Neutron Interactions with Matter (i.e. Neutron Cross Sections)

Introduction

Up to this point we have mentioned the term ***cross section*** many times. It is about time that we make a formal definition and give some insight into the behavior and use of neutron cross sections in describing a variety of neutron interactions with matter -- and this section of the Lecture Notes will give an overview of this relatively broad (and somewhat complex) subject. We should note that the text ***Introduction to Nuclear Engineering*** by Lamarsh and Baratta treats this subject in Chapter 3 and, towards the end of the chapter, also addresses the interaction of gamma rays and charged particles. These latter topics are extremely important and you should consult Lamarsh for an overview of these subjects -- in this section of notes we will only focus on neutron interactions…

In general, one can informally describe σ_x as follows:

σ_x = ***measure of the probability*** that interaction x will occur when neutrons strike a target

or σ_x = the ***effective cross sectional area*** of a nucleus for reaction x

Note that σ_x is referred to as the ***microscopic cross section*** and it has units of ***barns***, where 1 b = 10^{-24} cm^2). The x in the above description for σ_x refers to the type of neutron reaction that takes place, and some of the more important reactions and their common notations are given below:

absorption processes:

radiative capture -- $(n,\gamma) \rightarrow \sigma_c$ or σ_γ

charged particle -- (n,α) and $(n,p) \rightarrow \sigma_{n,\alpha}$ and $\sigma_{n,p}$

neutron producing -- $(n,2n)$ and $(n,3n) \rightarrow \sigma_{n,2n}$ and $\sigma_{n,3n}$

fission -- $(n,f) \rightarrow \sigma_f$

Note that a neutron absorption reaction causes the target atom to become something different. Thus, we refer to the total absorption cross section as the sum of all the individual partial cross sections for each of the absorption processes, or

$$\sigma_a = \sigma_c + \sigma_f + \sigma_{n,p} + \sigma_{n,\alpha} + \sigma_{n,\,2n} + \sigma_{n,\,3n} + \cdots \qquad (1)$$

and since capture and fission are dominant in most cases, one often writes,

$$\sigma_a \approx \sigma_c + \sigma_f \qquad (2)$$

scattering processes:

elastic scattering -- $(n,n) \rightarrow \sigma_{el}$

inelastic scattering -- $(n,n') \rightarrow \sigma_{in}$

Note that neutron scattering leaves the target atom in its original form -- that is, the mass number A doesn't change. As above, we refer to the total scattering cross section as the sum of elastic and inelastic processes, or

$$\sigma_s = \sigma_{el} + \sigma_{in} \qquad (3)$$

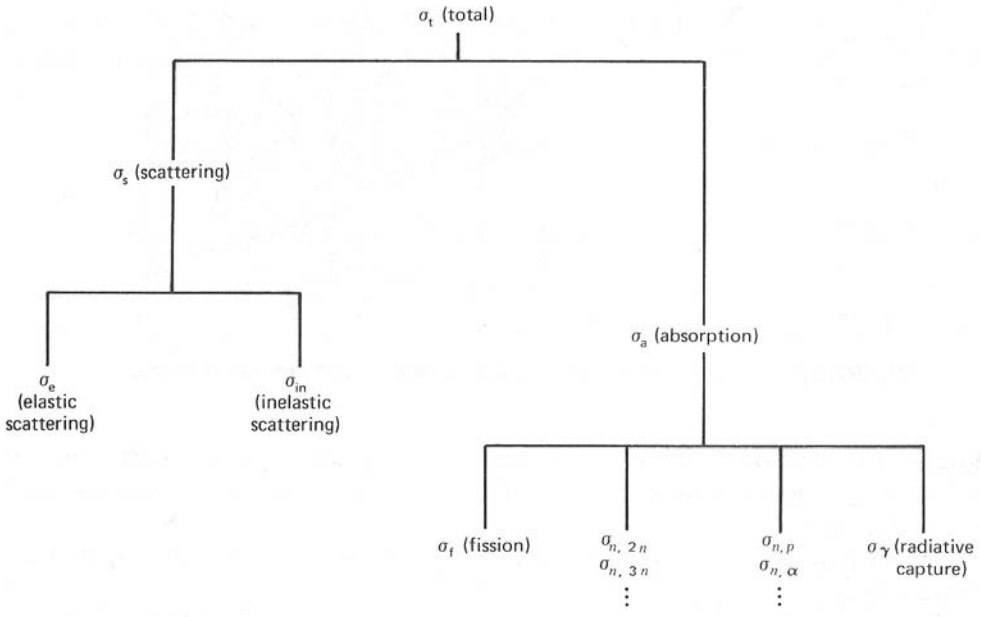
It is important to contrast the differences between elastic and inelastic scattering. Elastic scattering is effectively a classical two-body collision process, where momentum and energy are transferred in the collision, with the target nucleus remaining in its ground state. However, the kinetic energy and momentum of both the target nucleus and the neutron are changed in the process. In contrast, the energy loss in inelastic scattering is through the excitation of the target nucleus, with the subsequent emission of γ-rays as the nucleus decays to its ground state. For inelastic scattering the neutron's kinetic energy must be greater than the energy of the first excited state of the target. Thus, inelastic scattering is a threshold reaction, where the threshold energy decreases with increasing A (because of the large number of excited states at lower energies for large A materials). Inelastic scattering is an important energy transfer process for heavy nuclides (such as U238, for example), but it becomes less of the total contribution for light elements (hydrogen has no inelastic component, for example).

total cross section:

The total interaction rate is related to the sum of the absorption and scattering processes -- and we can write the total microscopic cross section as the sum of all the absorption and scattering cross sections, or

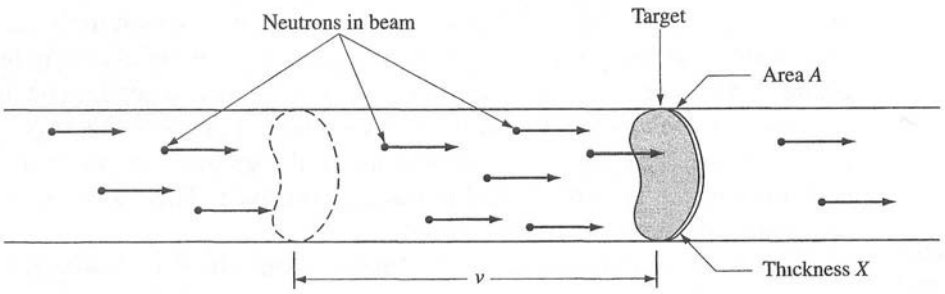
$$\sigma_t = \sigma_a + \sigma_s \qquad (4)$$

The relationships given in eqns. (1) - (4) are summarized nicely in the following sketch showing the various partial cross sections and cross section hierarchy.



Experimental Interpretation

Probably the best way to visualize the concept of a cross section is through a simple experiment. For example, consider a simple arrangement containing a thin target with a monoenergetic beam of neutrons directly bombarding the target. The configuration might look as indicated in the sketch given below.



Let's define some important terms:

n = neutron density in beam (neutrons/cm^3)

v = neutron speed = distance traveled per sec (cm/sec)

I = beam intensity = # of neutrons hitting target per cm^2-sec (note that I = nv)

A = surface area of target (cm^2)

Δx = target thickness (cm) [$\Delta x = X$ in the sketch]

N = atom density of target (atoms/cm^3)

Now, the goal of the experiment is to determine the number of collisions that take place in the target per unit time. The collision rate should be **proportional to the number of target atoms** (the larger the number of target atoms the greater the collision rate) and the **number of neutrons hitting the target per unit time** (the larger the neutron intensity the greater the interaction rate). For now, let's denote σ as the proportionality constant in this relationship. Thus, in equation form, we have

$$\begin{aligned} \begin{pmatrix} \text{\# of collisions} \\ \text{per sec in target} \end{pmatrix} &= \sigma \begin{pmatrix} \text{\# neutrons hitting} \\ \text{target per cm}^2 - \text{sec} \end{pmatrix} \begin{pmatrix} \text{\# of target} \\ \text{atoms} \end{pmatrix} \\ &= \sigma (nv)(NA\Delta x) = \sigma I (NA\Delta x) \end{aligned} \qquad (5)$$

Equation (5) is really the defining expression for the cross section σ. It is simply a proportionality constant that relates the observed quantity in the experiment (in this case, the collision rate) to the known variables that affect the desired quantity. From here we also note that, for consistency of units, σ must have units of cm^2, which gives units of collisions per second on both sides of the equation -- this is why we referred to σ as the "effective" cross sectional area of the target nucleus.

If eqn. (5) is divided by the target volume, we have

$$F = \text{collision rate density} = \begin{pmatrix} \text{\# collisions per} \\ \text{unit volume per sec} \end{pmatrix} = N\sigma_t I \qquad (6)$$

where we have now added the subscript t to indicate that the total number of interactions per sec is related to the total cross section. This expression for the ***collision rate density*** (or ***collision density*** for short) is very important and we will see this combination of terms quite frequently. In fact, the term $N\sigma$ occurs so often that it is given the symbol, Σ, and called the ***macroscopic cross section***. For example,

$$\Sigma_t = N\sigma_t, \qquad \Sigma_f = N\sigma_f, \qquad \text{and} \qquad \Sigma_c = N\sigma_c$$

refer to the macroscopic total, fission, and capture cross sections, respectively. These quantities have units of cm^{-1} -- that is, $(atoms/cm^3) \times (cm^2) = cm^{-1}$. Now, with this definition, the collision density can be written as

$$F = N\,\sigma_t\, I = \Sigma_t\, I \tag{7}$$

Also, in a similar vein, $\Sigma_f I$ would be the fission rate density, $\Sigma_c I$ would be the capture rate density, etc. all with units of interactions/cm^3-s.

Neutron Attenuation and Mean Free Path

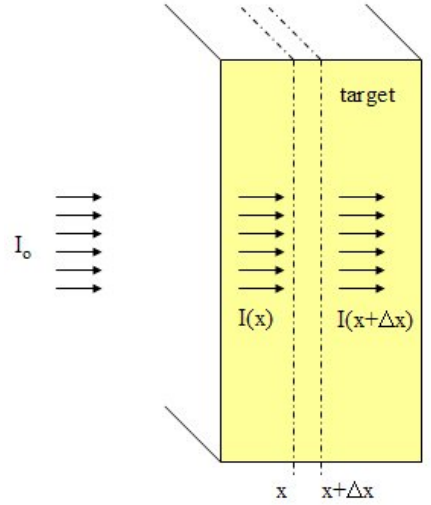
In the above experiment it was assumed that the target material was very thin. For thick targets, the neutron beam will be attenuated by neutron interactions as it passes through the target. To address this situation, let I(x) be the uncollided intensity of neutrons at position x within the target (here uncollided means that they have not had any type of collision as yet). If we perform a balance on a small segment of the target (see sketch), one has

$$\begin{matrix}\#\text{ of collisions} \\ \text{within } \Delta x\end{matrix} = \left[I(x) - I(x+\Delta x)\right] A$$

but

$$\begin{matrix}\#\text{ of collisions} \\ \text{within } \Delta x\end{matrix} = F_t A \Delta x \;=\; \Sigma_t I A \Delta x$$

We now equate these two expressions, divide by Δx, and take the limit as $\Delta x \to 0$. This procedure gives



$$\lim_{\Delta x \to 0} \frac{I(x+\Delta x) - I(x)}{\Delta x} = -\Sigma_t I(x) \tag{8a}$$

or $$\frac{dI(x)}{dx} = -\Sigma_t I(x) \qquad \text{with} \quad I(0) = I_o \tag{8b}$$

where I_o is the initial uncollided neutron beam intensity prior to interaction with the target. The solution to this first order differential equation is simply

$$I(x) = I_o e^{-\Sigma_t x} \tag{9}$$

which says that the uncollided intensity decreases exponentially with distance into the target. Equations (8) and (9) are important for a couple of reasons -- they give another interpretation of the macroscopic cross section and they also allow a quantitative measure of neutron attenuation (only in the sense of the uncollided intensity, however -- see below). For example, rearranging eqn. (8b), one has

$$-\frac{dI(x)}{I(x)} = \Sigma_t dx = \frac{\text{\# of collisions per sec in dx}}{\text{\# of neutrons per sec available at x}} = \begin{array}{c}\text{probability that a neutron which survives}\\ \text{up to x without interaction will}\\ \text{collide in the next dx}\end{array} \tag{10}$$

Or re-worded slightly, one has

$$\Sigma_t = \text{probability per unit path length that a neutron will have a collision} \tag{11}$$

Continuing this line of discussion, from eqn. (9) we have

$$\frac{I(x)}{I_o} = e^{-\Sigma_t x} = \begin{array}{c}\text{probability that a neutron can move through}\\ \text{a distance x without having a collision}\end{array} \tag{12}$$

From eqns (10) - (12), let's define a new probability distribution function as

$$p(x)dx = \begin{array}{c}\text{probability that a neutron will have}\\ \text{its first collision in dx around x}\end{array} = \left(\begin{array}{c}\text{probability that it}\\ \text{survives to x}\end{array}\right)\left(\begin{array}{c}\text{probability that it}\\ \text{collides in dx}\end{array}\right)$$

or $$p(x)dx = e^{-\Sigma_t x}\,\Sigma_t dx \tag{13}$$

Note that this is a properly normalized distribution function since

$$\int_0^\infty p(x)dx = \Sigma_t \left.\frac{e^{-\Sigma_t x}}{-\Sigma_t}\right|_0^\infty = -(0-1) = 1$$

With this distribution function we can define and evaluate an expression for the ***mean free path***, which is defined as the **average distance traveled between collisions**. The mean free path, typically denoted as λ, represents the average distance, $< x >$, that a neutron travels between interactions. As we have seen previously, we can find the desired average value by performing a weighted average, where the weight function, in this case, represents the chance that an uncollided neutron will make it to point x and then have its next collision in interval dx about x -- but, of course, this is the probability function, p(x), just defined.

Thus, we can compute the mean free path, λ, as

$$\lambda = < x > = \frac{\int_0^\infty x\,p(x)dx}{\int_0^\infty p(x)dx} = \Sigma_t \int_0^\infty x e^{-\Sigma_t x} dx = \Sigma_t \left[\frac{e^{-\Sigma_t x}}{\Sigma_t^2}(-\Sigma_t x - 1)\right]_0^\infty = -\frac{1}{\Sigma_t}(0-1) \tag{14}$$

or

$$\lambda = \frac{1}{\Sigma_t} \tag{15}$$

Thus, the average distance that a neutron moves between collisions is simply related to the inverse of the total macroscopic cross section. ***This quantity is useful in characterizing how big a system is from the point of view of the neutron***. If the system is large compared to the mean free path of the neutron at all energies, then the system may be considered infinite. In contrast, if the characteristic dimension of a heterogeneous unit within a large system is small compared to

λ, then the system may be treated as a homogeneous system. This point will be discussed again in conjunction with heterogeneous versus homogeneous critical systems later in this course. For now it is sufficient to comment that the mean free path is an important quantity for characterizing the overall neutronic behavior of a given region.

