Nuclear Reactor Theory

Lesson 2: The Multigroup Neutron Balance Equation

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Lesson 2 Objectives

Describe the notation and discretization process for the continuous energy variable -- which results in the standard multigroup formulation.

Write formal expressions for the appropriately averaged multigroup cross sections for all the important 1-D and 2-D processes.

List the key production and loss terms needed to develop the steady state multigroup neutron balance equation, and write formal expressions for each of these terms using standard multigroup notation.

Develop the differential form of the multigroup neutron balance equation and distinguish between the transport and diffusion theory formulations.

Lesson 2 Objectives (cont.)

Describe Fick's Law and identify some of its limitations.

Identify and explain the common boundary conditions needed when solving the multigroup balance equation.

Write the diffusion equation using matrix operator notation and specialize this to the three usual situations that occur for most applications.

Neutron Life Cycle

In a thermal system, fission neutrons are born at high energy, they slow down via inelastic and elastic neutron scattering, and then, as thermal neutrons, they cause additional fissions to continue the cycle…

Scattering only changes the neutron energy level (represents both production and loss mechanisms in a multigroup formulation).

Parasitic absorption and neutron leakage can occur in all energy groups. These are the ultimate loss mechanisms.

Fission is the primary neutron source in critical reactors, but there are a variety of other neutron sources used in subcritical systems.

Neutron Life Cycle (cont.)

When the neutron production and loss rates are in balance, then the neutron population remains constant or, in steady state, we have

for steady state

Rate of Change = Production Rate – Loss Rate = 0

external source
fission source
inscatter source

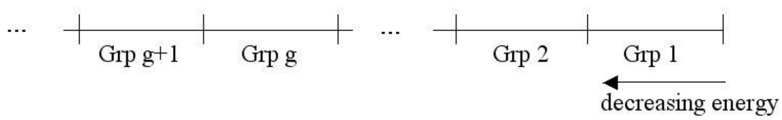
leakage
absorption
outscatter

The Multigroup Formulation

Within the multigroup formulation, the full continuous energy domain is broken into a finite number of energy bins.

An arbitrary energy bin or interval is usually given the symbol g and the groups are numbered from high energy to low energy.

Thus, the energy scale in the multigroup formulation can be represented as:



The Multigroup Formulation (cont.)

Since the full space-energy treatment is too complex for most realistic situations, our goal becomes one of finding out what happens, on the average, in each energy interval, E_{g+1} to E_g (there are G+1 energy boundaries for G total energy groups).

A neutron balance will be performed for an arbitrary energy group g, and this balance will be applied to each interval resulting in a set of G coupled differential equations…

Let's do this in detail…

The Multigroup Formulation (cont.)

In the multigroup formulation, the flux for energy group g is given as

$$\phi_g = \int_{E_{g+1}}^{E_g} \phi(E)dE$$

where it should be noted that this is an energy integrated value.

With this definition, several of the reaction rates that will be needed when writing the neutron balance equation are given as

$$\begin{matrix}\text{\# of absorptions/cm}^3\text{-sec} \\ \text{within energy group g}\end{matrix} = \int_{E_{g+1}}^{E_g} \Sigma_a(E)\phi(E)dE = \Sigma_{ag}\phi_g \quad \boxed{\text{absorption rate}}$$

$$\begin{matrix}\text{\# of fissions/cm}^3\text{-sec} \\ \text{within energy group g}\end{matrix} = \int_{E_{g+1}}^{E_g} \Sigma_f(E)\phi(E)dE = \Sigma_{fg}\phi_g \quad \boxed{\text{fission rate}}$$

The Multigroup Formulation (cont.)

The average cross sections are given by

$$\Sigma_{ag} = \frac{\int_{E_{g+1}}^{E_g} \Sigma_a(E)\phi(E)dE}{\int_{E_{g+1}}^{E_g} \phi(E)dE} \quad \text{and} \quad \Sigma_{fg} = \frac{\int_{E_{g+1}}^{E_g} \Sigma_f(E)\phi(E)dE}{\int_{E_{g+1}}^{E_g} \phi(E)dE}$$

etc… ϕ_g

Also, if we desire the energy integrated reaction rates, then

$$\#\text{ of absorptions/cm}^3\text{-sec} = \int_{\substack{\text{all}\\\text{energy}}} \Sigma_a(E)\phi(E)dE = \sum_{g=1}^{G} \Sigma_{ag}\phi_g$$

$$\#\text{ of fissions/cm}^3\text{-sec} = \int_{\substack{\text{all}\\\text{energy}}} \Sigma_f(E)\phi(E)dE = \sum_{g=1}^{G} \Sigma_{fg}\phi_g$$

The Multigroup Formulation (cont.)

The above reactions (absorption, fission, capture, total, etc.) are sometimes called one-dimensional (1-D) processes since they only involve neutrons at a single energy or within a given energy group.

Neutron scattering, on the other hand, is a 2-D process, since we must consider the final neutron energy as well as the initial neutron energy.

