the infinite slab case.

However, although the defining balance equation is the same,

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi = 0 \qquad x > 0$$

it is convenient to write the general solution for a finite geometry in terms of hyperbolic sines and cosines [instead of the real exponentials as used previously].

Thus, the general solution for the finite geometry case is usually written as

$$\phi(x) = A_1 \sinh x/L + A_2 \cosh x/L$$

where it should be emphasized that this form is simply better suited for finite geometry cases.

Planar Source in a Bare Slab (cont.)

To find A_1 and A_2 to give a unique solution, we need to apply two independent boundary constraints -- one at the extrapolated boundary of the slab and one at x = 0.

At the right boundary of the bare slab (i.e. at x = H + d), we say that the flux goes to zero at the extrapolated boundary, where d is the extrapolation distance (this is the standard vacuum boundary condition used in diffusion theory).

Mathematically this statement is written as

$$\phi(H+d) = 0$$

and, from the general solution for ϕ(x), we have

$$A_1 \sinh(H+d)/L + A_2 \cosh(H+d)/L = 0$$

or

$$A_1 = -A_2 \frac{\cosh(H+d)/L}{\sinh(H+d)/L}$$

Planar Source in a Bare Slab (cont.)

For the second boundary constraint at x = 0, we apply the same source condition as in the previous example.

For the present case, the net neutron current is given by

$$\vec{J}(x) = -D\frac{d\phi}{dx}\hat{i} = -\frac{D}{L}\left(A_1 \cosh x/L + A_2 \sinh x/L\right)\hat{i}$$

or

$$J(x) = -\frac{DA_2}{L}\left(-\frac{\cosh(H+d)/L}{\sinh(H+d)/L}\cosh x/L + \sinh x/L\right)$$

Thus, evaluating the source condition for this system,

$$\lim_{x\to 0} J(x>0) = \frac{Q}{2} \quad \text{(for a symmetric block)}$$

with the above expression for the net neutron current, gives

$$\lim_{x\to 0}\left\{-\frac{DA_2}{L}\left(-\frac{\cosh(H+d)/L}{\sinh(H+d)/L}\cosh x/L + \sinh x/L\right)\right\} = \frac{DA_2}{L}\frac{\cosh(H+d)/L}{\sinh(H+d)/L} = \frac{Q}{2}$$

Planar Source in a Bare Slab (cont.)

or

$$A_2 = \frac{QL}{2D}\frac{\sinh(H+d)/L}{\cosh(H+d)/L}$$

Substitution of this expression for A_2 into the equation for A_1 gives a simple result for A_1, or

$$A_1 = -\frac{QL}{2D}$$

Now putting the expressions for A_1 and A_2 back into the expression for $\phi(x)$ gives the flux due to a planar source in a finite medium as

$$\phi(x) = \frac{QL}{2D}\left[\frac{\sinh(H+d)/L}{\cosh(H+d)/L}\cosh x/L - \sinh x/L\right]$$

Planar Source in a Bare Slab (cont.)

And, from the definition of the net current, we can also write the current due to a planar source in a finite medium as

$$J(x) = \frac{Q}{2}\left[\cosh x/L - \frac{\sinh(H+d)/L}{\cosh(H+d)/L}\sinh x/L\right]$$

where, of course, the net current for x > 0 is pointed in the +x direction.

Planar Source in a Bare Slab (cont.)

To address the neutron balance within the finite bare slab, we again need to compute the leakage and absorption rates within the right half of the block using the flux and current expressions given on the previous slide.

Thus, for the finite geometry case, we have

$$\begin{aligned}\text{leakage per unit area} &= \int_A \vec{J}\cdot\hat{n}\,dA = J(H)(1) \\ &= \frac{Q}{2}\left[\cosh H/L - \frac{\sinh(H+d)/L}{\cosh(H+d)/L}\sinh H/L\right]\end{aligned}$$

$$\text{source per unit area} = \int_V Q\,\delta(x)\,d\vec{r} = \int_0^H Q\,\delta(x)\,dx = \frac{Q}{2}$$

Planar Source in a Bare Slab (cont.)

$$\begin{aligned}\text{absorption rate per unit area} &= \int_0^H \Sigma_a \phi(x)(1)dx \\ &= \frac{QL\Sigma_a}{2D}\int_0^H \left[\frac{\sinh(H+d)/L}{\cosh(H+d)/L}\cosh x/L - \sinh x/L\right]dx \\ &= \frac{Q}{2L}\left[\frac{\sinh(H+d)/L}{\cosh(H+d)/L} L\sinh x/L - L\cosh x/L\right]_0^H \\ &= \frac{Q}{2}\left[\frac{\sinh(H+d)/L}{\cosh(H+d)/L}\sinh H/L - \cosh H/L\right] + \frac{Q}{2}\end{aligned}$$