At this point we also note that the above discussion can be easily extended to materials containing a mixture of elements. In this case, the total probability per unit path length that a neutron interacts with one of the elements is given by

$$\Sigma_t = N_1\sigma_{t1} + N_2\sigma_{t2} + N_3\sigma_{t3} + \cdots = \Sigma_{t1} + \Sigma_{t2} + \Sigma_{t3} + \cdots = \sum_i N_i\sigma_{ti} = \sum_i \Sigma_{ti} \qquad (16)$$

where N_i is the homogenized atom density of the i^{th} component of the mixture and σ_{ti} is the corresponding total cross section. Note that this treatment is also applicable to all the partial cross sections, so the macroscopic fission cross section, Σ_f, macroscopic capture cross section, Σ_c, macroscopic scattering cross section, Σ_s, etc. for a mixture of nuclides can be written in a similar fashion. In particular, the macroscopic cross section for reaction x for a mixture of homogeneous materials is given as

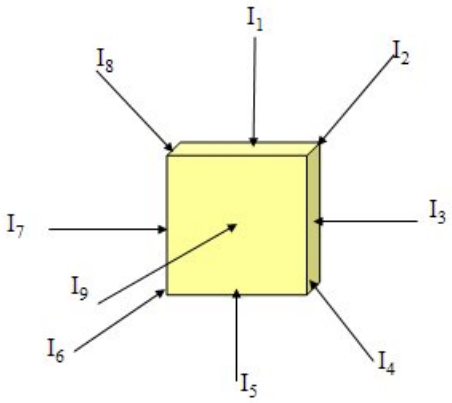
$$\Sigma_x = \sum_i N_i\sigma_{xi} = \sum_i \Sigma_{xi} \qquad (17)$$

Note that eqn. (17) assumes that all the nuclei act independent of one another. This is valid for all neutron interactions except for elastic scattering at low energy in bound molecules and various solids. For example, the scattering cross sections for a free hydrogen gas and for hydrogen bound within a water molecule at low energies are quite different, and this behavior can be an important consideration in modeling the overall behavior of low energy neutrons in thermal systems. This is treated, in practice, by simply having multiple cross sections tabulated for various uses -- for example, it is common to tabulate multiple cross section sets for hydrogen as a free gas, when bound in water, when bound in polyethylene, when bound in zirconium hydride, etc.. When these separate tabulations are available, then eqn. (17) can be used even for x $\rightarrow$ elastic scattering, as long as the appropriate density and cross-section combinations are used. As a final note here, since water is such a common element in many nuclear systems, most cross section tabulations and problem-independent libraries will give the cross section for hydrogen as bound hydrogen in water (but it doesn't hurt to be careful here)…

Although the formal definition of mean free path involves the total cross section [as defined in eqn. (15)], we often extend this concept to include other processes as well. For example, the mean free path for neutron capture is simply $1/\Sigma_c$, the average distance traveled between fission reactions is $1/\Sigma_f$, etc. In general, however, unless otherwise noted, the mean free path, $\lambda = 1/\Sigma_t$, implies the average distance traveled between collisions.

We also note that the cross sections are strong functions of energy (we will elaborate on this shortly), so the mean free path also varies with neutron energy -- that is, $\lambda(E) = 1/\Sigma_t(E)$. In particular, since the low energy cross sections tend to be much larger than at high energy (again, we will see this shortly), the mean free path at low energy tends to be much smaller than at higher energy. Thus, when considering the "size" of a system, the concept of large and small may be dependent on the neutron energy region of interest.

We should note that the above 1-D conceptual experiment can be easily generalized by allowing the small target to be bombarded simultaneously by a number of different beams that strike the target from different directions. This is what occurs in a reactor environment, or any time a target is placed in a general neutron field, where the neutrons are moving in a variety of directions (essentially all directions). Since the interaction of neutrons with nuclei is essentially independent of the incident angle, the total interaction rate within a small target from multiple monoenergetic beams would be



$$F_t = \Sigma_t \left(I_1 + I_2 + I_3 + \cdots\right) = \Sigma_t \left(n_1 + n_2 + n_3 + \cdots\right) v = \Sigma_t n v = \Sigma_t \phi \qquad (18)$$

where the intensity of each beam is related to the neutron density in the beam, $I_i = n_i v$, and the neutron speed, v, is assumed to be the same (here we are assuming that all the neutrons have the same energy -- that is, monoenergetic beams of the same energy). Finally, we write the total neutron density at a point, n, as the sum of the neutron densities in each beam, and this composite density times the neutron speed is referred to as the neutron flux, ϕ.

This situation is the usual case so that, unless we are referring explicitly to a beam experiment, all the collision rates that have been previously defined should use the neutron flux, ϕ, instead of the beam intensity, I. Thus, we can write a number of common reaction rate densities as

total interaction rate: $F_t = \Sigma_t \phi$ fission rate: $F_f = \Sigma_f \phi$ capture rate: $F_c = \Sigma_c \phi$ etc.

Finally, as indicated above, we note again that both the cross section and the flux are strong functions of energy. Thus, the above expressions are only valid at a specific energy, and the reaction rate density really has units of reactions/cm^3-sec per unit energy -- that is, it is also a density function with respect to the energy variable. In most cases, however, we are interested in the total reaction rate density integrated over some energy interval and, if this energy interval covers the full spectrum ($0 \le E \le \infty$), then the above reaction rates become

total interaction rate per unit volume: $F_t = \int_0^\infty \Sigma_t(E)\phi(E)dE$ (19a)

fission rate per unit volume: $F_f = \int_0^\infty \Sigma_f(E)\phi(E)dE$ (19b)

capture rate per unit volume: $F_c = \int_0^\infty \Sigma_c(E)\phi(E)dE$ (19c)

etc.

To formally do these integrals over energy, of course, we need to know the behavior of $\Sigma_x(E)$ and $\phi(E)$ -- and this is discussed, to some extent, in the next subsection…

Before continuing our discussions, however, Examples 1 - 2 first illustrate some of the basic terminology and concepts discussed thus far. In particular, they compute and contrast several cross sections and mean free paths for some important materials common to most nuclear systems.

Example 1: Mean Free Path of Thermal Neutrons in Graphite

To give an explicit example, let's compute the mean free path for thermal neutrons in carbon, where $E_o = 0.0253$ eV is considered as the "thermal" energy for this example. From the ENDFB-VI library in JANIS 3.0, the carbon scattering and absorption cross sections at E_o are $\sigma_s = 4.9$ b and $\sigma_a = 3.4$ mb, respectively. Also, assuming that the mass density of graphite is about 1.6 g/cm^3, the appropriate atom density is

$$N_C = \frac{1.6\,\text{g of C}}{\text{cm}^3} \times \frac{0.6022\times10^{24}\ \text{at. of C}}{12\,\text{g of C}} \times \frac{10^{-24}\,\text{cm}^2}{\text{b}} = 8.03\times10^{-2}\,\frac{\text{at. of C}}{\text{b}-\text{cm}}$$

Now we can compute the macroscopic cross sections at E_o as

$$\Sigma_a = N\sigma_a = \left(8.03\times10^{-2}\right)\left(3.4\times10^{-3}\right) = 2.73\times10^{-4}\ \text{cm}^{-1}$$

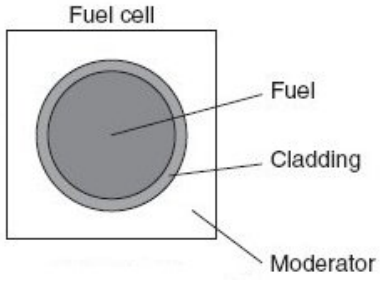
$$\Sigma_s = N\sigma_s = \left(8.03\times10^{-2}\right)\left(4.9\right) = 0.393\,\text{cm}^{-1}$$

and, since $\Sigma_s >> \Sigma_a$, the total cross section becomes $\Sigma_t = \Sigma_a + \Sigma_s \approx \Sigma_s = 0.393$ cm^{-1}. Finally, using eqn. (15), the mean free path in graphite at $E_o = 0.0253$ eV is $\lambda = 1/\Sigma_t \approx 2.5$ cm. Thus, on the average, a thermal neutron will travel about 2.5 cm between collisions in graphite.

Note also how small the absorption cross section is relative to its scattering cross section. Using the above data, we have that $\Sigma_s/\Sigma_a > 1400$, and this says that a thermal neutron in graphite will have over 1400 scattering collisions on the average before being absorbed -- and this is a nice feature of a thermal scattering material in nuclear systems!

Example 2: Thermal Cross Sections for a PWR Fuel Pin

This example re-visits Ex. #5 in the Lecture Notes: Basic Atomic and Nuclear Physics. In that example, we computed the appropriate region densities for the fuel, clad, and moderator (water) regions for a typical PWR fuel pin cell arrangement. In addition, the region volume fractions and a set of cell-averaged densities were also calculated -- and all these results are summarized in the two tables given below (see the referenced example for further details):



Region Atom Densities (atoms/b-cm)

Material	Fuel Pin	Clad	Water
U235	9.961e-4	--	--
U238	2.243e-2	--	--
O	4.686e-2	--	--
Zr	--	4.331e-2	--
H_2O	--		2.409e-2

Region Volume Fractions and Cell-Averaged Atom Densities (atoms/b-cm)

Region	Volume Fraction		Material	Average Density
fuel	0.4600		**U235**	4.582e-4
clad	0.1166		**U238**	1.032e-2
water	0.4234		**O**	2.156e-2
			Zr	5.050e-3
			H_2O	1.020e-2

Note, in particular, that the oxygen in the UO_2 fuel and the oxygen in the water have been separated here so that an explicit molecular density for water could be tabulated. This is needed because we will use a microscopic cross section for water that has already taken into account the bound nature of the hydrogen and oxygen within the water molecule (see previous discussion on this subject on page 6 of these Lecture Notes).

Using the given region and cell-averaged densities and the corresponding 2200 m/s (0.0253 eV) cross sections from the ENDFB-VI library in the JANIS 3.0 code and the water cross sections from Lamarsh, our goal here is to compute a series of region and cell-averaged macroscopic cross sections and mean free paths for this particular fuel pin configuration. Before doing this, let's tabulate the summary microscopic cross section data, as follows:

2200 m/s Microscopic Cross Sections for Common PWR Materials (barns)

Material	σ_c	σ_f	σ_s
U235	98.7	585	15.1
U238	2.72	--	9.38
O	--	--	3.98
Zr	0.185	--	6.41
H_2O	0.664	--	103

Now, with these microscopic cross section values and the corresponding region and cell averaged densities from above, we can compute the corresponding macroscopic cross sections using eqn. (17) and the thermal mean free paths using eqn. (15) to give the following results:

Macroscopic Thermal Cross Sections and Mean Free Paths for PWR Fuel Cell

Material Region	Σ_c (cm^{-1})	Σ_f (cm^{-1})	Σ_s (cm^{-1})	Σ_t (cm^{-1})	λ (cm)
fuel	0.159	0.583	0.412	1.15	0.87
clad	0.008	--	0.278	0.29	3.50
water	0.016	--	2.481	2.50	0.40
cell-averaged	0.081	0.268	1.272	1.62	0.62

Note that, in looking at these data, we see that the parasitic capture and fission in the fuel and the scattering in the water region are the dominant reactions here. In the fuel, the capture to fission ratio is about 0.159/0.583 ≈ 0.27, and about 65% (0.742/1.15 ≈ 0.645) of the interactions are absorptions (capture + fission). In the water region, over 99% of the interactions are scattering reactions!

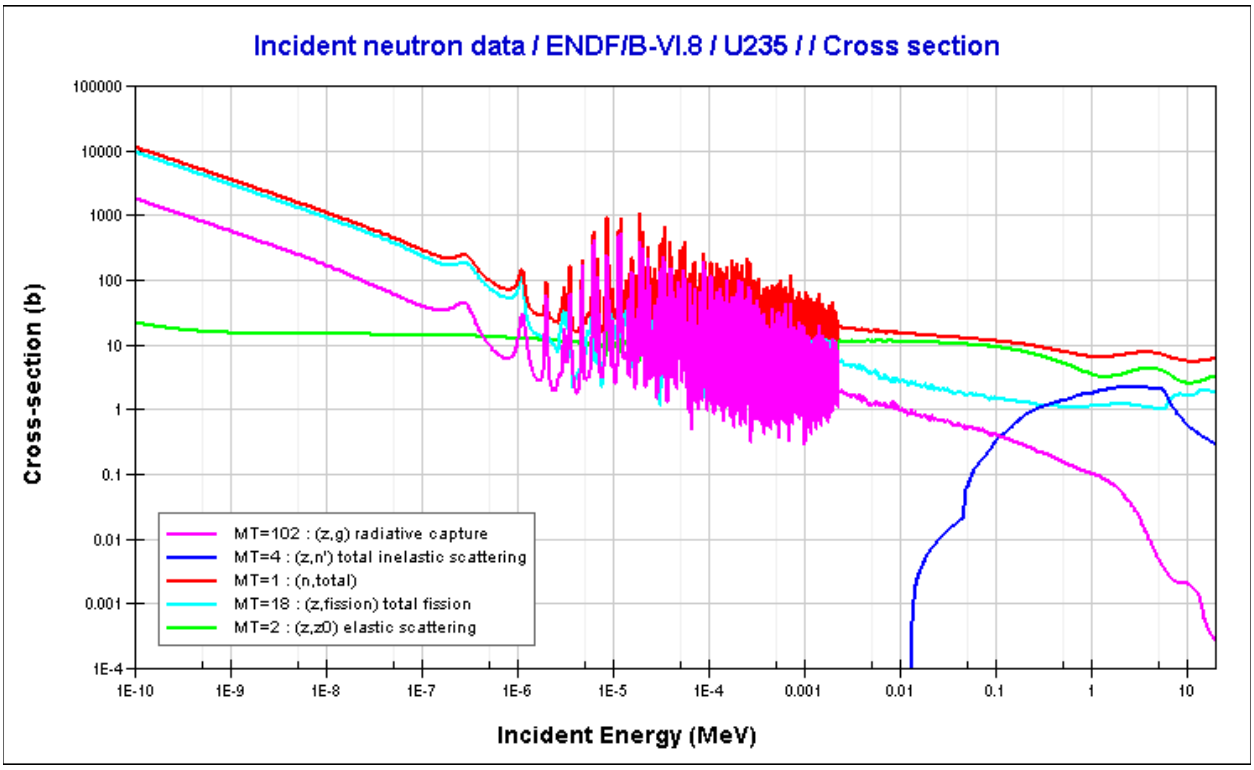
Note also that, in our basic understanding of the neutron life cycle in a thermal system, the fission neutrons are born in the fuel as fast neutrons, they primarily slow down to thermal in the water regions, and then get absorbed as thermal neutrons back in the fuel to start a new generation. Although we are only looking at the 2200 m/s data as part of this example, the numerical values of the thermal cross sections for this PWR fuel cell are fully consistent with our life-cycle interpretation…

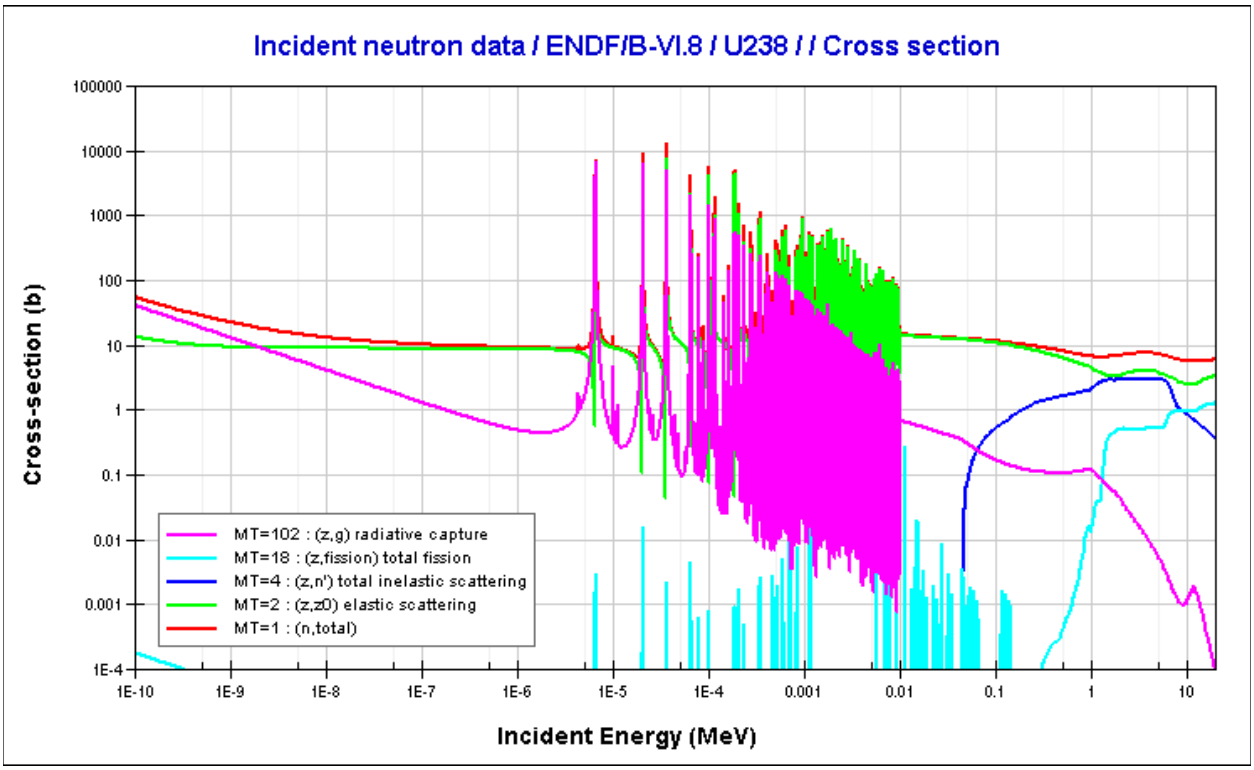
Finally we note that the pin diameter and pin pitch for this problem are 0.9566 cm and 1.25 cm, respectively, and we see that the region and cell-averaged mean free paths are somewhat smaller than these dimensions. Thus, over even the relatively small dimensions associated with a single pin cell, we might expect some noticeable neutron attenuation. For example, the average distance traveled between absorptions in the fuel is $\lambda_a = 1/\Sigma_a = 1/0.742 = 1.35$ cm. Thus, the pin radius is (0.9566/2)/1.35 ≈ 0.35 mean free paths -- and a simple exponential attenuation model gives $(1 - e^{-0.35}) \approx 0.3$, so we might expect a decrease of roughly 20% – 30% in the thermal neutron flux from the pin periphery to pin center. Note that detailed calculations of neutron attenuation in cylindrical fuel pins give somewhat different results, but this simple analysis puts us "in the ballpark"…