For example, the scattering rate from energy E' to energy interval dE can be written as

$$\begin{matrix}\text{scattering rate}\\ \text{from E' to dE}\end{matrix} = \begin{pmatrix}\text{scattering rate}\\ \text{at E'}\end{pmatrix}\begin{pmatrix}\text{probability of}\\ \text{scattering into dE}\end{pmatrix}$$

$$= \Sigma_s(E')\phi(E')f(E' \to E)dE$$

The Multigroup Formulation (cont.)

To simplify the notation slightly, we define the scattering cross section from E' to E as

$$\Sigma_s(E' \to E) = \Sigma_s(E')f(E' \to E)$$

and, with this definition, we have

$$\begin{matrix}\text{scattering rate} \\ \text{from E' to dE}\end{matrix} = \Sigma_s(E' \to E)\phi(E')dE$$

The Multigroup Formulation (cont.)

Now, when using a discrete energy group notation within the multigroup formulation, we can define the group-to-group scattering cross section as

$$\Sigma_{g' \to g} = \frac{\int_{E_{g'+1}}^{E_{g'}} \int_{E_{g+1}}^{E_g} \Sigma_s(E' \to E)\phi(E')dE\,dE'}{\int_{E_{g'+1}}^{E_{g'}} \phi(E')dE'}$$

and the scattering rate from g' to g as

$$\begin{matrix}\text{scattering rate} \\ \text{from g' to g}\end{matrix} = \Sigma_{g' \to g}\phi_{g'}$$

The Multigroup Formulation (cont.)

The total scattering rate out of a particular group g is given as

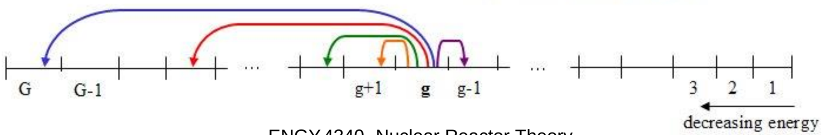
$$\begin{array}{c}\text{total scattering}\\ \text{rate out of g}\end{array} = \sum_{g' \neq g} \Sigma_{g \to g'} \phi_g \qquad \boxed{\text{outscatter rate}}$$

where the notation and ordering of the g and g' are important.

The reaction takes place in group g and the sum is over all groups g' that neutrons in group g can scatter into.

Note that the ≠ symbol indicates that group g is not included in the sum, since within group scattering, $\Sigma_{g \to g} \phi_g$, is not a removal (or source) mechanism for neutrons in group g.

Outscatter from group g



The Multigroup Formulation (cont.)

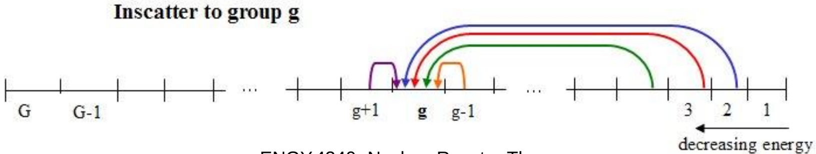
Similarly, the total scattering rate into group g is given by

$$\begin{array}{c}\text{total scattering}\\ \text{rate into g}\end{array} = \sum_{g' \neq g} \Sigma_{g' \to g} \phi_{g'} \qquad \boxed{\text{inscatter source}}$$

The reaction takes place in group g' and the neutron after scattering ends up in g -- and, for the total scattering rate into group g, the sum is over all groups, g', that scatter into group g.

Upscatter can only occur when there are several narrow low energy groups -- because, for the neutron to gain energy in a collision, the target must have a higher initial energy. This can only occur if the neutron energy is below a few eV.

Inscatter to group g



The Multigroup Formulation (cont.)

Another 2-D production process of interest is related to the neutrons produced from the fission event -- that is, the fission reaction takes place at some neutron energy, E', and the neutrons that are emitted have some different energy, E.

The term that describes this process is referred to as the fission source term, $S^{fis}(E)$ or S_g^{fis} .

This source term contains information about the fission rate, the average number of neutrons emitted per fission, and the fission spectrum [i.e. $\chi(E)$, which gives the distribution of energies for the emitted neutrons].

The Multigroup Formulation (cont.)

The fission source term was addressed in some detail in the Fundamentals of NSE course and, for steady state operation, this is given as

$$\begin{matrix}\text{neutrons emitted per} \\ \text{cm}^3\text{-sec in interval dE}\end{matrix} = \chi(E)dE\int \nu\Sigma_f(E')\phi(E')dE'$$

$$\begin{matrix}\text{neutrons emitted per} \\ \text{cm}^3\text{-sec in group g}\end{matrix} = \chi_g\sum_{g'}\nu\Sigma_{fg'}\phi_{g'}$$

fission source

Note that, for transient problems, the distinction between prompt and delayed neutrons is important. We will address this situation later in the semester and, at that time, modify the fission source representation given above to properly treat both prompt and delayed neutrons…

Neutron Leakage

In the multigroup formulation, neutrons that exist within group g can ultimately:

1. Be absorbed within group g (via capture, fission, etc.),
2. Scatter out of group g, or
3. Leak out of the spatial element of interest (while in group g).