And the overall neutron balance for the right side of the bare finite slab gives

Overall Neutron Balance
[leakage + absorption = source]

$$\left\{\frac{Q}{2}\left[\cosh H/L - \frac{\sinh(H+d)/L}{\cosh(H+d)/L}\sinh H/L\right]\right\} + \left\{\frac{Q}{2}\left[\frac{\sinh(H+d)/L}{\cosh(H+d)/L}\sinh H/L - \cosh H/L\right] + \frac{Q}{2}\right\} = \frac{Q}{2}$$

and, for a proper balance, Q/2 = Q/2, as expected!!!

Planar Source in a Bare Slab (cont.)

Summary:

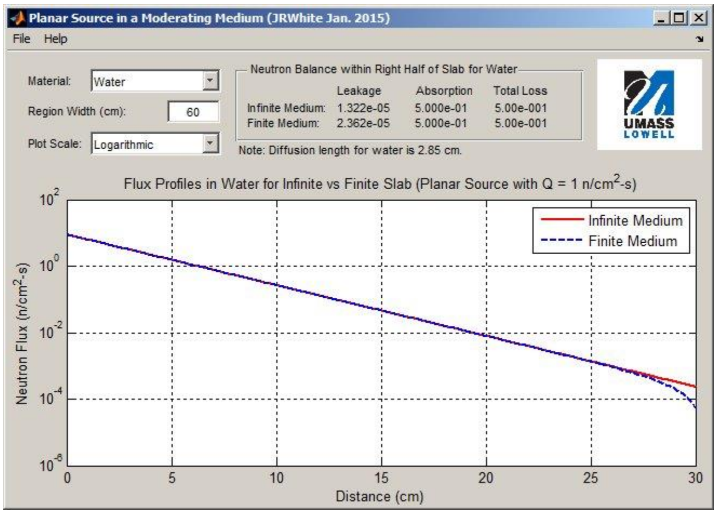
The previous two examples provide a detailed derivation of the 1-group flux, current, and neutron balance components for the case of a planar source in a non-multiplying medium -- with developments for both infinite and finite slab geometries centered on the planar source location.

The resultant equations for both cases have been implemented into the slabmm_gui code, which provides an easy-to-use GUI where one can explore and contrast the use of different materials and slab dimensions.

This formal development, coupled with a user-friendly computational tool, should allow the user to get a better understanding of the basic physics, and also get a good feel for both the qualitative and quantitative aspects of 1-group neutron diffusion in a simple 1-D Cartesian geometry.

ENGY.4340 Nuclear Reactor Theory
Lesson 3: Neutron Diffusion in Moderating Media I
(Sept. 2016)

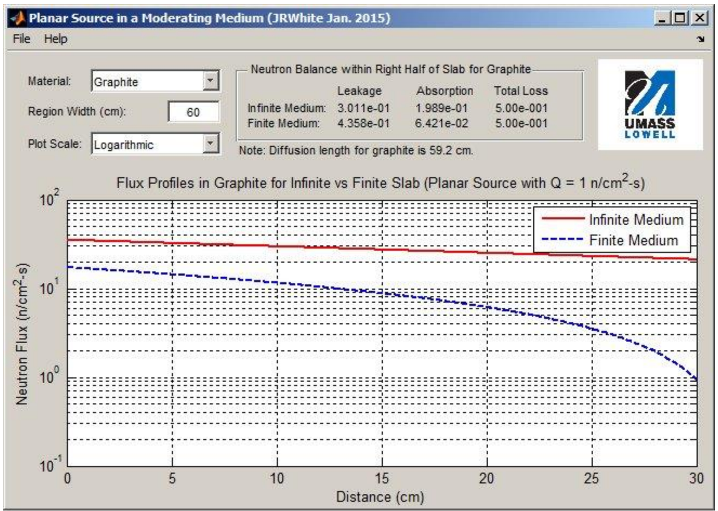
slabmm_gui Interface (water)



ENGY.4340 Nuclear Reactor Theory
Lesson 3: Neutron Diffusion in Moderating Media I

(Sept. 2016)

slabmm_gui Interface (graphite)



ENGY.4340 Nuclear Reactor Theory
Lesson 3: Neutron Diffusion in Moderating Media I

(Sept. 2016)

Point Source in an Infinite Medium

Another classical problem that is often discussed when studying typical solutions of the 1-group diffusion equation involves a point source of neutrons in a pure moderating medium.