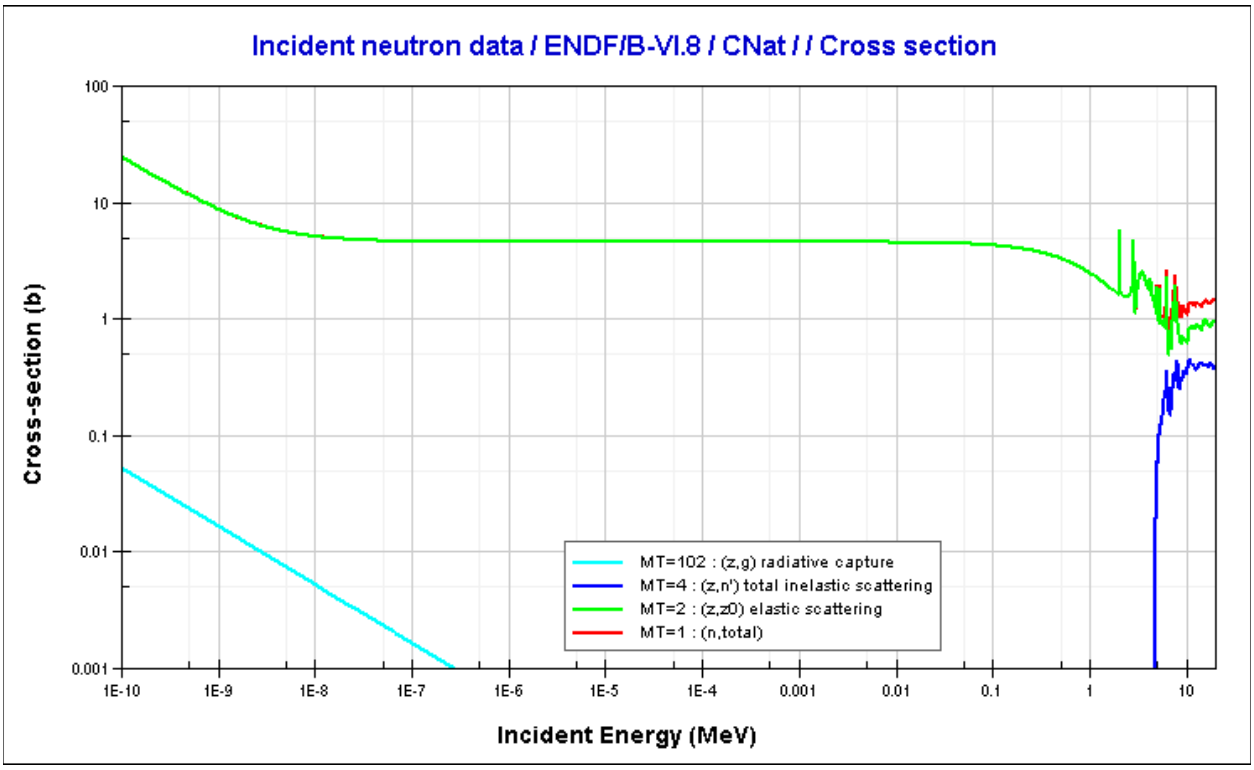
General Nature of σ(E)

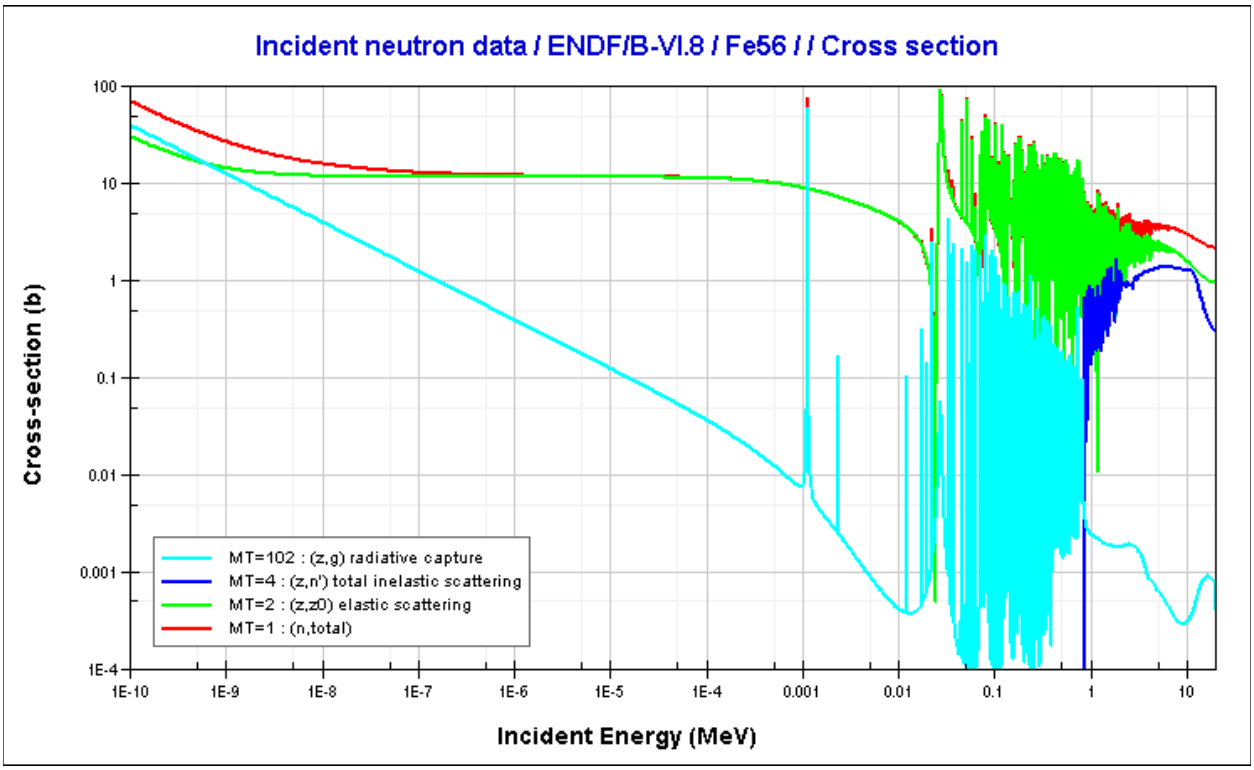
Neutron cross sections are strong functions of the incident energy of the neutron involved in the reaction. Although it is difficult to generalize the quantitative variation of σ(E), one can characterize the qualitative behavior versus energy quite easily. It is important to have a general understanding of this variation as a point of reference for explaining many important phenomena. For example, a statement that "the mean free path of fast neutrons is much larger than that of thermal neutrons" becomes meaningful only if it is clear that σ(E) tends to decrease with increasing energy. Therefore, since $\lambda(E) = 1/\Sigma_t(E)$, we would expect λ to be smaller at lower energies and larger at high energies where $\Sigma_t(E)$ becomes a slowly decreasing function of energy (for most nuclides).

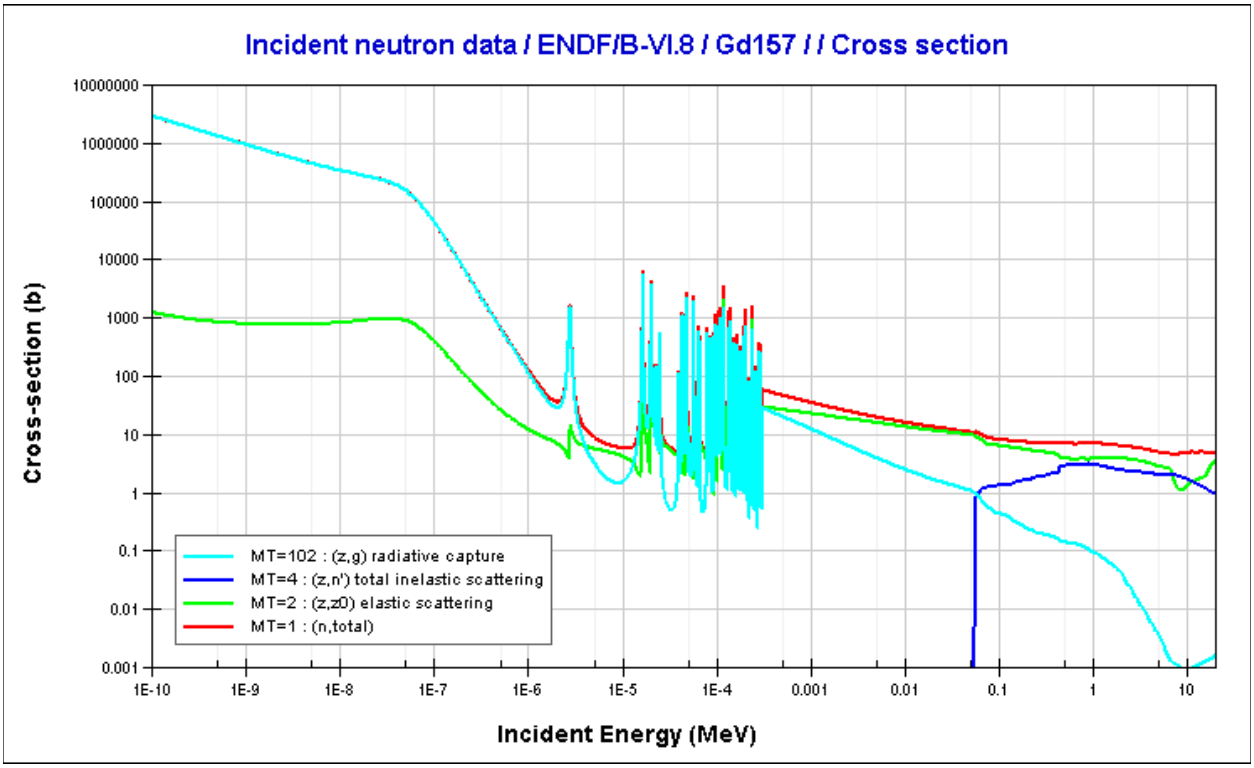
For most isotopes and reaction types, the energy spectrum can be broken into three ranges; low, resonance, and high energies. The specific boundary between regions varies considerably. For example, the interface between "low" and "resonance" energies for C12 is in the range of 1 MeV, and this same boundary for U235 is around 1 eV. In general, the resonance region tends downward in energy as the mass number of the target becomes large. Thus, the boundaries between the three generic energy regions tend to shift with the mass number of the target nucleus. The best way to illustrate these tendencies is through actual examples. Included in the next few pages are several figures which illustrate the behavior of σ(E) versus E. These figures were generated with the JANIS 3.0 program using ENDFB-VI data. These examples show some of the features discussed above, as well as several other general characteristics.


Incident neutron data / ENDF/B-VI.8 / U235 / / Cross section
Cross-section (b)
100000
10000
1000
100
10
1
0.1
0.01
0.001
1E-4
1E-10
1E-9
1E-8
1E-7
1E-6
1E-5
1E-4
0.001
0.01
0.1
1
10
Incident Energy (MeV)
MT=102 : (z,g) radiative capture
MT=4 : (z,n') total inelastic scattering
MT=1 : (n,total)
MT=18 : (z,fission) total fission
MT=2 : (z,z0) elastic scattering


Incident neutron data / ENDF/B-VI.8 / U238 / / Cross section
Cross-section (b)
100000
10000
1000
100
10
1
0.1
0.01
0.001
1E-4
1E-10
1E-9
1E-8
1E-7
1E-6
1E-5
1E-4
0.001
0.01
0.1
1
10
Incident Energy (MeV)
MT=102 : (z,g) radiative capture
MT=18 : (z,fission) total fission
MT=4 : (z,n') total inelastic scattering
MT=2 : (z,z0) elastic scattering
MT=1 : (n,total)


Incident neutron data / ENDF/B-VI.8 / CNat / / Cross section
Cross-section (b)
Incident Energy (MeV)
MT=102 : (z,g) radiative capture
MT=4 : (z,n') total inelastic scattering
MT=2 : (z,z0) elastic scattering
MT=1 : (n,total)


Incident neutron data / ENDF/B-VI.8 / Fe56 / / Cross section
Cross-section (b)
Incident Energy (MeV)
MT=102 : (z,g) radiative capture
MT=4 : (z,n') total inelastic scattering
MT=2 : (z,z0) elastic scattering
MT=1 : (n,total)



The five nuclides selected here for demonstration purposes -- U235, U238, C_{nat}, Fe56, and Gd157 -- correspond to typical fissile and fertile fuel materials, and to representative moderator, structure, and neutron poison materials, respectively. In all cases, we can easily see the three distinctly different behaviors of the cross sections in the low, resonance, and high energy ranges -- but, clearly, the specific boundary between the different regions varies considerably with nuclide (i.e. with mass number, A). Notice also, in particular, that just below the resonance region the elastic scattering cross section is relatively constant over a fairly wide energy range, that the capture and fission processes have a signature 1/v behavior over most of the thermal energy region, that the inelastic scattering cross is zero below a specific threshold energy, and that fission in fertile materials (such as U238) only has an appreciable cross section at very high neutron energies (i.e. this is also a threshold reaction).

Also of interest is the relative magnitude of specific cross sections for the different materials. Clearly, only the fuel materials fission, but we see that in fissile materials (such as U235) the fission cross section is indeed the dominant partial cross section. Also, in moderating materials (such as carbon), the capture cross sections are extremely low, and elastic scattering is the dominant partial cross section. Also, neutron poisons (such as Gd157 – which is one of several stable Gd isotopes used as burnable absorbers in many BWR and some PWR fuel designs) have very large capture cross sections (especially in the low energy regions). Finally, we note that the structural materials chosen for nuclear systems tend to have relatively low parasitic capture cross sections so that they affect, as little as possible, the overall neutron economy within the system.

Each of these general observations will be highlighted in the somewhat expanded discussions below -- however, you should take a good look at the sample data given in the above figures, since they are quite representative of many of the nuclides of interest for nuclear applications…

Scattering Cross Sections

The key differences between elastic and inelastic scattering were discussed previously. The important difference for the present discussion is that inelastic scattering is a threshold reaction. Thus, $\sigma_{in}(E)$ is essentially zero over much of the energy spectrum and, after the threshold energy associated with the first excited level of the target nucleus, the cross section tends to be a slowing varying function of energy (as shown in the above illustrative plots). In contrast, elastic scattering occurs at all energies, and the general behavior indicated above is quite characteristic. In particular, at energies just below the resonance region, the scattering cross section is nearly constant -- this is referred to as the potential scattering region, with $\sigma_s \approx 4\pi R^2$ where R is the nuclear radius. At resonance energies there are large discrete variations over very narrow energies intervals. These resonances occur at energies associated with the differences in the discrete nuclear energy levels of the compound nucleus. Finally, at high energy (i.e. above the resonance region), $\sigma(E)$ is generally a smoothly varying function of energy and, below the potential scattering region, the cross sections tend to vary as 1/v(E), where v(E) is the neutron speed at energy E.

This general qualitative description is appropriate for scattering in most nuclei. Also, for many nuclei of interest (especially the light elements), the scattering term is the major component of the total cross section. Therefore, for these nuclei the variation of $\sigma_t(E)$ is very similar to the behavior of $\sigma_s(E)$ -- see the C_{nat} profiles, for example.

Absorption Cross Sections (Capture and Fission)

A similar qualitative discussion can be made concerning the energy behavior of the absorption cross section. Recall that $\sigma_a \approx \sigma_c + \sigma_f$. Therefore, for most nuclei, absorption and radiative capture are identical. For fissionable isotopes, however, one should isolate the fission and capture components. In practice, the variation of $\sigma_c(E)$ and $\sigma_f(E)$ are similar for the fissionable isotopes. It is only the threshold fission reactions that behave differently (as shown above for U238, for example). Thus, ignoring fast fission in the non-fissile isotopes (Th232, U238, Pu240, and Pu242) for the moment, we will lump the discussion of $\sigma_c(E)$ and $\sigma_f(E)$ together.