From our previous definitions and equations, we know how to describe the absorption and scattering rates, but we have not yet looked at the third loss mechanism, neutron leakage.

To address this topic, let's define the net neutron current, $\vec{J}(\vec{r},E)$, as

$$\vec{J}(\vec{r},E) = \int_{\substack{\text{all}\\\text{angles}}} n(\vec{r},E,\hat{\Omega})\vec{v}(E)d\hat{\Omega} = \int_{\substack{\text{all}\\\text{angles}}} n(\vec{r},E,\hat{\Omega})v(E)\hat{\Omega}d\hat{\Omega}$$

where n is the neutron density, v is the neutron speed, and $\hat{\Omega}$ is a unit vector that describes the direction of travel.

Neutron Leakage (cont.)

Since ϕ = nv, we can also write the net neutron current, $\vec{J}(\vec{r},E)$, as

$$\vec{J}(\vec{r},E) = \int_{\substack{\text{all}\\\text{angles}}} \hat{\Omega}\phi(\vec{r},E,\hat{\Omega})d\hat{\Omega} = \int_{\substack{\text{all}\\\text{angles}}} \vec{J}(\vec{r},E,\hat{\Omega})d\hat{\Omega}$$

Note that $\vec{J}(\vec{r},E)$, or $\vec{J}_g(\vec{r})$ if one integrates over energy interval $\Delta E_g = E_g - E_{g+1}$, is a vector quantity and it represents the net neutron current density, since $\vec{J}(\vec{r},E,\hat{\Omega})$ has been integrated over all angles.

The direction of $\vec{J}_g(\vec{r})$ is not that of any specific collection of neutrons -- it has the direction of the net flow of neutrons.

Note also that the units of net neutron current are the same as neutron flux, neutrons/cm^2-sec.

However, the current density is a vector quantity, and it describes the net directional behavior of the neutrons.

Neutron Leakage (cont.)

Since $\vec{J}_g$ is associated with the net flow of neutrons in energy group g, then

$$\vec{J}_g \cdot d\vec{A} = \vec{J}_g \cdot \hat{n}\, dA = \begin{array}{c}\text{net rate at which neutrons in group g} \\ \text{pass through a surface area dA normal} \\ \text{to the outward pointing unit vector}\end{array}$$

where we note that current density times area gives neutrons/sec -- that is, a neutron flow rate across differential area dA.

With this interpretation, we can now define neutron leakage as the net number of neutrons/sec that leave a given volume V enclosed by surface area A, or

$$\begin{array}{c}\text{leakage rate out} \\ \text{of volume V for} \\ \text{energy group g}\end{array} = \int_A \vec{J}_g \cdot \hat{n}\, dA$$

Neutron Leakage (cont.)

At times, it is convenient to convert the surface integral to a volume integral over V using the well-known Divergence Theorem, or

$$\begin{array}{c}\text{leakage rate} \\ \text{from V and g}\end{array} = \int_A \vec{J}_g \cdot \hat{n}\, dA = \int_V \vec{\nabla} \cdot \vec{J}_g\, d\vec{r}$$

surface and volume integral formulations for neutron leakage

Neutron Balance Equation

We now have all the pieces required to write the general multi-group neutron balance equation.

For some arbitrary volume V and energy group g, one has

accumulation rate = production rate – loss rate

or

rate of change of neutrons within V and g	=	production rate of neutrons within V and g	–	loss rate of neutrons within V and g
Term 1	=	Term 2	–	Term 3
accumulation rate of neutrons in V and g		external source fission source inscatter source		leakage absorption outscatter

Neutron Balance Equation (cont.)

Writing Term 1 in full detail gives

$$\text{Term 1} = \frac{d}{dt}\int_V \int_E n(\vec{r},E,t)\,d\vec{r}\,dE = \frac{d}{dt}\int_V \int_E \frac{n(\vec{r},E,t)v(E)\,d\vec{r}\,dE}{v(E)}$$

or

$$\text{Term 1} = \frac{d}{dt}\int_V \frac{1}{v_g}\phi_g(\vec{r},t)\,d\vec{r} = \int_V \frac{1}{v_g}\frac{\partial}{\partial t}\phi_g(\vec{r},t)\,d\vec{r}$$

implies a fixed volume element

Neutron Balance Equation (cont.)

Also writing Terms 2 and 3 in detail gives

Term 2 = $\int_V Q_g($ [redacted]

external source fission source inscatter source

$$\text{Term 3} = \int_V \vec{\nabla}\cdot\vec{J}_g(\vec{r})d\vec{r} + \int_V \Sigma_{ag}(\vec{r})\phi_g(\vec{r})d\vec{r} + \sum_{g'\neq g}\int_V \Sigma_{g\to g'}(\vec{r})\phi_g(\vec{r})d\vec{r}$$

leakage absorption outscatter

where we note that Term 2 (the production term) includes the steady state fission source expression [not valid for time dependent problems].

Since the integrals in Terms 1 - 3 are over the same arbitrary volume element, one can simply equate the integrands to obtain a pointwise or space continuous neutron balance equation (per unit volume) for group g.

Neutron Balance Equation (cont.)