The solution of this 1-D spherical geometry problem follows a similar development as just completed for the 1-D slab geometry, so we will not go through all the details in class.

However, there are two key differences, as follows:

1. The 1-D spherical geometry balance equation has variable coefficients, but a simple substitution can convert the original variable-coefficient ODE into one that has constant coefficients -- which can then be solved as done previously.

2. The resultant flux profile decreases with distance from the source, but now there is a geometric attenuation term in addition to the expected neutron diffusion and absorption term.

Point Source in an Infinite Medium (cont.)

To elaborate on the first point, we note that, for 1-D spherical geometry, the Laplacian becomes

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)$$

In addition, since the isotropic point source is only non-zero at r = 0, we can write the diffusion equation as a homogeneous equation for r > 0, or

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) - \frac{1}{L^2}\phi = 0 \qquad \text{for } r > 0$$

Since this is a variable-coefficient ODE, let's first make the substitution $\omega = r\phi$ to put this into a more manageable form.

Thus, letting

$$\omega = r\phi \qquad \text{or} \qquad \phi = \frac{\omega}{r}$$

Point Source in an Infinite Medium (cont.)

gives
$$\frac{d\phi}{dr} = -\frac{\omega}{r^2} + \frac{1}{r}\frac{d\omega}{dr}$$

and
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(-\omega + r\frac{d\omega}{dr}\right) = \frac{1}{r^2}\left[-\frac{d\omega}{dr} + r\frac{d^2\omega}{dr^2} + \frac{d\omega}{dr}\right] = \frac{1}{r}\frac{d^2\omega}{dr^2}$$

and, after formal substitution of this latter result into the defining ODE, we get the following balance equation for $\omega(r)$,

$$\frac{d^2}{dr^2}\omega - \frac{1}{L^2}\omega = 0 \qquad \text{for } r > 0$$

This ODE now has constant coefficients and it can be solved via simple analytical methods (see the Lecture Notes for a complete development).

Point Source in an Infinite Medium (cont.)

To address the second key difference, we note that the solutions to the infinite-region and finite-region point source problems addressed here are:

$$\phi(r) = \frac{Q}{4\pi D}\frac{1}{r}e^{-r/L}$$

flux due to a point source in an infinite medium

$$\phi(r) = \frac{Q}{4\pi D \sinh(R+d)/L}\frac{1}{r}\sinh\left(\frac{R+d-r}{L}\right)$$

flux due to a point source in a finite medium

Focusing, in particular, on the infinite medium problem, we notice an $e^{-r/L}$ factor that accounts for the neutron diffusion and absorption within the medium of interest (as characterized by the diffusion length, L) -- this is similar to the slab geometry problem.

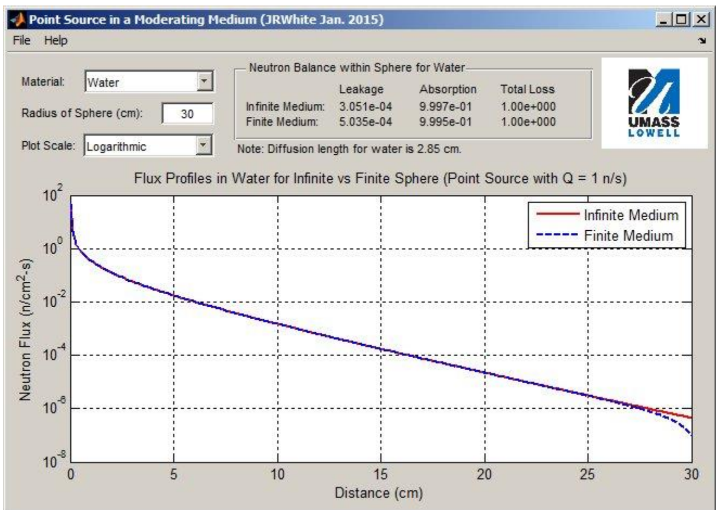
Point Source in an Infinite Medium (cont.)

In addition, however, there is also a 1/r term, and this term is a result of the spherical geometry, since the surface area increases with increasing distance from the source -- this is referred to as the geometric attenuation term.