The general behavior of the capture and fission processes is illustrated above for U235 fission and for capture in all the nuclides shown. The low, resonance, and high energy regions are quite distinct. The wild resonance variations and the smoothly decreasing function at high energy are similar in nature to the elastic scattering cross section that was just discussed. In contrast, however, the low energy component has an approximate 1/v behavior over all energies below the resonance region (as compared to the constant potential scattering cross section region that occurs for elastic scatter).

This 1/v behavior for capture and fission at low energies is very important and it justifies further analysis here. In fact, we can make the following bold statement: "For 1/v-absorbers, the thermal absorption rate is independent of the energy distribution of the neutrons and it is determined by the cross section at an arbitrary energy (usually $E_o = 0.0253$eV)." We can justify this statement as follows:

If 1/v behavior is valid then

$$\sigma_a(E) = \frac{C_1}{v(E)} = \sigma_a(E_o)\frac{v_o}{v(E)} = \frac{C_2}{\sqrt{E}} = \sigma_a(E_o)\left(\frac{E_o}{E}\right)^{1/2} \qquad (20)$$

Also recall that the absorption rate (denoted as F_a) can be written as

$$F_a = \int N\sigma_a(E)\phi(E)dE = \int N\sigma_a(E)n(E)v(E)dE = \int \Sigma_a(E)n(E)v(E)dE \quad (21)$$

but, from eqn. (20), we have

$$\Sigma_a(E) = \Sigma_a(E_o)\frac{v_o}{v(E)} \quad (22)$$

Therefore

$$F_a = \Sigma_a(E_o)v_o \int n(E)dE = \Sigma_a(E_o)v_o n_{tot} \quad (23)$$

where n_{tot} is the total (energy integrated) neutron density with units of neutrons/cm^3. Since, in general, the neutron density times the neutron speed is the neutron flux, $\phi(E) = n(E)v(E)$, the $n_{tot}v_o$ product is often referred to as the 2200 m/s flux, or $\phi_o = n_{tot}v_o$. With this expression eqn. (23) becomes

$$F_a = \Sigma_a(E_o)\phi_o \quad (24)$$

This supports the above statement that the absorption rate is independent of the energy distribution and can be characterized by a single E_o. Since much of the available data is tabulated at 0.0253 eV, E_o is usually taken to be this value (note that the neutron speed, $v_o = \sqrt{2E_o / m}$, that corresponds to E_o = 0.0253 eV is approximately 2200 m/s -- and this is why we called ϕ_o the 2200 m/s flux).

In some cases, the assumption of 1/v behavior is not precise. If one assumes that n(E) in eqn. (21) is given by the Maxwellian distribution (see below) characterized at some temperature T, then a correction factor or ***non- 1/v factor*** can be defined as follows:

$$F_a(T) = \int \Sigma_a(E)n(E,T)v(E)dE = g_a(T)\Sigma_a(E_o)\phi_o \quad (25)$$

or

$$g_a(T) = \frac{F_a(T)}{\Sigma_a(E_o)\phi_o} \quad (26)$$

where $g_a(T)$ is called the non- 1/v factor.

In practice, values of $g_a(T)$ are computed by numerically evaluating the first part of eqn. (25) with n(E,T) given by the Maxwellian distribution and also by calculating $\Sigma_a(E_o)\phi_o$ (which assumes 1/v behavior). The ratio of the two quantities is $g_a(T)$ as given in eqn. (26). Once tabulated, this value can be used in the last part of eqn. (25) to "correct" for any non- 1/v behavior that may be present. For example, a small set of tabulated values for $g_a(T)$ for several nuclides are given in Table 3.2 of the Introduction to Nuclear Engineering textbook by Lamarsh and Baratta (pg. 75 in the 3rd Ed.), and this can be used to correct for actual non- 1/v behavior in a given problem.

Distribution Functions

Up to this point, we have noted several times, without any real justification, that a neutron energy of 0.0253 eV -- which corresponds to a neutron speed of 2200 m/s -- takes on special importance when dealing with thermal systems. In addition, in the above paragraphs, we have referred to the so-called thermal Maxwellian distribution without much explanation at all. Well, in fact, these two subjects are directly related, and this subsection will try to shed some light on these concepts as well as on a more general picture of the relationship between neutron flux and neutron density.

In general, the study of the neutron behavior within a reactor is a very complicated process. In a typical situation, the neutron density is usually a function of three spatial coordinates, three velocity coordinates, and time. This seven-dimensional phase space is usually too complicated to work with for normal everyday applications. As part of a practical model reduction process, the functional dependence of the angular neutron flux, $\phi(\vec{r},\vec{v},t) = n(\vec{r},\vec{v},t)\,v(E)$, is usually simplified by assuming steady state (removes the time dependence) and by assuming separability of the geometry and velocity variables. In addition, for many cases of practical interest, we will assume that the angular dependence can be neglected -- that is, $\varphi(\vec{r},E,\hat{\Omega},t)$ can be integrated over the angular variable $\hat{\Omega}$, where $\vec{v} = v\hat{\Omega}$ and the neutron speed, v, is directly related to the neutron energy via the classical expression for kinetic energy, $E = \frac{1}{2}\,mv^2$ (thus, we can refer to the flux as being a function of energy E or as a function of neutron speed v, depending upon which view is more instructive in a given situation). Thus, in most applications we are interested in the spatial and energy distribution of the scalar neutron flux, $\varphi(\vec{r},E)$ (where we are assuming time independence and that the angular dependence has been integrated out already).

Now, the scalar flux, $\varphi(\vec{r},E)$, can be written in terms of the scalar neutron density, $n(\vec{r},E)$, which has units of neutrons per unit volume per unit energy, and the neutron speed, v(E), written in units of distance per unit time. Doing this gives

$$\varphi(\vec{r},E) = n(\vec{r},E)\,v(E) \qquad (27)$$

We can define the neutron density precisely in differential terms as,

$n(\vec{r},E)d\vec{r}dE$ = number of particles (neutrons) in $d\vec{r}$ around $\vec{r}$ and in dE around E

If we can make a case for separability of space and energy, then

$$n(\vec{r},E) = n(\vec{r})\,f(E) \qquad (28)$$

where

$n(\vec{r})\,d\vec{r}$ = number of particles in volume element $d\vec{r}$ around $\vec{r}$

f(E) dE = probability of finding any given particle in energy level dE around E

It is this last term, f(E)dE, that is an example of a ***distribution function***.

Note that if eqn. (28) is integrated over all space and energy, the result is simply the total number of neutrons in the volume of interest, or

$$\int n(\vec{r}) f(E)\, d\vec{r} dE = \int n(\vec{r}) d\vec{r} \int f(E)\, dE = \text{number of particles in volume}$$

Thus, a ***properly normalized distribution function*** integrates to unity, and simply indicates that the probability of finding a particle in the energy region between 0 and ∞ is unity. Therefore,

$$\int_0^\infty f(E) dE = 1 \tag{29}$$

To put the above discussion into perspective, let's consider the case of a very large system. As discussed previously, neutrons must eventually be absorbed or leak out of the system. As the system becomes large, the leakage term decreases and, in the limit, the leakage goes to zero as the volume becomes infinite. When we develop the full neutron balance equation later in the semester, it will become clear that it is the differential leakage term that accounts for the spatial redistribution of neutrons among different regions. If the system is homogeneous (i.e. with constant material properties) and infinite in size (relative to the distance traveled by the neutron), there is no leakage and thus the spatial distribution of the neutrons is simply a constant. Thus, in an infinite homogeneous system, the neutron density becomes

$$n\left(\vec{r}, E\right) = n_{tot}\, f(E)$$

where n_{tot} is a constant (with units of neutrons per cm^3). Therefore, in an infinite homogeneous critical reactor, the separability approximation used above becomes exact.

In a real reactor of finite proportions, we are interested in both the spatial and energy distribution of the neutrons. However, in determining this behavior, it is convenient to first approximate the energy behavior with the infinite reactor distribution function, f(E). Then one uses this distribution function to formally break the continuous energy variable into a discrete multigroup formulation. This is done by averaging all the variables over discrete energy ranges.

Recall that when averaging any quantity over energy, one needs to weight that quantity by the probability of finding the particles in any given energy region. In essence, it is necessary to know the relative importance of the various energies before taking the average. Thus, the average value of any quantity, g(E), over some energy interval, ΔE, is given by

$$\overline{g} = \langle g \rangle = \frac{\int_{\Delta E} g(E) f(E) dE}{\int_{\Delta E} f(E) dE} \tag{30}$$

where the distribution function, f(E), is the importance or weight function that properly accounts for the fact that there may be more particles at some energies than at others. If f(E) is not constant within ΔE, then the average of g(E) over this interval must account for the fact that some energies are more important than others. Equation (30) accounts for this effect -- this is the same type of averaging process we have already seen...

Now, in the context of reactor physics applications, we are interested in calculating average multigroup cross sections for use in solving the multigroup diffusion equation (to be derived later). The multigroup constants are determined by evaluating an expression similar to eqn. (30). If we denote σ_g as the average value of σ(E) over energy interval ΔE_g, then σ_g is usually given by

$$\sigma_g = \frac{\int_{\Delta E_g} \sigma(E)\varphi(E)dE}{\int_{\Delta E_g} \varphi(E)dE} \tag{31}$$

where in many applications, the weight function is simply the energy dependent neutron flux. Recalling that $\phi(E) = n(E)v(E) \approx n_{tot} f(E)v(E)$, we see that knowing the distribution function, f(E), will allow determining the desired averages.

Much of the above discussion will be reiterated several times as we discuss how to develop and solve the neutron balance equation. However, at present, it is convenient to introduce a particular distribution function that is of interest in describing the low energy behavior of neutrons in a thermal reactor. It has been found that, in a dilute gas, the energies of the gas atoms or molecules are distributed according to the well-known ***Maxwellian distribution function***. In a thermal reactor, the low energy neutrons can be thought of as a ***dilute neutron gas*** and, to a good approximation, the energy distribution of these neutrons is given by the same distribution function that describes real gas particles.

In particular, the Maxwellian distribution is given as

$$f(E) = C E^{\frac{1}{2}} e^{-\frac{E}{kT}} \tag{32}$$

where k is Boltzmann's constant (8.6170×10^{-5} eV/K) and T is the absolute temperature of the (neutron) gas. For this to be a properly normalized distribution function, we require that the integral over all energy be unity, or

$$\int_0^\infty f(E)dE = C\int_0^\infty E^{\frac{1}{2}} e^{-\frac{E}{kT}} dE = 1 \tag{33}$$

Performing the integration and solving for C gives,

$$C = \frac{2\pi}{(\pi kT)^{\frac{3}{2}}} \tag{34}$$

Given that the energy distribution of the low energy neutrons is described by eqn. (32), it is appropriate to ask, "What is the average particle energy?". From the above discussion of importance functions, we see immediately that the average energy, $< E >$, is given by

$$< E > = \frac{\int_0^\infty E f(E)\, dE}{\int_0^\infty f(E)\, dE} \tag{35}$$

Since f(E) is properly normalized, this becomes

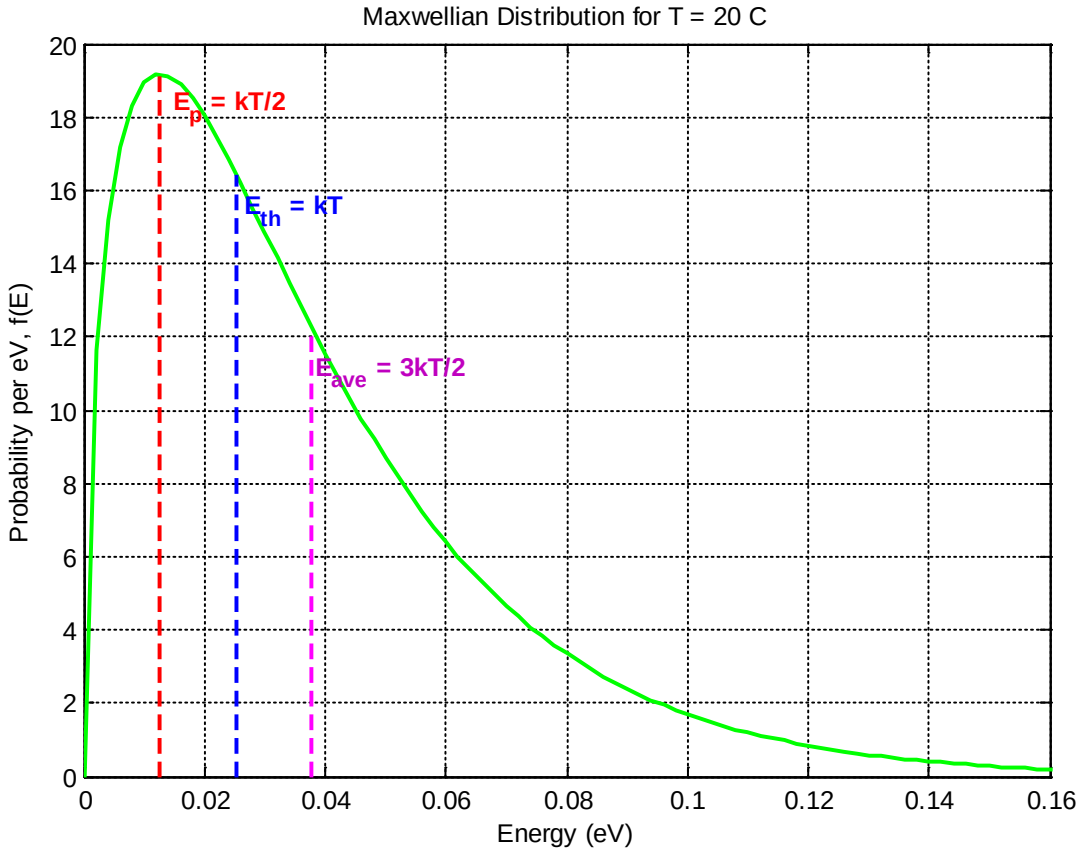
$$< E > = C\int_0^\infty E^{\frac{3}{2}} e^{-\frac{E}{kT}} dE = \frac{3}{2} kT \tag{36}$$

Thus, the ***average energy*** in a thermal system operating at an average system temperature of T is given by $E_{ave} = < E > = \frac{3}{2} kT$.

To determine the most probable energy, one simply differentiates the function and sets the derivative to zero. This is the standard procedure for determining the maxima/minima of any function, since the first derivative is zero at any local maxima/minima. Setting df(E)/dE = 0 gives E_p = ½ kT. Thus, the most probable energy in a thermal system operating at an average system temperature of T is E_p = ½ kT.

The combination of terms, kT, appears frequently in reactor physics work. In fact, if one transforms the Maxwellian distribution function, f(E), to a velocity space distribution, f(v), and then finds the most probable speed (using a procedure similar to that described above), the result is $v_p = \sqrt{2kT/m}$. Since E = ½ mv^2, we see that the energy associated with the most probable speed is E_{th} = kT (which is usually associated with the term thermal energy). At room temperature (T = 20°C or 293°K), E_{th} = 0.0253 eV and $v_p = v_{th}$ = 2200m/s -- and this analysis finally shows where we obtained the E_o = 0.0253 eV value. This value of energy, $E_o = E_{th}$ = kT, is that associated with the most probable speed in a thermal system at room temperature.

A plot of the Maxwellian distribution function, with the specific energies noted above highlighted on the plot, is shown below. This specific plot is associated with T = 20 C (i.e. room temperature,) and thus we see that E_{th} is indeed about 0.025 eV. Note also that, since f(E) goes to zero very quickly (well before 1 eV), there is essentially no contribution to the integrals above the thermal energy region. Thus, we can say that the average energy of ³⁄₂ kT is the average thermal energy.



One final note before leaving our discussion of the Maxwellian distribution concerns the procedure for actually performing the integrals in eqns. (33) and (36). There are several ways to tackle this mathematical problem; integration by parts, use of integral tables, etc. One specific technique that is frequently used for integrals involving the exponential function is the use of the ***gamma function***, $\Gamma(n)$. The student is referred to any good table of mathematical functions for a complete discussion of the gamma function. Some properties of $\Gamma(n)$ that are useful for the specific task of evaluating eqns. (33) and (36) are:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \tag{37}$$

with $\Gamma(n+1) = n\Gamma(n)$ $\Gamma(n) = (n-1)!$ for n = positive integer $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (38)

These relationships are not proved here, but these properties of the gamma function are indeed helpful in working with integral expressions similar to those given above. Example 3 illustrates the use of eqns. (37) and (38) for determining the normalization constant, C, in the Maxwellian distribution, f(E).