Thus, the space-continuous neutron balance equation is

$$\vec{\nabla}\cdot\vec{J}_g(\vec{r}) + \Sigma_{Rg}(\vec{r})\phi_g(\vec{r}) - S_g(\vec{r}) = -\frac{1}{v_g}\frac{\partial}{\partial t}\phi_g(\vec{r})$$

where

$$\Sigma_{Rg}(\vec{r}) = \Sigma_{ag}(\vec{r}) + \sum_{g'\neq g}\Sigma_{g\to g'}(\vec{r}) \quad \boxed{\text{removal cross section}}$$

$$S_g(\vec{r}) = Q_g(\vec{r}) + \chi_g\sum_{g'}\nu\Sigma_{fg'}(\vec{r})\phi_{g'}(\vec{r}) + \sum_{g'\neq g}\Sigma_{g'\to g}(\vec{r})\phi_{g'}(\vec{r}) \quad \boxed{\text{total source}}$$

Notice that this expression has been multiplied by -1. The reason for this is for consistency of notation for steady state cases (i.e. when $\partial\phi_g/\partial t = 0$).

Also note that these equations define the removal cross section and total steady state neutron source.

The above equations completely describe the general neutron balance equation (for no delayed neutrons).

Neutron Balance Equation (cont.)

The time derivative term was introduced and retained up to this point for generality.

However, the balance equation developed here is usually applied to steady state systems.

For steady state, the time derivative vanishes and the steady-state space-continuous neutron balance equation becomes

$$\vec{\nabla}\cdot\vec{J}_g(\vec{r})+\Sigma_{Rg}(\vec{r})\phi_g(\vec{r})=S_g(\vec{r})$$

This simply states that, in steady state, the loss rate (leakage + removal) exactly matches or balances the production rate on a per unit volume basis -- that is, the production and loss terms must balance out at every spatial point and energy group.

This expression, along with the definitions of the removal cross section and the total neutron source, is the desired steady state multigroup neutron balance equation.

Transport vs. Diffusion Theory

Before we can solve the steady state balance equation for a specific system, something needs to be done to relate the neutron current, $\vec{J}_g$, to the neutron flux, ϕ_g, since we now have a single equation with two dependent variables.

There are two approaches for doing this -- one leading to the Transport Equation and the other giving the Diffusion Equation.

The transport equation is derived using the basic definition of neutron current,

$$\vec{J}(\vec{r},E,\hat{\Omega})=\hat{\Omega}\phi(\vec{r},E,\hat{\Omega})$$

Therefore, the $\vec{\nabla}\cdot\vec{J}_g$ term in the general balance equation becomes

$$\vec{\nabla}\cdot\vec{J}_g=\vec{\nabla}\cdot\hat{\Omega}\phi_g=\hat{\Omega}\cdot\vec{\nabla}\phi_g$$

Transport vs. Diffusion Theory (cont.)

Also note that, if self scattering is included in both the inscatter and outscatter terms (so that the balance equation is unaffected), then the removal cross section becomes the total cross section,

$$\Sigma_{tg} = \Sigma_{Rg} + \Sigma_{g\to g} = \Sigma_{ag} + \sum_{g'} \Sigma_{g\to g'}$$

With these substitutions, the steady state Boltzmann transport equation becomes

$$\hat{\Omega}\cdot\vec{\nabla}\phi_g(\vec{r},\hat{\Omega}) + \Sigma_{tg}(\vec{r})\phi_g(\vec{r},\hat{\Omega}) = S_g(\vec{r},\hat{\Omega})$$

where

$$S_g(\vec{r},\hat{\Omega}) = Q_g(\vec{r},\hat{\Omega}) + \chi_g \sum_{g'} \int_{4\pi} \nu\Sigma_{fg'}(\vec{r})\phi_{g'}(\vec{r},\hat{\Omega}')d\hat{\Omega}' + \sum_{g'} \int_{4\pi} \Sigma_{g'\to g}(\vec{r},\hat{\Omega}'\to\hat{\Omega})\phi_{g'}(\vec{r},\hat{\Omega}')d\hat{\Omega}'$$

This expression takes into account the angular dependence of the scattering cross sections and the neutron flux.

Transport vs. Diffusion Theory (cont.)

The formal solution of the transport equation is relatively complicated.

Discrete Ordinates, Monte Carlo, or Integral Transport Theory methods are usually employed for its solution (using one of many of its equivalent representations).

However, a discussion of these methods is beyond the scope of the introductory treatment of Nuclear Reactor Theory given in this course.

For the present discussion, the existence of the Boltzmann Transport Equation and its distinction from diffusion theory are the key points of interest.

Transport vs. Diffusion Theory (cont.)

The Diffusion Equation uses an approximate relationship between the neutron flux and net current density based on the observation that neutrons tend to diffuse from regions of high concentration to regions of low concentration.

Fick's Law states this observation in mathematical terms as

$$\vec{J}_g(\vec{r}) = -D_g(\vec{r})\vec{\nabla}\phi_g(\vec{r})$$

which says that the net neutron current is proportional to the negative gradient of the neutron flux.