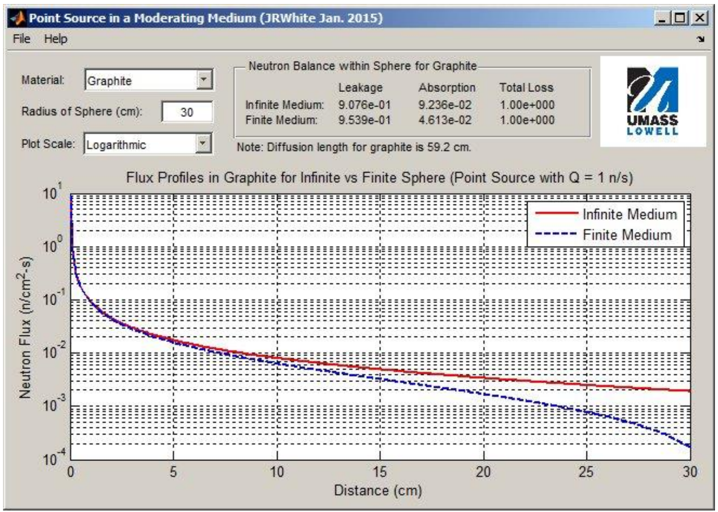
Thus, even if the absorption term was very small (so that L is very large and $e^{-r/L} \to 1$), the flux will still decrease with distance from the source because of the geometric attenuation that is inherent in curvilinear geometries (this behavior occurs in spherical and cylindrical geometry, but not in Cartesian geometry problems).

Except for the two key differences noted here, the point source and planar source problems are solved using similar techniques and they give similar results -- and a Matlab GUI, spheremm_gui, is also available to do various material and size comparison studies…

spheremm_gui Interface (water)



spheremm_gui Interface (graphite)



Treating Multiregion Problems

The examples thus far have focused on 1-region configurations.

However, for problems involving multiple homogeneous regions, we can apply the same overall methodology, as follows:

1. Write the 1-group diffusion equation for each region, being careful to distinguish between the material properties and flux solutions that are appropriate to each region.

2. Find the general solution applicable to each region -- there will be two arbitrary coefficients within each region.

3. In addition to the normal symmetry and outer boundary conditions, apply the continuity of flux and continuity of current interface conditions at each interface between dissimilar materials (you still need to treat a discrete discontinuous source as a separate source condition).

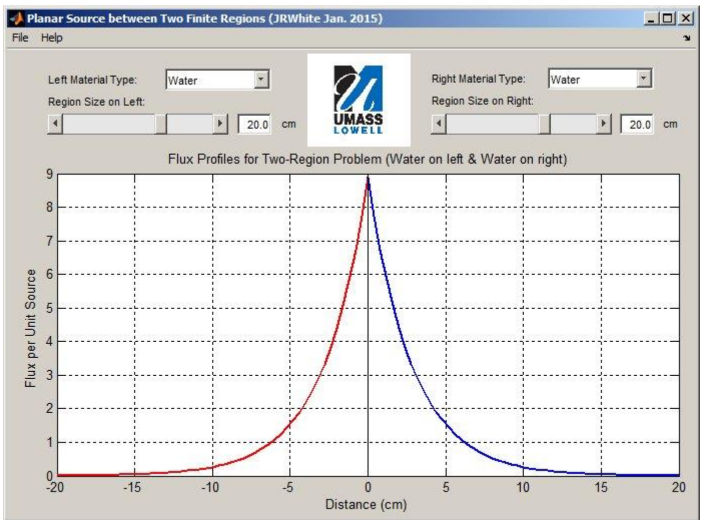
Treating Multiregion Problems

4. **Step 3 should lead to 2*N equations for the unknown coefficients for a problem containing N regions -- and solution of these gives the desired unique solution throughout the full geometry.**

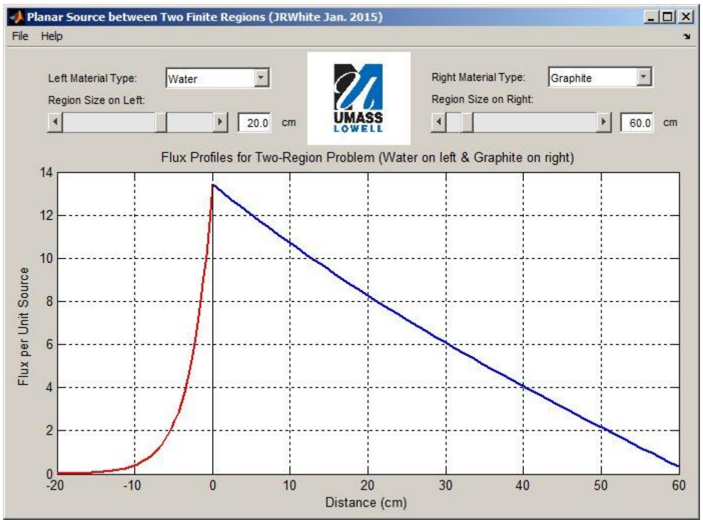
5. **Plot and analyze the solution as usual…**

see the two_regions_gui code for a specific example

two_regions_gui Interface (water-water)



two_regions_gui Interface (water-graphite)