Example 3: Normalization of the Maxwellian using the Gamma Function

As an example of the use of eqns. (37) and (38), we proceed to evaluate the integral in eqn. (33) and actually derive the expression for the normalization constant, C, in the Maxwellian distribution. In particular, we proceed to evaluate the integral, I, as follows:

$$I = \int_0^\infty E^{\frac{1}{2}} e^{-\frac{E}{kT}} dE = \int_0^\infty (kT)^{\frac{1}{2}} x^{\frac{1}{2}} e^{-x} (kT dx) = (kT)^{\frac{3}{2}} \int_0^\infty x^{\frac{1}{2}} e^{-x} dx$$

where we have substituted x = E/(kT) and dx = dE/(kT) into the original integral expression.

Now, from comparison to eqn. (37), we see that the actual integral expression in the last part of the above relationship is simply $\Gamma(3/2)$ -- where n–1 = 1/2 or n = 3/2. But, from the additional relationships given in eqn. (38), we have

$$\Gamma\left(\tfrac{3}{2}\right) = \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

Thus, the desired integral becomes

$$I = (kT)^{\frac{3}{2}} \frac{\sqrt{\pi}}{2} = (kT)^{\frac{3}{2}} \frac{\sqrt{\pi}}{2} \left(\frac{\pi}{\pi}\right) = \frac{(\pi kT)^{\frac{3}{2}}}{2\pi}$$

Finally, from eqn. (33), we have

$$C = \frac{1}{\int_0^\infty E^{\frac{1}{2}} e^{-\frac{E}{kT}} dE} = \frac{1}{I} = \frac{2\pi}{(\pi kT)^{\frac{3}{2}}}$$

which is the desired result of this example…

The Fission Reaction

Early in the semester we talked in general about the fission process. Also, in the last subsection, we addressed the behavior of $\sigma_f(E)$ versus energy. However, since fission is the primary source of neutrons in a reactor, it is particularly important that we highlight all the essential features of the process.

Considering U235 fission, we can write an expression for the fission process as

$$U235 + n \rightarrow (U236)^* \rightarrow 2\,FPs + \nu\,n's + 200\ MeV \qquad (39)$$

The fission process is usually induced by neutron absorption. The compound nucleus that is formed (U236 in our example) is in an excited state with an energy equal to the kinetic energy (KE) of the incident neutron plus the binding energy of the last neutron (BE_n) in nuclide ^{A+1}Z (for absorption in ^{A}Z). The liquid drop model of the nucleus suggests that as the excitation energy of the compound nucleus increases, the nuclear drop can become elongated -- possibly to the point where the electrostatic repulsive force exceeds the short range attractive nuclear forces between nucleons in the two lobes of the elongated droplet -- thus causing the compound nucleus to actually break into two intermediate mass fission fragments. This is represented as states A → B → C as shown in the sketch. For this to happen, a critical energy (E_{crit}) associated with the compound nucleus must be reached -- see Table 3.3 in Lamarsh for the critical energies for fission (as reproduced below). Thus, if the KE + BE_n is greater than E_{crit} for isotope ^{A+1}Z, then fission can occur. If the BE_n, by itself, is greater than E_{crit}, then fission can occur with any incident neutron energy, and such a nuclide is denoted as a fissile isotope.

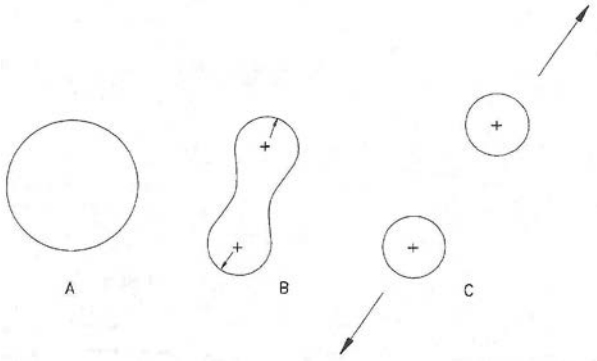


TABLE 3.3 CRITICAL ENERGIES FOR FISSION, IN MeV

Fissioning Nucleus ^{A}Z	Critical Energy	Binding Energy of Last Neutron in ^{A}Z
^{232}Th	5.9	*
^{233}Th	6.5	5.1
^{233}U	5.5	*
^{234}U	4.6	6.6
^{235}U	5.75	*
^{236}U	5.3	6.4
^{238}U	5.85	*
^{239}U	5.5	4.9
^{239}Pu	5.5	*
^{240}Pu	4.0	6.4

It should be noted that it is common practice to say that U235 fissions. In fact, however, U235 absorbs a neutron and the U236 compound nucleus that is formed is what actually fissions. Also,

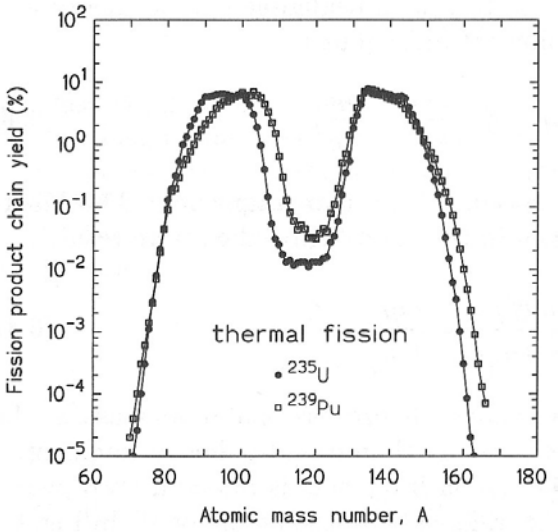
from Table 3.3 in Lamarsh, we have for this process that E_{crit} = 5.3 MeV and BE_n = 6.4 MeV for U236. Since $BE_n > E_{crit}$, U235 (not U236) is said to be a fissile isotope.

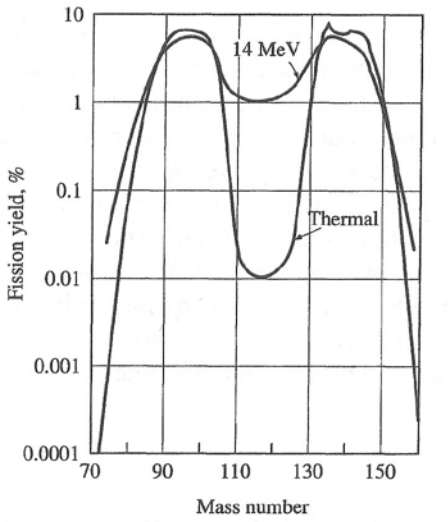
As another example, notice that Table 3.3 lists E_{crit} = 5.5 MeV and BE_n = 4.9 MeV for U239. Thus, U238 is not fissile and it requires a threshold kinetic energy of about 0.6 MeV for fission ($KE_{thres} = E_{crit} - BE_n$). Thus, U238 is said to be a fissionable isotope.

In addition to understanding the basic process of fission, it is also important to address the products of fission. As seen in the example for U235 fission, there are usually two fission products, ν neutrons (ν = average number of neutrons per fission), and about 200 MeV excess energy that are released per fission event. We will address each of these briefly.

Fission Products

Fission is almost always asymmetric. That is, the masses of the two fission fragments are usually substantially different. Referring to the sketches shown below, the plot on the left shows the actual fission product yield distribution for thermal fission in both U235 and Pu239. These curves certainly show that the specific yields are a function of the fissioning nuclide but, in both cases, the characteristic double hump shape is also observed (which supports the above statement concerning asymmetry). A similar figure (i.e. the one on the right) displays the fission product yields for thermal and 14-MeV fission in U235 -- thus, highlighting the fact that the fission product yields are also functions of the energy of the incident neutron that causes the fission.





The fission product yield is typically written as γ_{ij} which is defined as

γ_{ij} = number of atoms of isotope i produced per fission in nuclide j

where

$\sum_i \gamma_{ij} = 2$ for any j (that is, there are two fission products per fission event)

The energy dependence noted above is often averaged for a particular reactor type (i.e. a thermal reactor, fast reactor, etc.), giving a single yield for each fission product i and fissioning nuclide j -- this can be justified because the actual energy dependence is a rather weak over a small energy range (note that the above figure compares 0.025 eV and 14 MeV neutrons!). The dependence on fissionable isotope is of more importance and this effect can be significant in

many applications. For example, let's consider the specific yields for I-135 and Xe-135 -- where we shall see later that Xe-135 is an extremely important fission product in thermal systems (since it has a very large thermal absorption cross section). For now, we will simply tabulate some yields for Xe-135 and I-135 (which decays to Xe-135) as an illustration of the dependence of γ_{ij} on the fissioning nuclide. This variation in FP yield among the fissile isotopes is also important when considering the production of fission products that decay via neutron emission (see below).

Fission Yield (Atoms/Fission) from Thermal Fission

Isotope	U233	U235	Pu239
I-135	0.0475	0.0639	0.0604
Xe-135	0.0107	0.0024	0.0105

Fission products are not only important from a neutron balance point of view. In addition, most fission product nuclides are radioactive. These isotopes are usually neutron rich and decay via β^- decay. This radioactivity and its associated decay heat cause many of the problems associated with the use of nuclear energy. In general, the radioactive fission products must be contained, cooled after shutdown (requires emergency core cooling), and stored as waste after removal from the reactor. Thus, the subject of fission products is extremely important. In this course, however, we will concentrate primarily on their neutronic poisoning effect, with some further discussion on the cooling needs associated with the decay heat issue (later in the 2nd semester).

As a preview to the decay heat concern, consider the table given below showing the average distribution of recoverable energies per fission. Clearly, the dominant component is the kinetic energy of the FP fragments. However, this energy contribution, as well as the energy associated with the prompt fission and capture gammas and neutrons emitted from the fission reactions go away very quickly after the reactor is shutdown. In contrast, the 7-8 percent (15/200 = 7.5%) of the total energy released due the fission product decay particles (betas and gammas), do not vanish immediately and, in fact, this decay heat represents a significant contribution for a relatively long time after shutdown.

Distribution of Recoverable Energy from the Fission Process.

Energy Component	Approx. MeV per fission
kinetic energy of FPs	**168**
kinetic energy of neutrons	**5**
prompt fission gammas	**7**
prompt capture gammas	**5**
FP decay -- beta particles	**8**
FP decay -- gamma rays	**7**
total	**200**

For example, for a reactor operating at a power level of about 3000 MW_{th}, the power level shortly after shutdown is about $0.075 \times 3000 = 225$ MW_{th} -- which is still a very significant

power source. Also, as shown in Fig. 8.3 from the 3rd Ed. of Lamarsh (as reproduced below), even after 2.75 hours ≈ 10^4 seconds, the decay heat is still almost 1% of the initial reactor power before shutdown -- which corresponds to about 30 MWth for our specific example. This represents a lot of energy deposition within the core region. Thus, the core requires continuous cooling capability long after the system has been shut down -- and providing this cooling under all possible accident scenarios is a key aspect of the design of all power reactor systems!!!

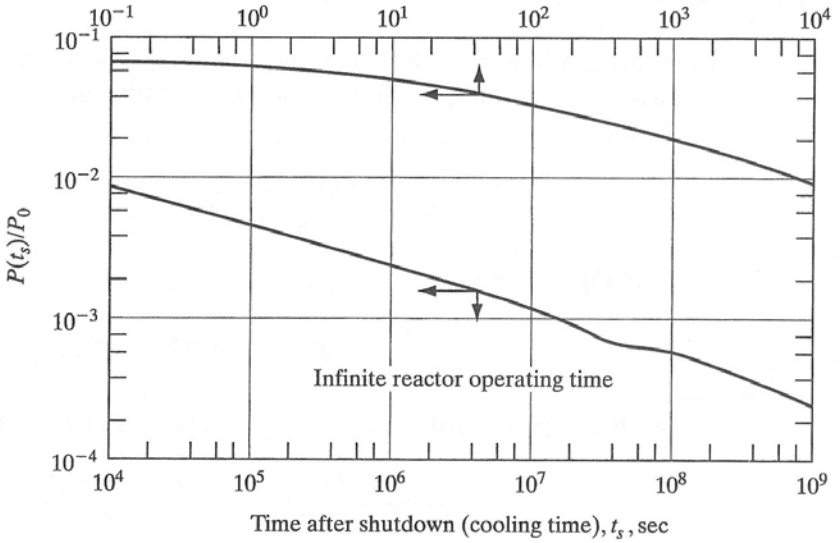


Figure 8.3 The ratio $P(t_s)/P_0$ of the fission product decay power to the reactor operating power as a function of time t_s after shutdown.

Fission Neutrons

The neutrons emitted from fission are important since they are the mechanism for keeping the chain reaction going. The average number of neutrons emitted per fission is given the symbol ν_T (usually just ν). The contributions to ν_T are made up from prompt neutrons and delayed neutrons. Prompt neutrons are emitted directly in the fission event. The delayed component occurs from the neutrons that are released in the decay of certain fission products. These delayed neutron precursors are produced from fission and the neutrons they emit can be thought of as resulting from fission with a characteristic delay before release.

The delayed neutrons are usually grouped into six separate groups with effective decay constants λ_i (or half-life $T_{1/2_i} = \ln 2 / \lambda_i$) and yields β_i, where

$$\beta_i = \frac{\nu_i}{\nu_T} = \frac{\text{delayed neutrons from precursor group i per fission}}{\text{total neutrons emitted per fission}} \tag{40}$$

Note also that

$$\nu_d = \sum_i \nu_i = \text{total delayed neutrons per fission} \tag{41}$$

and $$\beta = \sum_i \beta_i = \frac{\nu_d}{\nu_T} = \text{total delayed neutron fraction} \qquad (42)$$

Note also that β is quite small -- only about 0.00685 for U235 and about 0.0023 for Pu239.

In writing an expression for the neutron source in a reactor, it is often important to consider whether the neutrons emitted are prompt or delayed. The total neutrons emitted per cm^3-sec can be written as the product of the average number of neutrons per fission, $(1-\beta)\nu_T$ and $\beta\nu_T$ for the prompt and delayed components, respectively, times the fission rate density, $\Sigma_f\phi$, or

$$\frac{\text{neutrons emitted}}{\text{cm}^3\text{-sec}} = \int \nu_T \Sigma_f(E)\phi(E)dE$$

$$= \int (1-\beta)\nu_T \Sigma_f(E)\phi(E)dE + \int \beta\nu_T \Sigma_f(E)\phi(E)dE$$

$$\text{prompt} \qquad + \qquad \text{delayed}$$

The above expression separates the total source into prompt and delayed components. However, a key point that is not addressed here is the time at which the neutrons are emitted -- thus, ***this statement is only valid for the steady state (time-independent) case***, where the different time dependence of the prompt and delayed neutrons is not important. However, in writing a neutron balance equation that includes the time variable (that is, for a non-steady state problem), the time dependence plays an important role and a different expression will need to be developed (***the above expression can not be used for time dependent problems***).

The above introductory concepts concerning prompt and delayed neutrons are extremely important. In fact, without the small fraction of delayed neutrons that exist, we would not be able to control the chain reaction. However, we will postpone a detailed treatment of this subject until next semester when we study reactor kinetics (the time dependent reactor) in some depth. Nevertheless, just to wet your appetite for more information on this subject, Example 4 provides a simple illustration that highlights the importance of the delayed neutron component of the fission source when considering the control of nuclear systems. However, for now, we simply want you to be aware of the prompt and delayed components of the overall neutron source and their different roles when considering reactor operations and control -- all the formal details on this subject will come later when we discuss the time-dependent reactor…

Thus, for our current work, we will consider only the steady state fission source and write this as

$$S_{fis} = \int \nu_T \Sigma_f \phi \, dE = \int \nu \Sigma_f \phi \, dE \qquad (43)$$

Also, we should note that the symbol η is used to refer to the average number of neutrons emitted per absorption in the fuel -- Lamarsh, in particular, often uses this notation. With this definition, we can also write the total steady state fission source as

$$S_{fis} = \int \nu \Sigma_f \phi \, dE = \int \eta \Sigma_a^F \phi \, dE \qquad (44)$$

where $$\eta(E) = \frac{\nu \Sigma_f(E)}{\Sigma_a^F(E)} \qquad (45)$$

and the F superscript denotes that only absorption in the fuel material is treated here.