The variable D_g is the proportionality constant and it is typically called the diffusion coefficient for group g.

Transport vs. Diffusion Theory (cont.)

Substitution of Fick's Law into the basic steady state neutron balance equation gives the standard multigroup diffusion formulation,

$$-\vec{\nabla}\cdot D_g(\vec{r})\vec{\nabla}\phi_g(\vec{r}) + \Sigma_{Rg}(\vec{r})\phi_g(\vec{r}) = S_g(\vec{r})$$

Multigroup Diffusion Equation

where, for completeness, we rewrite the explicit definitions of Σ_{Rg} and S_g as given previously:

$$\Sigma_{Rg}(\vec{r}) = \Sigma_{ag}(\vec{r}) + \sum_{g'\neq g} \Sigma_{g\to g'}(\vec{r})$$

removal cross section

$$S_g(\vec{r}) = Q_g(\vec{r}) + \chi_g \sum_{g'} \nu\Sigma_{fg'}(\vec{r})\phi_{g'}(\vec{r}) + \sum_{g'\neq g} \Sigma_{g'\to g}(\vec{r})\phi_{g'}(\vec{r})$$

total source

This set of equations (i.e. the multigroup diffusion approximation to neutron transport) will be the basis for the remainder of our study of steady state reactor theory…

Limitations of Fick's Law

It should be noted that Fick's Law is only an approximation. In particular, it is not strictly valid:

a. in a medium that strongly absorbs neutrons (i.e. near control rods),

b. within a few mean free paths of either a neutron source or the exterior surface of a medium (the neutron flux has a strong angular dependence in these regions), and

c. when the scattering of neutrons is strongly anisotropic (has a strong angular dependence).

In general, when the angular dependence is not extreme, then Fick's Law represents a good approximation.

However, even with its limitations, the diffusion equation is a reasonable mathematical representation of the neutron behavior within a reactor core.

Limitations of Fick's Law (cont.)

The validity of Fick's law weakens as one approaches the core periphery and shield regions, but even in these situations it gives a rough estimation of neutron diffusion and attenuation.

For core physics studies, diffusion theory is often used for modeling multidimensional systems.

On the other hand, transport theory is usually used for modeling fuel cell and assembly configurations in cross section collapsing codes, and in treating shielding analysis problems. The angular dependence of the neutron flux is usually a key consideration in these applications.

However, in large homogenized core regions, where the isotropic fission source is the dominant source of neutrons, the diffusion equation (and Fick's Law) is usually an adequate approximation.

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The Diffusion Coefficient

As we have seen, the diffusion coefficient appears as a proportionality constant in an approximate expression for the neutron current in terms of the neutron flux (i.e. within Fick's Law).

Using transport theory methods, one can compute ϕ_g and J_g directly, and then determine consistent values for D_g.

This procedure shows that a good approximation to D_g is given by

units of D_g are cm

$$D_g = \frac{1}{3\Sigma_{trg}} \quad \text{where} \quad \Sigma_{trg} = \Sigma_{tg} - \bar{\mu}_0 \Sigma_{sg}$$

where Σ_{trg} is the transport cross section for group g.

The Diffusion Coefficient (cont.)

Note that $\Sigma_{ag} << (1-\bar{\mu}_0)\Sigma_{sg}$ for many materials (especially for moderators). When this is true, then the transport cross section is often written as

$$\Sigma_{trg} \approx (1-\bar{\mu}_0)\Sigma_{sg}$$

The transport cross section is a derived quantity which is written in terms of the total cross section and the scattering cross section.

Also, recall that $\bar{\mu}_0$ is the average value of the scattering angle in the laboratory system for isotropic scattering in the center-of-mass (CM) system.

In few-group cross section libraries, σ_{trg} (note that $\Sigma_{trg} = N\sigma_{trg}$) is usually tabulated with the other basic data (σ_{ag}, σ_{fg}, etc.).

Boundary Conditions

Before we can actually solve the diffusion equation, one must address what happens at the boundaries of the system.

The diffusion equation is a second-order differential equation in the spatial variable (because of the leakage term, $-\vec{\nabla} \cdot D_g \vec{\nabla} \phi_g$).

Therefore, it requires two boundary conditions for each spatial dimension to obtain the complete solution for a particular problem situation. There are three types of conditions discussed here:

General Boundary Conditions

1. **The neutron flux must be real and non-negative.**
2. **The neutron flux must be finite (except at artificial singular points of a source distribution).**
3. **The neutron current is zero at symmetry boundaries (no net current across boundary).**

Boundary Conditions (cont.)