Problems with Multiple Discrete Sources

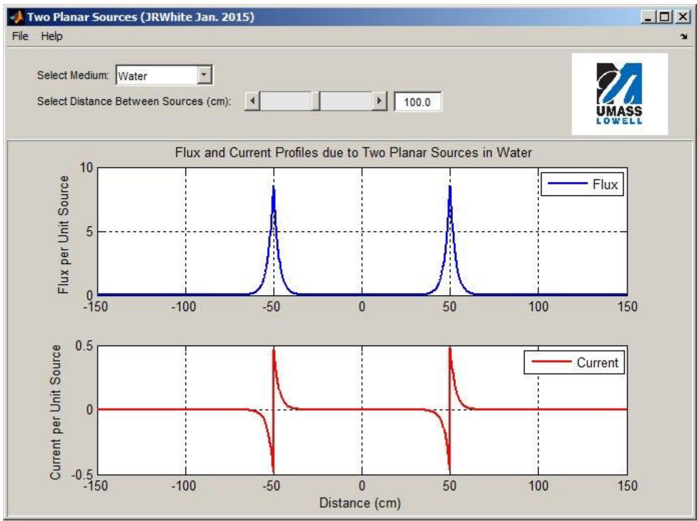
For problems containing multiple discrete discontinuous sources, one simply needs to remember that the neutron flux is a scalar quantity and that the neutron current is a vector quantity.

Thus, the composite flux at a point due to multiple discrete sources is simply the scalar addition of the fluxes due to each individual source.

For the net neutron current, the same basic statement is true, but vector addition must be performed, where the sign of the vector components due to each source may add or subtract, depending on the direction of the current due to individual sources.

see the two_planar_sources_gui code for a specific example

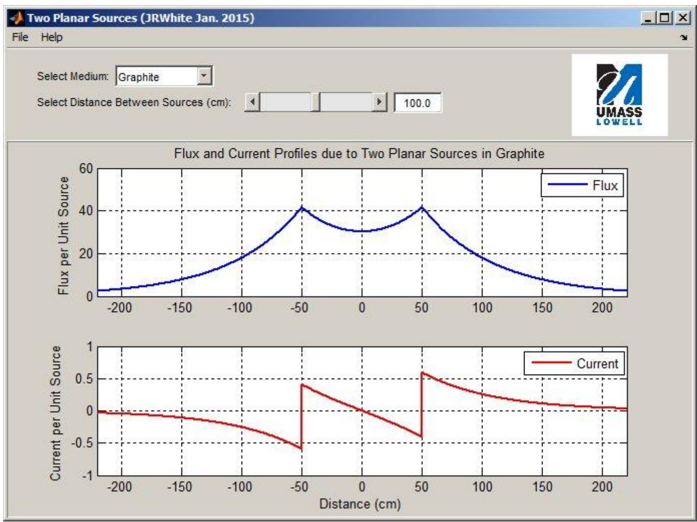
two_planar_sources_gui Interface (water)



ENGY.4340 Nuclear Reactor Theory
Lesson 3: Neutron Diffusion in Moderating Media I

(Sept. 2016)

two_planar_sources_gui Interface (graphite)



ENGY.4340 Nuclear Reactor Theory
Lesson 3: Neutron Diffusion in Moderating Media I

(Sept. 2016)

Problems with Distributed Sources

If the neutron sources are continuously distributed within a given region, then the original problem is usually solved as an inhomogeneous ODE with corresponding homogeneous and particular solutions (and the particular component is directly related to the given source distribution).

See the HW problems associated with this Lesson for an example containing a distributed source.

In addition, a worked-out 2-group example is also given later in Lesson 5.

Lesson 3 Summary

In this Lesson we have briefly discussed the following subjects:

The basic procedure for solving 2nd order, linear, constant coefficient, source-driven ODEs.

The setup and solution of the 1-group diffusion equation for a variety of moderating media problems.

The appropriate expressions for the absorption and leakage rates and how to validate the neutron balance equation in a variety of simple geometries using the 1-group diffusion theory approximation.

The fundamental difference between Cartesian geometry and curvilinear geometries when interpreting the spatial flux profiles in source-driven problems.

Lesson 3 Summary (cont.)

The term diffusion length and its importance in neutron diffusion problems.

The general procedure for treating multiregion problems and configurations containing multiple discrete sources and continuously distributed sources.