Example 4: A Note on the Control of Nuclear Systems

This example illustrates, in a simple way, the importance of the delayed neutron component to the control of reactor systems. We will use typical numerical values of various quantities without formal justification (more details are given later in our treatment of reactor kinetics in the 2nd semester of this course). In particular, for a U235 fueled and water moderated critical reactor, we have the following typical parameters:

$\beta = 0.0068$ = total delayed neutron fraction

$l_p \approx \Lambda_p = 10^{-4}$ s = prompt neutron generation time

$l_d = 12.7$ s = average delayed neutron lifetime

Also recall that the multiplication factor, k, is given by

$$k = \frac{\text{fission source in one generation}}{\text{fission source in preceeding generation}} = \frac{\text{neutrons produced in one generation}}{\text{neutrons produced in preceeding generation}}$$

Thus, if n_o is the neutron population at time zero, then the total population after N generations is simply

$$n_N = k^N n_o$$

Now, let's assume that we only have prompt neutrons in the system (i.e. all the neutrons produced from fission are consider to be prompt neutrons). If $n_o = 1$ and $k = 1.001$ (just slightly above critical), then in 1 second there will be about 10000 neutron generations, where

$$N = \frac{t}{\Lambda_p} = \frac{1\text{sec}}{10^{-4}\,\text{sec/ gen}} = 10,000 \text{ generations}$$

and, after this many generations, the neutron population will be

$$n_{10,000} = (1.001)^{10,000}(1) \approx 21,900$$

Thus, the power level would increase by a factor of more than 21,000 in just 1 second. Clearly, this is not a controllable situation!!!

Now, assume that the multiplication factor is still 1.001, but that only $(1-\beta) = 0.9932$ of the neutrons produced are prompt -- the remaining 0.68% are delayed neutrons. In this case, if we considered only the prompt neutrons, the system would be subcritical -- that is, $k_{prompt} = (0.9932)(1.001) \approx 0.9942$ -- and, in 1 second, the new neutron population produced from these prompt neutrons would be

$$n_{10,000} = (0.9942)^{10,000}(1) \approx 5.5\times10^{-26} \approx 0$$

Thus, in 1 second, all the original neutrons are gone -- that is, the system is subcritical on prompt neutrons alone.

However, if we consider the delayed neutron contribution, with an effective neutron lifetime given by $l_{eff} = (1-\beta)\,\Lambda_p + \beta\, l_d = (0.9932)(10^{-4}) + (0.0068)(12.7) \approx 0.086$ s, then, in 1 second, there will only be about 12 neutron generations, where

$$N = \frac{t}{l_d} = \frac{1\text{sec}}{0.086\,\text{sec/ gen}} = 11.6 \text{ generations}$$

and, after this many generations, the neutron population will be

$$n_{12} = (1.001)^{12}(1) \approx 1.012$$

Thus, in 1 second, there is only about a 1% increase in neutron level and overall reactor power -- and we see that, with delayed neutrons, we have a much more manageable control scenario!!!

Note: Clearly this example is not overly rigorous since, in practice, we need to treat the time dependence of the prompt and various delayed neutron groups separately. However, this "back of the envelope calculation" gives us a rough picture of the effects of delayed neutrons, and it clearly illustrates that, without the small fraction of delayed neutrons present, we would not be able to control a nuclear system. This subject -- Reactor Kinetics and Control -- is extremely important, and we will give a more detailed and more rigorous treatment of this topic in a later section of this sequence of Lecture Notes (also see Chapter 7 in Lamarsh)…

Not only is it important to determine the number of neutrons emitted per fission, but we also need to know the energy of these neutrons. Fission neutrons are emitted with a continuous energy spectrum which is Maxwellian in nature but shifted in energy such that the peak of the curve is slightly less than 1 MeV and the average energy is about 2 MeV. This distribution of fission neutron energies is known as the fission spectrum (typically called the prompt fission spectrum), where

$$\chi(E)dE = \begin{array}{l}\text{probability that a fission neutron will be} \\ \text{born within energy interval dE around E}\end{array}$$

Also since this must be a properly normalized distribution function (i.e. the probability of finding the neutron with energies between 0 and ∞ is unity), then

$$\int_0^\infty \chi(E)dE = 1 \tag{46}$$

In a multigroup formulation, χ_g is defined as

$$\chi_g = \int_{\Delta E_g} \chi(E)dE \tag{47}$$

and

$$\sum_g \chi_g = 1 \tag{48}$$

For prompt neutrons, a modified Maxwellian distribution known as the Watt fission spectrum is often used to describe $\chi(E)$. The general form of the Watt fission spectrum is given by

$$\chi(E) = a\, e^{-bE} \sinh\sqrt{cE} \tag{49}$$

where the parameters a, b, and c are found by fitting eqn. (49) to appropriate experimental data for each fissionable nuclide of interest. For example, for U235 fission, Lamarsh gives

$$\chi(E) = 0.453 e^{-1.036E} \sinh\sqrt{2.29E} \quad \text{(for U235 fission with E in MeV)} \qquad (50)$$

Now, if we combine the discussion of the neutron energy with the treatment of the number of neutrons emitted per fission, we can formally define the fission source as

$$\begin{pmatrix}\text{number of neutrons emitted per}\\ \text{cm}^3\text{- sec in energy interval dE}\end{pmatrix} = \begin{pmatrix}\text{fraction of fission neutrons}\\ \text{emitted in energy interval dE}\end{pmatrix} \times \begin{pmatrix}\text{total neutron production}\\ \text{rate from all fission}\end{pmatrix}$$

or $$S_{fis}(E) = \begin{matrix}\text{number of neutrons emitted per}\\ \text{cm}^3 - \text{sec in energy interval dE}\end{matrix} = \chi(E)\left[\int_{\text{all E'}} \nu\Sigma_f(E')\phi(E')dE'\right]dE \qquad (51)$$

and, in a discrete multigroup energy formulation, this becomes

$$S_g^{fis} = \begin{matrix}\text{number of neutrons emitted per}\\ \text{cm}^3\text{-sec in energy group g}\end{matrix} = \chi_g \sum_{g'} \nu\Sigma_{fg'}\phi_{g'} \qquad (52)$$

This latter representation for the fission source will be used as one of the primary source terms when we develop the full multigroup balance equation later in the semester…

Finally, we should note that delayed neutrons also a have a spectrum of energies, but the distribution tends to be peaked at a lower energy -- typically in the several hundred keV range instead of near 1 MeV as for prompt neutrons (see the sketches given below from Duderstadt and Hamilton). However, since the delayed neutron fraction, β, is so small, this difference in delayed versus prompt neutron spectra is negligible for steady state analyses -- so the $\chi(E)$ and χ_g in eqns. (49) and (50) usually refer to the prompt fission spectrum. In contrast, for time dependent problems, the differences between $\chi_p(E)$ and $\chi_d(E)$ are very important and this distinction will need to be treated explicitly (again, this subject will be treated later when we formally develop the time-dependent neutron balance equation in its full detail)…

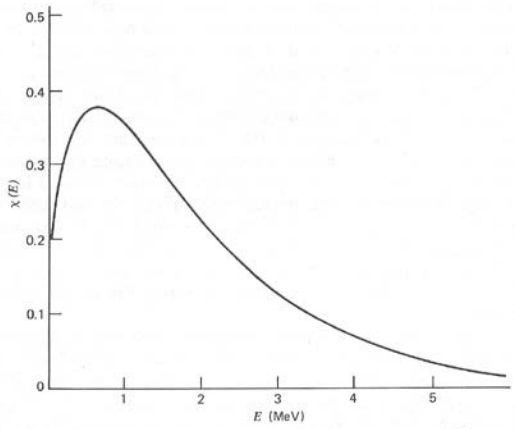


FIGURE 2-21. Fission spectrum for thermal neutron induced fission in ^{235}U.

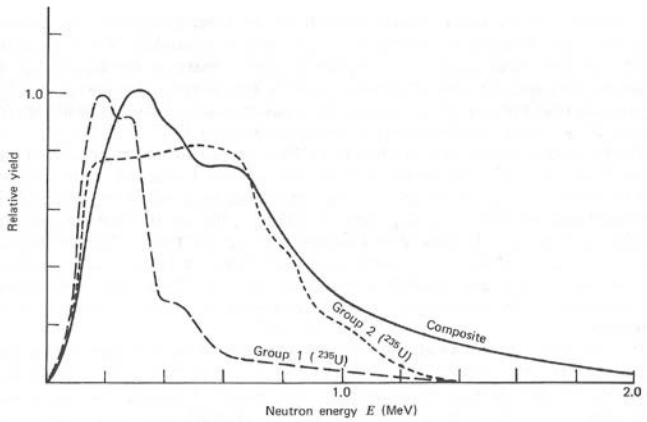


FIGURE 2-23. Composite delayed neutron spectrum.

Fission Energy

The last and probably most important product from fission is the energy released. The practical application of this available energy is one of the primary reasons for studying nuclear reactor physics and engineering. Up to now we have been using a nominal value of 200 MeV per fission event. This value is not only nuclide dependent, but it is also made up of several components (see previous table on page 23). The dominant source of energy is in the form of the kinetic energy of the fission fragments. These fragments deposit their energy locally (essentially at the

spot of fission). Fission product decay heat and the emission of γ-rays following radiative capture (i.e. the (n,γ) reaction) are also important contributions to the total. In detailed computations, many of the individual components can be treated explicitly, giving a more accurate representation of energy deposition. However, most preliminary work simply uses 200 MeV per fission.

We have already noted that the flux magnitude in the reactor is determined by the imposed power level (recall that power and criticality are decoupled at least to first order). If we denote κ (note that Lamarsh calls this term E_R) as the recoverable energy per fission, then the reactor power can be expressed in terms of the fission rate as

$$P = \kappa \int \Sigma_f(\vec{r},E)\phi(\vec{r},E)d\vec{r}dE \qquad (53)$$

where the integration is over all space and energy. A check on the consistency of the units used here gives

$$\text{watts} = \left(\frac{\text{joules}}{\text{fission}}\right)\left(\frac{\text{fissions}}{\text{cm}^3 - \text{sec} - \text{energy}}\right)\left(\text{cm}^3\right)\left(\text{energy}\right) = \frac{\text{joules}}{\text{sec}} = \text{watts}$$

Equation (53) is typically used as a normalization of the flux distribution. The nominal value of κ is 3.204×10^{-11} joules/fission, which corresponds to 200 MeV per fission event.

Elastic Scattering

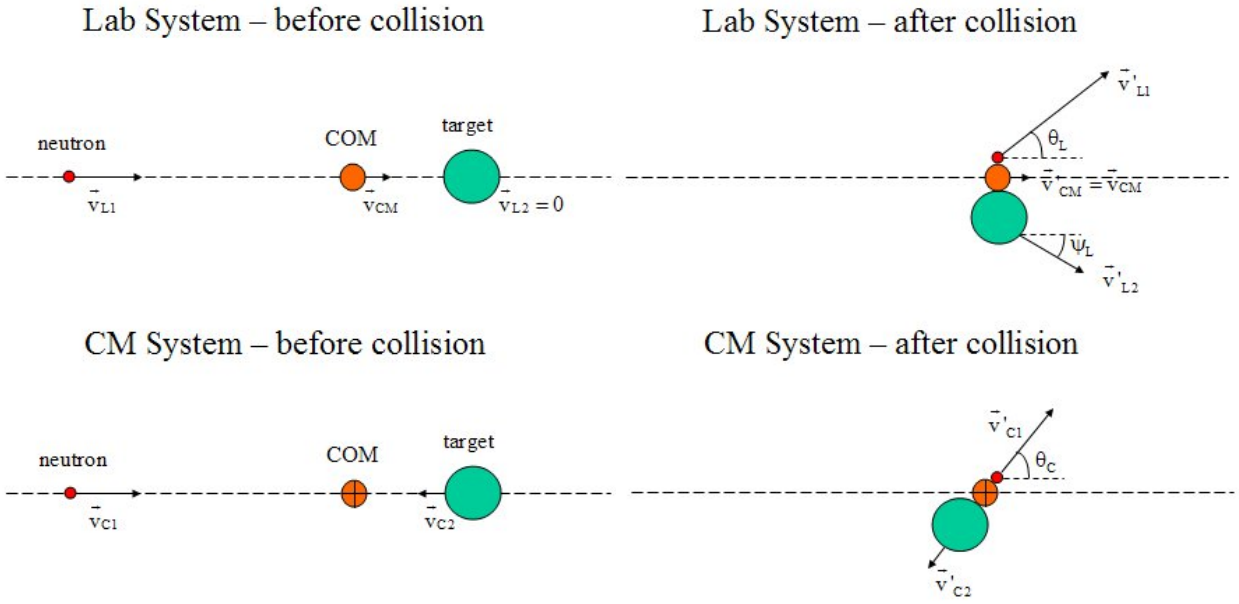
In this subsection we concentrate on elastic scattering, which is described nicely by the two-body collision process. Our goal here will be to describe the momentum and energy transfer that takes place in individual collisions, and then average these over all interactions to find out what happens to the average or typical neutron. There is a fair amount of new terminology and certainly a bunch of equations that are needed to accomplish this goal, but you should try not to get hung up on all the equations -- that is, try to keep your eye on the big picture. In the end, we will have a set of relatively straightforward relationships that have been derived in a formal and insightful manner. In addition, we will bring everything together to define some new terminology that is useful in characterizing the effectiveness of neutron moderators within a thermal system.

The momentum and energy transfer that occurs in elastic scattering between neutral particles can be described with the help of the diagrams shown below. The definitions of the terms in these sketches are as follows:

1. In the lab coordinate system, $\vec{v}_{L1}$ and $\vec{v}_{L2}$ are the particle velocities before collision and $\vec{v}'_{L1}$ and $\vec{v}'_{L2}$ are the particle velocities after collision, where particle 1 is assumed to be the neutron and particle 2 is initially a stationary target nucleus (i.e. the target's initial velocity is assumed to be zero, or $\vec{v}_{L2} = 0$)

2. In the center-of-mass (CM) coordinate system, the same notations apply to $\vec{v}_{C1}$ and $\vec{v}_{C2}$ (before) and $\vec{v}'_{C1}$ and $\vec{v}'_{C2}$ (after). Note that the collision in the CM systems is viewed from an observer located on the center of mass (where the observer thinks he or she is stationary).

3. The velocity of the CM is a constant of the motion -- that is $\vec{v}'_{CM} = \vec{v}_{CM}$. Note that this velocity is from the perspective of an external stationary observer who is viewing the whole stationary laboratory reference frame.
4. θ_L is the scattering angle of particle 1 (the neutron) in the lab system relative to the direction of the CM. ψ_L is the scattering angle of the target in the lab system (note that we won't do much with the target's scattering angle, since our primary interest is with the scattered neutron).
5. θ_C is the neutron scattering angle in the CM system relative to the direction of the CM. Note that, from the perspective of an observer on the center of mass, the system has no net momentum before or after collision. Thus, in the CM system, the neutron and target velocities are always co-linear and directed in opposite directions.



Now let's write a series of formal mathematical relationships using the variables defined above. In particular, from considering the momentum before the collision, we have that the momentum associated with the center of mass must be equal to the sum of the individual particle momentums, or

$$\left(m_1 + m_2\right)\vec{v}_{CM} = m_1\vec{v}_{L1} + m_2\vec{v}_{L2} \tag{54}$$

However, we have already stated that the target is initially stationary (i.e. $\vec{v}_{L2} = 0$), and we usually write the mass of the particles in terms of their mass number -- that is, the mass of the neutron is m_1 = 1 amu and the mass of the target is approximately m_2 = A amu. Thus, using these simplifications, eqn. (54) becomes

$$\vec{v}_{CM} = \frac{m_1}{m_1 + m_2}\vec{v}_{L1} + 0 = \frac{1}{1+A}\vec{v}_{L1} \tag{55}$$

Also, by inspection of the Lab and CM diagrams before collision, we have

$$\vec{v}_{C1} = \vec{v}_{L1} - \vec{v}_{CM} \qquad \text{and} \qquad \vec{v}_{C2} = -\vec{v}_{CM} \tag{56}$$

which, from eqn. (55), can also be written as

$$\vec{v}_{C1} = \vec{v}_{L1} - \frac{1}{1+A}\vec{v}_{L1} = \frac{A}{1+A}\vec{v}_{L1} \qquad \text{and} \qquad \vec{v}_{C2} = -\frac{1}{1+A}\vec{v}_{L1} \tag{57}$$

Noting that the total momentum in the CM system is given by the vector sum of the momentum of the individual particles, we have

$$\vec{p}_{\text{in CM}} = (1)\vec{v}_{C1} + A\vec{v}_{C2} = \frac{A}{1+A}\vec{v}_{L1} - \frac{A}{1+A}\vec{v}_{L1} = 0 \tag{58}$$

and this is consistent with item #5 noted above.