Interface Boundary Conditions

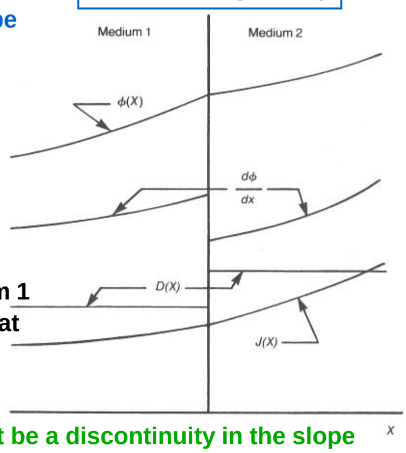
1-D Cartesian geometry

1. Continuity of Flux -- the flux must be continuous across a material interface:

$$\lim_{\varepsilon \to 0} \phi(x_o - \varepsilon) = \lim_{\varepsilon \to 0} \phi(x_o + \varepsilon)$$

2. Continuity of Current -- the current must be continuous across a material interface:

$$\lim_{\varepsilon \to 0} J(x_o - \varepsilon) = \lim_{\varepsilon \to 0} J(x_o + \varepsilon)$$



Using the subscript 1 to denote medium 1 and 2 to refer to medium 2, this says that

$$-D_1 \frac{d}{dx}\phi\bigg|_{x_o\,(\text{from left})} = -D_2 \frac{d}{dx}\phi\bigg|_{x_o\,(\text{from right})}$$

Thus, if $D_1 \neq D_2$, then clearly there must be a discontinuity in the slope of the flux at the interface.

Boundary Conditions (cont.)

External (Vacuum) Boundary Conditions

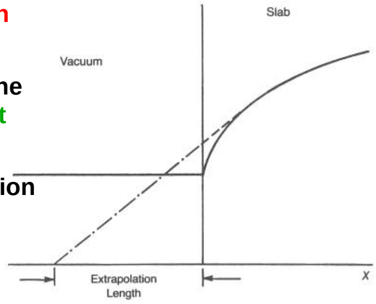
1-D Cartesian geometry

First we note that Fick's Law is not valid in this situation.

The true vacuum boundary condition is one which specifies that there are no reentrant neutrons.

Using 1-D Cartesian geometry, this condition can be written mathematically as

$$\phi_g(x,\hat{\Omega})\Big|_{x=x_o} = 0 \quad \text{for } \hat{\Omega} > 0 \text{ for left boundary}$$



For use with Fick's Law, one can modify this condition such that the diffusion theory flux approximates the transport theory result:

> At an external boundary, the diffusion theory flux vanishes at some small distance, d, beyond the external surface, where d is referred to as the extrapolation distance.

Boundary Conditions (cont.)

Using transport theory as the true solution, a good approximation for the extrapolation length for 1-D Cartesian geometry is

$$d = 0.71\lambda_{tr} = \frac{0.71}{\Sigma_{tr}} = 0.71(3D) = 2.13D$$

However, for large power reactors, d is usually small compared to realistic reactor dimensions and can often be ignored -- but, for small bare critical systems, d cannot be neglected!!!

The vacuum boundary condition for use in diffusion theory can be written as (again, we use 1-D Cartesian geometry to simplify the notation):

$$\phi_g(x)\Big|_{x=x_o \pm d} = 0 \qquad \text{or} \qquad \phi_g(x)\Big|_{x=x_o} = 0 \quad \text{for } |x_o| >> d$$

where x_o is the location of the external boundary of the system and $x_o \pm d$ refers to the extrapolated boundary (the plus sign is used for a right boundary and the negative sign is used on the left side).

Operator Form

The steady state multigroup diffusion equation represents a set of G coupled 2nd order differential equations.

With the discrete g and g′ notation and the summation symbol to represent an integration over energy occurring in many of the terms, it becomes a little tedious to write out these relationships in full detail every time one wants to discuss the basic balance equation.

Thus, it certainly would be convenient to be able to write the general steady state diffusion equation using some simplified notation.

Towards this goal, let's define the following matrix operators,

$$\underline{\underline{L}} = -\vec{\nabla} \cdot \underline{\underline{D}} \vec{\nabla} + \underline{\underline{\Sigma}}_R - \underline{\underline{\Sigma}}_S^I \quad \text{and} \quad \underline{\underline{F}} = \underline{\chi} \nu \underline{\underline{\Sigma}}_f$$

(leakage + removal – inscatter) (fission)

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Operator Form (cont.)

where

$$\underline{\underline{D}} = \begin{bmatrix} D_1 & & & & \\ & D_2 & & 0 & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & \cdot & \\ & & & & D_G \end{bmatrix} \qquad \underline{\underline{\Sigma}}_R = \begin{bmatrix} \Sigma_{R1} & & & & \\ & \Sigma_{R2} & & 0 & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & \cdot & \\ & & & & \Sigma_{RG} \end{bmatrix}$$

$$\underline{\underline{\Sigma}}_S^I = \begin{bmatrix} 0 & \Sigma_{2\to1} & \Sigma_{3\to1} & \cdots & \Sigma_{G\to1} \\ \Sigma_{1\to2} & 0 & \Sigma_{3\to2} & \cdots & \Sigma_{G\to2} \\ \Sigma_{1\to3} & \Sigma_{2\to3} & 0 & \cdots & \Sigma_{G\to3} \\ & & \vdots & & \\ \Sigma_{1\to G} & \Sigma_{2\to G} & \Sigma_{3\to G} & \cdots & 0 \end{bmatrix}$$

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Operator Form (cont.)

and

$$\underline{\underline{\chi}} = \begin{bmatrix} \chi_1 & & & & \\ & \chi_2 & & 0 & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & & \cdot \\ & & & & \chi_G \end{bmatrix} \qquad \underline{\underline{\nu\Sigma_f}} = \begin{bmatrix} \nu\Sigma_{f1} & \nu\Sigma_{f2} & \nu\Sigma_{f3} & \cdots & \nu\Sigma_{fg} \\ \nu\Sigma_{f1} & \nu\Sigma_{f2} & \nu\Sigma_{f3} & \cdots & \nu\Sigma_{fG} \\ \nu\Sigma_{f1} & \nu\Sigma_{f2} & \nu\Sigma_{f3} & \cdots & \nu\Sigma_{fG} \\ & & \vdots & & \\ \nu\Sigma_{f1} & \nu\Sigma_{f2} & \nu\Sigma_{f3} & \cdots & \nu\Sigma_{fG} \end{bmatrix}$$

Finally, defining the group flux vector, $\underline{\phi}$, and external source vector, $\underline{Q}$, as

$$\underline{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_G \end{bmatrix} \qquad \text{and} \qquad \underline{Q} = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_G \end{bmatrix}$$

Operator Form (cont.)

one can write the operator form of the diffusion equation as

$$\left[-\vec{\nabla}\cdot\underline{\underline{D}}\vec{\nabla} + \underline{\underline{\Sigma}}_R - \underline{\underline{\Sigma}}_S^I\right]\underline{\phi} - \left[\underline{\underline{\chi}}\,\underline{\underline{\nu\Sigma_f}}\right]\underline{\phi} = \underline{Q} \quad \text{or} \quad \boxed{\left(\underline{\underline{L}} - \underline{\underline{F}}\right)\underline{\phi} = \underline{Q}}$$

To see these definitions at work, let's look at the specifics of the 2-group problem.

With full expansion of Σ_{Rg} and S_g, for the group 1 equation, we let g = 1 and vary g' = 1,2 accordingly and, for the group 2 equation, we simply let g = 2 and again vary g' over all appropriate groups. Doing this gives

$$-\vec{\nabla}\cdot D_1\vec{\nabla}\phi_1 + \left(\Sigma_{a1} + \Sigma_{1\to 2}\right)\phi_1 - \Sigma_{2\to 1}\phi_2 - \chi_1\left(\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2\right) = Q_1$$

$$-\vec{\nabla}\cdot D_2\vec{\nabla}\phi_2 + \left(\Sigma_{a2} + \Sigma_{2\to 1}\right)\phi_2 - \Sigma_{1\to 2}\phi_1 - \chi_2\left(\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2\right) = Q_2$$

Operator Form (cont.)

Now, putting these two equations into matrix form gives

$$\begin{bmatrix} -\vec{\nabla}\cdot D_1\vec{\nabla} & 0 \\ 0 & -\vec{\nabla}\cdot D_2\vec{\nabla} \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} + \begin{bmatrix} \Sigma_{a1}+\Sigma_{1\to2} & 0 \\ 0 & \Sigma_{a2}+\Sigma_{2\to1} \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} - \begin{bmatrix} 0 & \Sigma_{2\to1} \\ \Sigma_{1\to2} & 0 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$-\begin{bmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{bmatrix}\begin{bmatrix} \nu\Sigma_{f1} & \nu\Sigma_{f2} \\ \nu\Sigma_{f1} & \nu\Sigma_{f2} \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

However, for most 2-group thermal reactor problems, $\chi_1 = 1.0$ and $\chi_2 = 0.0$ (that is, there is no fission source in group 2) and upscatter is usually negligible (i.e. $\Sigma_{2\to1} = 0$).

With these conditions, the above expressions can be written as

$$\left(\underline{\underline{L}} - \underline{\underline{F}}\right)\underline{\phi} = \underline{Q}$$

Operator Form (cont.)

where

$$\underline{\underline{L}} = \begin{bmatrix} -\vec{\nabla}\cdot D_1\vec{\nabla} & 0 \\ 0 & -\vec{\nabla}\cdot D_2\vec{\nabla} \end{bmatrix} + \begin{bmatrix} \Sigma_{a1}+\Sigma_{1\to2} & 0 \\ 0 & \Sigma_{a2} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \Sigma_{1\to2} & 0 \end{bmatrix}$$

or

$$\underline{\underline{L}} = \begin{bmatrix} -\vec{\nabla}\cdot D_1\vec{\nabla}+\Sigma_{R1} & 0 \\ -\Sigma_{1\to2} & -\vec{\nabla}\cdot D_2\vec{\nabla}+\Sigma_{a2} \end{bmatrix}$$

and

$$\underline{\underline{F}} = \begin{bmatrix} \nu\Sigma_{f1} & \nu\Sigma_{f2} \\ 0 & 0 \end{bmatrix}$$

Thus, we see that the use of the expression $\left(\underline{\underline{L}} - \underline{\underline{F}}\right)\underline{\phi} = \underline{Q}$ or $(L - F)\phi = Q$ is simply a matter of convenience, as long as one understands the precise meaning and significance of the individual terms.