Since the net momentum in the CM system is always zero (before and after the collision), we have

$$\vec{v}_{C1} = -\frac{m_2}{m_1}\vec{v}_{C2} = -A\vec{v}_{C2} \qquad \text{and} \qquad \vec{v}\,'_{C1} = -A\vec{v}\,'_{C2} \tag{59}$$

Finally, ***conservation of linear momentum in the lab system*** using the combined mass and the center of mass velocity, shows that this velocity is a constant of the motion (as noted in item #3 above), or

$$(m_1 + m_2)\vec{v}_{CM} = (m_1 + m_2)\vec{v}\,'_{CM} \qquad \text{or} \qquad \vec{v}_{CM} = \vec{v}\,'_{CM} = \text{constant} \tag{60}$$

Now, recalling that elastic scattering leaves the nucleus in its ground state, we can write the ***conservation of energy in the center of mass system*** as

$$\frac{1}{2}m_1 v_{C1}^2 + \frac{1}{2}m_2 v_{C2}^2 = \frac{1}{2}m_1 v'^2_{C1} + \frac{1}{2}m_2 v'^2_{C2} \tag{61a}$$

before collision after collision

or $$v_{C1}^2 + A v_{C2}^2 = v'^2_{C1} + A v'^2_{C2} \tag{61b}$$

But, from eqn. (59), and the fact that $\vec{v}\bullet\vec{v} = v^2$, we have

$$A^2 v_{C2}^2 + A v_{C2}^2 = A^2 v'^2_{C2} + A v'^2_{C2}$$

or $$v_{C2} = v'_{C2} \tag{62a}$$

and, of course, we can also show a similar relationship between the neutron speeds before and after collision in the CM system , or

$$v_{C1} = v'_{C1} \tag{62b}$$

These last two expressions say that the particle speeds in the CM system are identical before and after collision -- that is, only the neutron and target directions change, not the magnitude of the velocities. This is an important concept!

Now we have enough inter-relationships so that we can compute any number of parameters. In particular, one parameter of interest is the particle speeds in the CM system after collision as a function of the initial neutron speed in the lab system -- and this can be easily determined since eqn. (62) relates the CM speeds after collision to the COM speeds before collision, and eqn. (57) relates the CM speeds before collision to the initial neutron speed, v_{L1}, in the lab. Obviously, several other relationships can also be determined from eqns. (55) – (62).

Now, our real goal here is to interrelate the scattering angle in the two systems -- that is, to obtain an expression for θ_L vs. θ_C -- and to determine the neutron energy after collision as a function of θ_C -- that is, to relate $E'_{L1} = \frac{1}{2} m_1 v'^2_{L1}$ vs. θ_C. To do this, we note that, from eqn. (56), the neutron velocities in the center of mass and lab systems are related by the velocity of the center of mass, $\vec{v}_{CM}$ -- and this relationship is true both before and after the collision.

Thus, rearranging eqn. (56) slightly gives

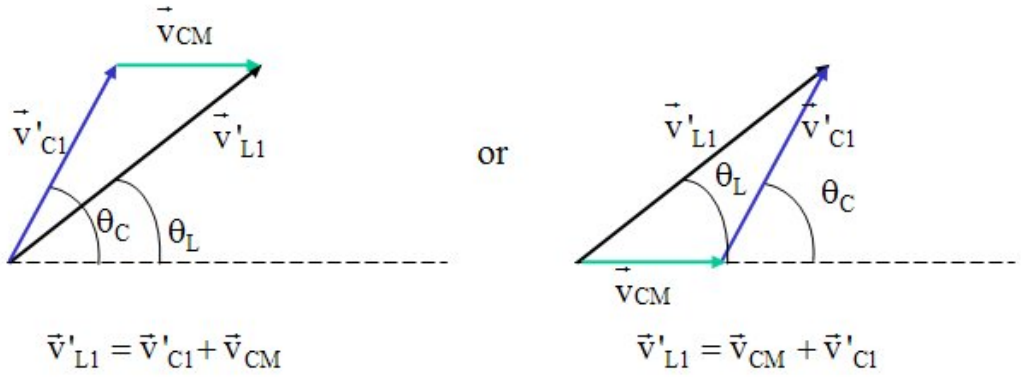
$$\vec{v}_{L1} = \vec{v}_{C1} + \vec{v}_{CM} \qquad \text{(before collision)} \qquad (63)$$

and, from a similar relationship after collision, we have

$$\vec{v}'_{L1} = \vec{v}'_{C1} + \vec{v}'_{CM} = \vec{v}'_{C1} + \vec{v}_{CM} \qquad \text{(after collision)} \qquad (64)$$

where we have used the fact that the center of mass velocity is constant and we have illustrated the last relationship (from two perspectives) in the sketch given below.

Relationship between Lab and CM Velocities



$\vec{v}'_{L1} = \vec{v}'_{C1} + \vec{v}_{CM}$ $\qquad$ $\vec{v}'_{L1} = \vec{v}_{CM} + \vec{v}'_{C1}$

The vector relationships depicted here can be broken into component form, with one component in the direction of $\vec{v}_{CM}$ and the other perpendicular to $\vec{v}_{CM}$, or

$$v'_{L1} \cos\theta_L = v_{CM} + v'_{C1} \cos\theta_C \qquad (65)$$

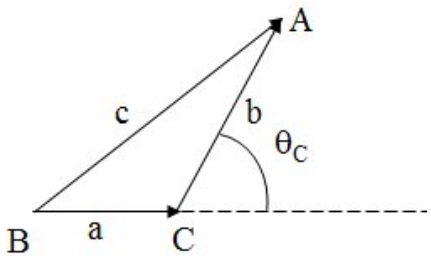
$$v'_{L1} \sin\theta_L = v'_{C1} \sin\theta_C \qquad (66)$$

Also, from the law of cosines (see sketch for notation),

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad (67)$$

we have

Law of Cosines



$$v'^2_{L1} = v^2_{CM} + v'^2_{C1} - 2v_{CM}v'_{C1}\cos(180-\theta_C)$$

but $\cos(180-\theta_C) = -\cos\theta_C$, since $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. Thus,

$$v'^2_{L1} = v^2_{CM} + v'^2_{C1} + 2v_{CM}v'_{C1}\cos\theta_C \tag{68}$$

Equation (68) is used to get a relationship for the amount of energy loss in a single scattering event, and eqn. (65) is used to relate the scattering angle in lab to the scattering angle in center of mass system.

Energy Loss per Collision

First let's develop a relatively simple expression for the neutron energy (particle 1) after a single scattering event. To do this, we start with eqn. (68) and substitute the equalities, as needed, given in eqns. (55) - (62), or

$$v'^2_{L1} = v^2_{CM} + v'^2_{C1} + 2v_{CM}v'_{C1}\cos\theta_C$$

with $v'_{C1} = v_{C1} = \dfrac{A}{1+A}v_{L1}$ and $v_{CM} = \dfrac{1}{1+A}v_{L1}$

gives

$$v'^2_{L1} = \left(\frac{1}{1+A}\right)^2 v^2_{L1} + \left(\frac{A}{1+A}\right)^2 v^2_{L1} + \frac{2A}{(1+A)^2}v^2_{L1}\cos\theta_C$$

or

$$\frac{v'^2_{L1}}{v^2_{L1}} = \frac{A^2 + 2A\cos\theta_C + 1}{(A+1)^2} \tag{69}$$

But, since

$$E'_1 = \frac{1}{2}mv'^2_{L1} \quad \text{and} \quad E_1 = \frac{1}{2}mv^2_{L1}$$

we have

$$\frac{E'_1}{E_1} = \frac{A^2 + 2A\cos\theta_C + 1}{(A+1)^2} \tag{70}$$

This is the final expression of interest here -- it gives the ratio of the neutron energy after collision to the neutron energy before collision as a function of the mass number, A, of the target nuclide and the cosine of the scattering angle in the center of mass system. From here we can ask questions concerning the magnitude of the maximum and minimum energy loss that is possible in a single scattering event and also, with some further information concerning the probability of scattering into a particular differential scattering angle $d\theta_C$ around angle θ_C, we can also address the average energy loss per collision (and this is really what we desire from this analysis).

Angular Dependence in Elastic Scatter

Now, to address the relationship between the two scattering angles (i.e., θ_L and θ_C), let's start with eqn. (65) and again substitute the relationships given in eqns. (55) - (62) as needed. Doing this gives

$$v'_{L1}\cos\theta_L = v_{CM} + v'_{C1}\cos\theta_C$$

with $\quad v'_{C1} = \dfrac{A}{1+A}v_{L1} \qquad$ and $\qquad v_{CM} = \dfrac{1}{1+A}v_{L1}$

or,

$$v'_{L1}\cos\theta_L = \frac{1}{1+A}v_{L1} + \frac{A}{1+A}v_{L1}\cos\theta_C = \frac{1+A\cos\theta_C}{1+A}v_{L1}$$

Solving this for $\cos\theta_L$ along with the use of eqn. (69) gives

$$\cos\theta_L = \frac{1+A\cos\theta_C}{1+A}\left(\frac{v_{L1}}{v'_{L1}}\right) = \frac{1+A\cos\theta_C}{1+A}\frac{1+A}{\sqrt{A^2+2A\cos\theta_C+1}}$$

or

$$\cos\theta_L = \frac{1+A\cos\theta_C}{\sqrt{A^2+2A\cos\theta_C+1}} \qquad (71)$$

This is the desired relationship and, as discussed above for the ratio E'_1/E_1, we can now address limiting conditions and, in particular, the average value of $\cos\theta_L$.

Limiting Conditions

Before addressing the average energy loss and average scattering angle questions (since we need a little further information before we can do this), let's first look at the limiting conditions associated with eqns. (70) and (71). In particular, from eqn. (71), we have the following limits:

If $\theta_C \approx 0$, then $\cos\theta_C \approx 1$, and eqn. (71) gives

$$\cos\theta_L = \frac{1+A}{\sqrt{A^2+2A+1}} = \frac{1+A}{A+1} = 1 \qquad \text{or} \qquad \theta_L \approx 0 \quad \text{(glancing collision)}$$

If $\theta_C \approx \pi$, then $\cos\theta_C \approx -1$, and eqn. (71) gives

$$\cos\theta_L = \frac{1-A}{\sqrt{A^2-2A+1}} = \frac{1-A}{\sqrt{(A-1)^2}} = \frac{1-A}{A-1} = -1 \qquad \text{or} \qquad \theta_L \approx \pi \quad \text{(backscatter collision)}$$

However, we note that there is a problem with the development of this last expression for the case of A = 1 (hydrogen). For this case, we get a divide by zero -- and, to address this situation, we start back with eqn. (71) with A = 1. Doing this gives

$$\cos\theta_L = \frac{1+\cos\theta_C}{\sqrt{1+2\cos\theta_C+1}} = \frac{1}{\sqrt{2}}\frac{1+\cos\theta_C}{\sqrt{1+\cos\theta_C}} = \frac{\sqrt{1+\cos\theta_C}}{\sqrt{2}} \qquad (\text{for } A=1) \qquad (71a)$$

Now, looking at the limits of this expression for the possible range of θ_C, we see that $\theta_L \to 0$ as $\theta_C \to 0$ (as above) and that $\theta_L \to \pi/2$ as $\theta_C \to \pi$. This means that, when scattering in hydrogen, the maximum scattering angle in the lab system is 90^o (so a 180^o backscatter is not possible in hydrogen!).

Concerning the limiting energy relationships, if there is only a glancing collision (or no collision), then the incident neutron essentially continues along its original path with $\theta_L = \theta_C \approx 0$ (as noted above). In this case, putting $\theta_C = 0$ into eqn. (70) gives

$$\frac{E'_1}{E_1} = \frac{A^2 + 2A(1) + 1}{(A+1)^2} = \frac{(A+1)^2}{(A+1)^2} = 1 \qquad \text{(no energy loss)} \tag{72a}$$

or $\quad E'_{1max} = E_1 \quad$ and $\quad \Delta E_{min} = E_1 - E'_{1max} = 0 \tag{72b}$

In contrast, when $\theta_C = \pi$, $\cos\theta_C = -1$, and we get the maximum possible energy loss, where

$$\frac{E'_1}{E_1} = \frac{A^2 + 2A(-1) + 1}{(A+1)^2} = \frac{(A-1)^2}{(A+1)^2} = \alpha \tag{73a}$$

or $\quad E'_{1min} = \alpha E_1 \quad$ and $\quad \Delta E_{max} = E_1 - E'_{1min} = (1-\alpha)E_1 \tag{73b}$

where we have defined a new parameter, α, as

$$\alpha = \frac{\text{collision}}{\text{parameter}} = \left(\frac{A-1}{A+1}\right)^2 \tag{74}$$

This term (the collision parameter) appears in many analyses and discussions of elastic scattering. In particular, we note some specific values of α for a few common nuclides:

for ^1H: $\quad A = 1 \quad \alpha = 0$

for ^{12}C: $\quad A = 12 \quad \alpha = \left(\frac{11}{13}\right)^2 = 0.716$

for ^{238}U: $\quad A = 238 \quad \alpha = \left(\frac{237}{239}\right)^2 = 0.983$

Thus, we see that it is possible to lose all the neutron energy in a single scattering collision with hydrogen but, less than 2% of the initial energy can be lost in each collision with a U238 nucleus!

Empirical Scattering Law

At present, we have eqn. (70) for a description of the energy change and eqns. (71) and (71a) for the relationship between the lab and CM scattering angles for a given scattering interaction (characterized by θ_C). The reason for describing the scattering process in this way is that, ***to a good approximation, scattering in the CM system is isotropic***. This is an extremely important statement. Isotropic means evenly distributed or that there is no preferred direction. This indicates that scattering into any particular scattering angle, θ_C, occurs with equal probability.

The above discussion describes what is known as the **Empirical Scattering Law**. It simply states that scattering in the CM system is isotropic (i.e. no preferred direction). This law is valid

for scattering in light elements to high energies and approximately true for heavy elements at relatively low energies. However, in thermal systems, most of the energy loss while slowing down occurs with light nuclei. Thus, the assumption of isotropic scattering in the CM system is an appropriate first approximation for most core physics applications of interest -- especially those dealing with thermal systems -- and the rest of our treatment of elastic scattering is tied to the assumption of isotropic scattering in the CM system (if this assumption is not valid, then all the subsequent manipulations can not be done analytically, and one has to resort to numerical evaluation of the integral relationships that follow).