Typical Applications

The above development and notation for the steady state multigroup diffusion equation is quite general. For any specific application, however, only the applicable terms are used.

In most cases of interest, one of the following three situations arise:

1. Subcritical non-multiplying system (no fission source):

$$L\phi = Q$$

2. Subcritical multiplying system (fission & external sources):

$$(L - F)\phi = Q$$

3. Critical system (no external sources):

$$(L - \lambda F)\phi = 0$$

Typical Applications (cont.)

Subcritical non-multiplying system: This first case is applicable primarily in shield design applications and for non-multiplying fusion blanket design. This situation represents a subcritical geometry with no fission source. The neutron balance equation for this case states that leakage + removal – inscatter = external source.

Using our condensed notation, this can be written as $L\phi = Q$.

Subcritical multiplying system: This second case must be considered in situations where both the fixed source and fission source are important.

This is given in equation form as $(L - F)\phi = Q$.

The most common situation where this case arises is during reactor startup and shutdown periods.

Typical Applications (cont.)

Clearly, a reactor core has a substantial fission potential, but it may be arranged in a subcritical configuration (either by having some assemblies missing or by having large amounts of control inserted).

Without an external source, there would be no steady state flux in this subcritical arrangement.

In most fuel there is an inherent neutron source due to the spontaneous fission and (α,n) reactions that are associated with the higher actinides.

The neutrons emitted from these reactions, or from an externally applied fixed source, undergo subcritical multiplication (they cause fission in the fuel material) and give rise to a steady state neutron distribution throughout the system.

Typical Applications (cont.)

Critical system: The third and most important class of problems that arises is the critical reactor problem. In this situation, the total leakage and absorption rates exactly balance the neutron production from fission, and any inherent neutron source that may be present in the fuel is totally dominated by the fission source. Since the fixed source is negligible, it is simply dropped from the defining equations.

For a steady state critical system, there has to be a very precise balance between the neutron production rate from fission and the total loss rate (absorption + leakage). Any arbitrary mixture of materials will not satisfy this constraint.

To emphasize this required (i.e. "critical") balance, one usually includes a mathematical eigenvalue, λ, before the fission source term, giving $(L - \lambda F)\phi = 0$.

Significance of the Eigenvalue, λ

In a critical operating reactor, λ is unity (exactly unity!!!).

In design analysis, however, we often want to know if a particular combination of materials will give a critical reactor.

Thus, for any given material distribution, λ is computed as part of the solution procedure.

It is allowed to vary from unity in the calculation so that the equation can be balanced mathematically (i.e. λ * production = loss) -- this is necessary to have a steady state configuration.

However, in an operating critical system, λ must be unity!!!

To see the significance of λ, let's integrate the operator form of the diffusion equation over all space and energy, giving

$$\langle \mathbf{L}\phi \rangle - \lambda \langle \mathbf{F}\phi \rangle = 0 \qquad \text{or} \qquad \lambda = \frac{\langle \mathbf{L}\phi \rangle}{\langle \mathbf{F}\phi \rangle} = \frac{\text{loss rate}}{\text{production rate}}$$

Significance of the Eigenvalue, λ

The term < Fϕ > represents the total neutron production rate from fission. Also, when performing integration over all energy, the inscatter and outscatter components within the < Lϕ > term exactly cancel. Therefore, < Lϕ > represents the total loss rate (leakage + absorption).

From the definition of the multiplication factor, k,

$$k_{eff} = \frac{\text{production rate}}{\text{loss rate}} \qquad \text{we have} \qquad \lambda = \frac{1}{k_{eff}}$$

Thus, we see that the addition of the eigenvalue within the defining equation is quite justifiable -- but, at operating conditions, $k_{eff} = 1/\lambda = 1.000$.

If the calculated k_{eff} is not unity for a given geometry/material configuration, this just tells the designer that a material or geometry modification is required for criticality (e.g. more control may need to be inserted or removed).

Neutron Balance Equation: A Summary

This lecture has focused on the development of the multigroup neutron balance equation, with special emphasis on the diffusion theory approximation.

It first identified all the neutron production and loss mechanisms that can occur within a nuclear system, and then put these together to give the desired steady state neutron balance relationship.

In addition, a shorthand operator notation was also introduced and utilized to overview the three primary classes of problems that can be addressed using the equations developed here.

However, the material presented here only represents a foundation for further study, since no specific examples were attempted.

Our goal here was to develop a strong base to support further work in reactor theory…

Lesson 2 Summary

In this Lesson we have briefly discussed/reviewed the following subjects:

The notation and discretization process for the continuous energy variable -- which results in the standard multigroup formulation.

Formal expressions for the appropriately averaged multigroup cross sections for all the important 1-D and 2-D neutron interaction processes.

The key production and loss terms needed to develop the steady state multigroup neutron balance equation, and formal expressions for each of these terms using standard multigroup notation.

The differential form of the multigroup neutron balance equation and the difference between the transport and diffusion theory formulations.

Lesson 2 Summary (cont.)

Fick's Law and some of its limitations.

The common boundary conditions needed when solving the multigroup balance equation.

The diffusion equation written using matrix operator notation and the three usual situations that occur for most applications.