Let's define a specific angular probability distribution function, $P(\hat{\Omega})$, as

$$P(\hat{\Omega})d\hat{\Omega} = \text{probability of scattering into solid angle } d\hat{\Omega} \text{ around } \hat{\Omega} \tag{75}$$

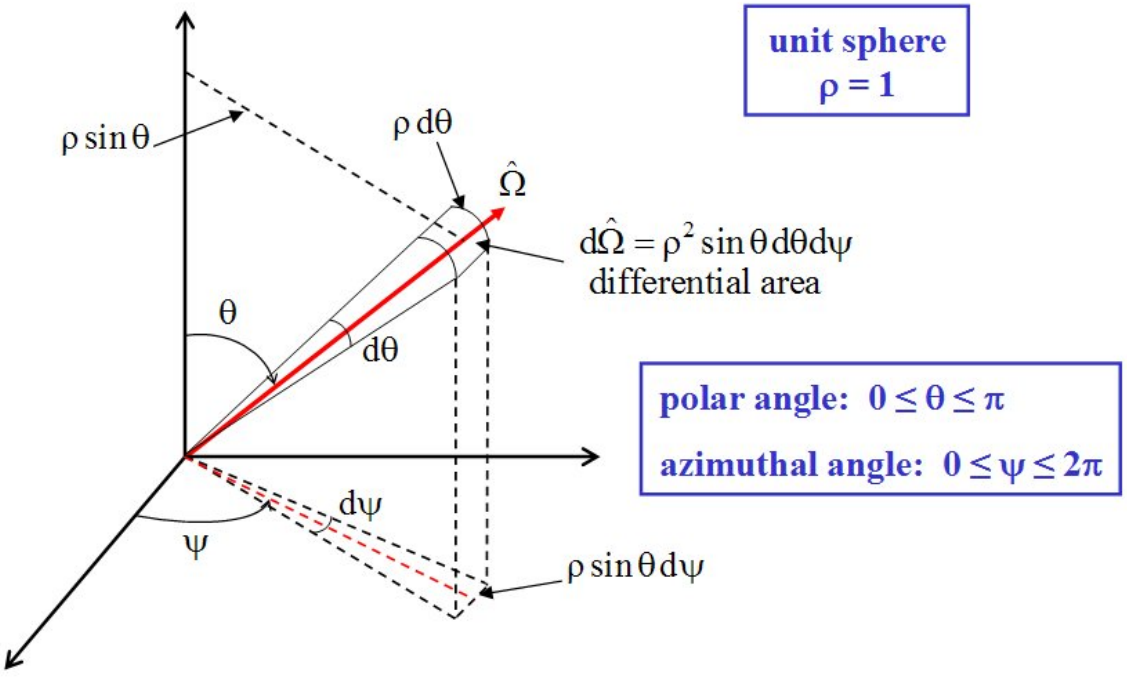
where

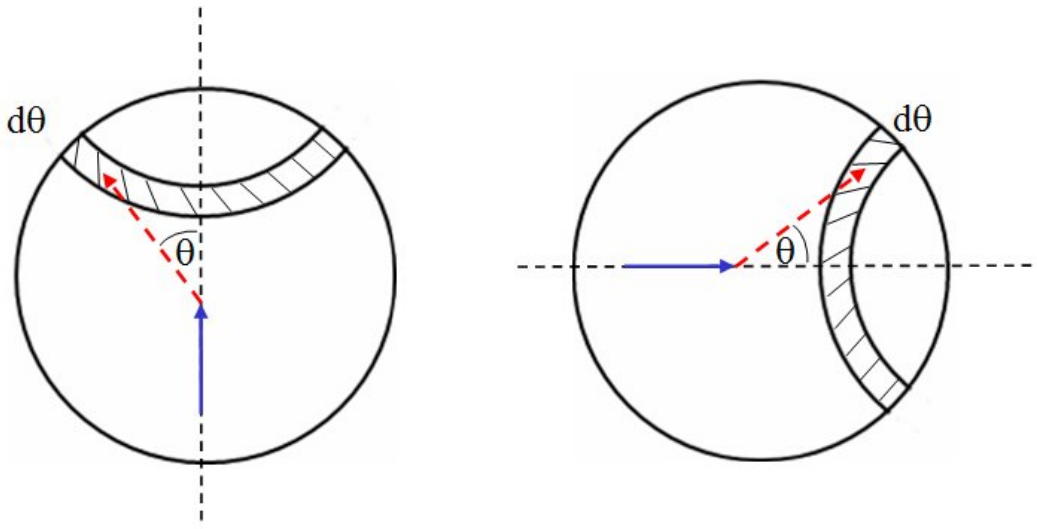
$$d\hat{\Omega} = \sin\theta\, d\theta d\psi = \text{differential area on a unit sphere } (\rho = 1) \tag{76}$$

To visualize this angular behavior, a series of two diagrams are shown below. The first sketch simply defines the notation associated with spherical coordinates, where we are interested in the case of a unit sphere (with $\rho = 1$). This diagram also clearly shows how the expression for the differential surface area is developed, and the range of both the polar and azimuthal angles. In all cases, elastic scattering is independent of the azimuthal angle, so the integral over $d\theta$ is usually performed to give a 2π factor in the expression for the differential area. Upon doing this integration, eqn. (76) becomes

$$d\hat{\Omega} = \sin\theta\, d\theta \int_0^{2\pi} d\psi = 2\pi \sin\theta\, d\theta \tag{77}$$

Spherical Coordinates





The second sketch simply shows that the scattering angle that we have been discussing can be viewed as the polar angle from spherical coordinates -- where the left side of the sketch can be compared directly to the usual orientation for spherical coordinates, and the right side of the diagram is this same picture rotated 90^o to coincide with our previous description of elastic scattering of a neutron off a target nucleus -- the θ shown here is simply the scattering angle in the CM system, θ_C, as discussed above. Note, in these sketches, $d\hat{\Omega} = 2\pi \sin\theta \, d\theta$, since the shaded area includes the integration over ψ.

Now, returning to our discussion of the angular probability distribution function, we require that $P(\hat{\Omega})$ be properly normalized, or

$$\int_{\substack{\text{all} \\ \text{angles}}} P(\hat{\Omega}) d\hat{\Omega} = 1 \tag{78}$$

But, for isotropic scattering in the CM system, $P(\hat{\Omega})$ is a constant -- that is, scattering into any polar angle between in 0 and π occurs with equal probability. Therefore,

$$\int_0^{\pi} P(\hat{\Omega}) d\hat{\Omega} = P(\hat{\Omega}) \int_0^{\pi} 2\pi \sin\theta d\theta = 1$$

or since $\int_0^{\pi} 2\pi \sin\theta d\theta = 4\pi$, then

$$P(\hat{\Omega}) = \text{constant} = \frac{1}{4\pi} \tag{79}$$

Now, recall that the two key results from our analysis of the two-body collision process -- that is, eqns. (70) and (71) -- both include cos θ_C instead of θ_C directly. Thus, one usually defines a new angular variable, μ, where

$$\mu = \cos\theta \tag{80}$$

where we have dropped the C subscript (for CM system) for convenience.

To find $P(\mu)$, we require that $P(\mu)d\mu = P(\hat{\Omega})d\hat{\Omega}$. Since $d\mu = -\sin\theta \, d\theta$, we have

$$P(\mu)d\mu = -P(\mu)\sin\theta\,d\theta = P(\hat{\Omega})d\hat{\Omega} = \frac{2\pi}{4\pi}\sin\theta\,d\theta = \frac{1}{2}\sin\theta\,d\theta$$

Thus, from this relationship, we see that $P(\mu) = -1/2$, and we note that this is already a properly normalized distribution function, since

$$\int_1^{-1} -\frac{1}{2}d\mu = -\frac{1}{2}\mu\Big|_1^{-1} = -\frac{1}{2}(-1-1) = 1$$

However, by convention, one usually lets

$$P(\mu) = \text{constant} = \frac{1}{2} \tag{81}$$

and then integrates from −1 to 1. Thus, the normalization becomes

$$\int_{-1}^{1} P(\mu)d\mu = \int_{-1}^{1}\frac{1}{2}d\mu = \frac{1}{2}\mu\Big|_{-1}^{1} = \frac{1}{2}(1+1) = 1$$

As noted above, the significance of this discussion is related to the fact that the energy after collision and the resultant scattering angle in the lab system are only a function of a single variable, $\mu = \cos\theta_C$. And for isotropic scattering, the probability per unit angle of scattering into angle $d\mu$ around μ is a constant, with $P(\mu) = 1/2$. Recalling that one of our goals in this overall analysis was to determine what happens "on the average", we see that we now have all the tools necessary to do this averaging.

Average Quantities for Isotropic Scattering

In particular, there are three average quantities associated with neutron scattering that are extremely important in characterizing the overall process, as follows:

Average Energy of Scattered Neutron

$$< E' > = \int_{-1}^{1} E'(\mu)P(\mu)d\mu = \frac{1}{2}(1+\alpha)E \tag{82}$$

Average Logarithmic Energy Decrement

$$\xi = < \ln\frac{E}{E'} > = \int_{-1}^{1} \ln\frac{E}{E'(\mu)}P(\mu)d\mu = 1 + \frac{\alpha}{1-\alpha}\ln\alpha \tag{83}$$

Average Cosine of the Scattering Angle in the Lab System

$$\bar{\mu}_0 = < \cos\theta_L > = \int_{-1}^{1} \cos\theta_L(\mu)P(\mu)d\mu = \frac{2}{3A} \tag{84}$$

In general, the integrals in these expressions for the average quantities are evaluated numerically using a known probability distribution function $P(\mu)$. However, if we assume isotropic scattering in the CM system, $P(\mu)$ is simply a constant. Thus, if isotropic scattering is a valid assumption (as it is in many cases), the integrals given in eqns. (82) - (84) can be evaluated analytically -- and these results are given as part of the above expressions. These quantities give us an overall appreciation for what happens on the average in the elastic scattering process, and they will also prove useful in the next subsection to quantify the neutronic behavior of neutron moderators.

For completeness, the derivations of the above summary results follow:

Derivation of <E'> for P(μ) = ½

The average neutron energy after collision is given by

$$< E' > = \int_{-1}^{1} E'(\mu)\, P(\mu) d\mu = \frac{1}{2}\int_{-1}^{1} E'(\mu)\, d\mu = \frac{1}{2}\int_{-1}^{1} \left[\frac{A^2 + 2A\mu + 1}{(A+1)^2} \right] E\, d\mu$$

$$= \frac{E}{2(A+1)^2} \left[\left(A^2+1\right)\mu + 2A\frac{\mu^2}{2} \right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{2(A^2+1)}{(A+1)^2} \right] E$$

However, the term in brackets is simply related to the collision parameter, α, given eqn. (74). In particular, expanding 1 + α gives

$$1+\alpha = 1+\left(\frac{A-1}{A+1}\right)^2 = \frac{(A^2+2A+1)+(A^2-2A+1)}{(A+1)^2} = \frac{2A^2+2}{(A+1)^2} = \frac{2(A^2+1)}{(A+1)^2}$$

Thus, the above expression for the average neutron energy after collision reduces to

$$< E' > = \frac{1}{2}(1+\alpha)E$$

which is the summary result given above in eqn. (82).

From this result, we can also determine the average energy loss < ΔE > as

$$< \Delta E > = E - < E' > = \left[1 - \frac{1}{2}(1+\alpha)E \right]$$

or $$< \Delta E > = \frac{1}{2}(1-\alpha)E \qquad (85)$$

Derivation of ξ = < ln E/E′ > for P(μ) = 1/2

Since the kinetic energy of the neutrons covers such a wide range (several MeV all the way down to 10^{-4} eV = 10^{-10} MeV), we often look at the energy grid on a logarithmic scale. For example, we have already seen that the energy dependence of the neutron cross sections is almost always plotted on a logarithmic scale (note that, for plotting purposes, this is usually a $\log_{10}$ scale). However, we could also work with natural logarithms which, in some cases, might facilitate our ability to analytically manipulate various expressions and might help us gain more insight into various processes -- such as the energy loss associated with elastic scattering.

In particular, the term ***lethargy*** is used to describe the relative neutron energy on a natural logarithmic scale. It is formally defined in terms of a change in the natural logarithmic energy of the neutron relative to the natural log of some arbitrarily high energy, E_o, or

$$u = \ln E_o - \ln E = \ln \frac{E_o}{E} \qquad (86)$$

where $u(E) = \ln (E_o/E)$ is said to be the lethargy of a neutron with energy E. With this definition, the change in lethargy, Δu, associated with going from energy E_1 to energy E_2 is given by

$$\Delta u = u_2 - u_1 = \ln \frac{E_o}{E_2} - \ln \frac{E_o}{E_1} = \ln E_1 - \ln E_2 = \ln \frac{E_1}{E_2} \tag{87}$$

Thus, we see that a change in lethargy is simply related to a change in the logarithmic energies and it is often referred to as the ***logarithmic energy decrement***.

The word lethargy can be used to describe the level of exhaustion or sluggishness of an object. Within the context of elastic scattering, this term is used to describe the slowness of the neutron -- that is, as the neutron slows downs from high energy to low energy, it becomes more lethargic, or its lethargy increases. Thus, as the neutron energy decreases, its lethargy increases!

Now, in a single elastic scattering interaction, the neutron before collision has energy E and the energy after collision is denoted as E′ -- thus the change in lethargy in a single scattering event is simply $\Delta u = \ln (E/E')$, where $E' = E'(\mu)$ is a function of the scattering angle in the CM system. And, as above, our real interest here is in the average value of this quantity -- that is, the ***average change in lethargy*** or the ***average logarithmic energy decrement***. In the literature, the average logarithmic energy decrement is given the symbol ξ and its development from the basic definition is as follows:

$$\xi = <\Delta u> = <\ln \frac{E}{E'}> = \int_{-1}^{1} \ln \frac{E}{E'(\mu)} P(\mu)\, d\mu = \frac{1}{2}\int_{-1}^{1} \ln \frac{E}{E'(\mu)}\, d\mu$$

In order to perform the integration, we make a change of variables. In particular,

$$E' = \frac{(A^2 + 2A\mu + 1)E}{(A+1)^2} \qquad \text{and} \qquad dE' = \frac{2AE}{(A+1)^2}\, d\mu$$

Also for $\mu = -1$, $\quad E' = \frac{(A-1)^2}{(A+1)^2} E = \alpha E$

and for $\mu = 1$, we have $\quad E' = \frac{(A+1)^2}{(A+1)^2} E = E$

Therefore, we can write the expression for ξ as

$$\xi = \frac{1}{2}\frac{(A+1)^2}{2AE}\int_{\alpha E}^{E} \ln \frac{E}{E'}\, dE' = \frac{1}{(1-\alpha)E}\int_{\alpha E}^{E} \ln \frac{E}{E'}\, dE' \tag{88}$$

where, in the last step, we have used the following equality,

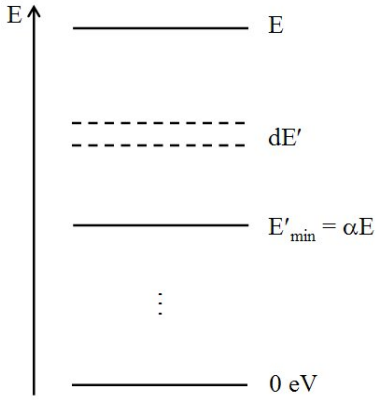
$$\frac{1}{(1-\alpha)} = \frac{1}{1 - \frac{(A-1)^2}{(A+1)^2}} = \frac{(A+1)^2}{(A+1)^2 - (A-1)^2} = \frac{(A+1)^2}{4A}$$

As a side note, it is interesting to observe that, in the transformation that leads to eqn. (88), we have effectively defined another probability distribution function, P(E→E'), where

P(E→E') dE' = probability that a neutron with energy E will scatter into dE' around E'

with

$$P(E \to E')\,dE' = \frac{dE'}{(1-\alpha)E} \qquad (89)$$



This says that the probability of scattering into interval dE' is simply the interval width dE' divided by the total possible scattering interval $E - E'_{min} = (1 - \alpha)E$ (see the sketch for a graphical representation).

With this new (properly normalized) probability distribution, the average value of any function of E', say f(E'), is then

$$< f(E') > = \int_{\alpha E}^{E} f(E')P(E \to E')\,dE' \qquad (90)$$

For example, the average value of E' (already calculated above) could be found in this way. We shall see that this form of energy transfer -- that is, the E → E' notation -- will be used again in the form of g → g' when we discuss the formal multigroup balance equation later in the course.

Now, getting back to evaluating ξ, we let x = E'/E and dx = dE'/E in eqn. (88), giving

$$\xi = \frac{1}{(1-\alpha)E}\int_{\alpha E}^{E} \ln\frac{E}{E'}\,dE' = \frac{1}{(1-\alpha)}\int_{\alpha}^{1} \ln\frac{1}{x}\,dx = \frac{-1}{(1-\alpha)}\int_{\alpha}^{1} \ln x\,dx$$

$$= \frac{-1}{1-\alpha}\left[x\ln x - x\right]_{\alpha}^{1} = \frac{-1}{1-\alpha}\left[-1-\left(\alpha\ln\alpha-\alpha\right)\right]$$

$$= \frac{1}{1-\alpha}\left[1-\alpha+\alpha\ln\alpha\right]$$

or $$\xi = 1 + \frac{\alpha}{1-\alpha}\ln\alpha$$

which is the summary result given in eqn. (83). Note that, for A = 1, α = 0, and ξ = 1 (this comes about because the term α ln α → 0 as α → 0).

In closing our discussion relative to the average change in lethargy per collision, we note that a particularly useful quantity is the average number of collisions to slow down from E_1 to E_2 (or from E to some E′). This is simply the change in lethargy associated with scattering from E_1 to E_2 divided by the average increase in lethargy per collision. In equation form, we have

$$\text{average number of collisions to go from } E_1 \text{ to } E_2 = \frac{\Delta u}{<\Delta u>} = \frac{1}{\xi}\ln\frac{E_1}{E_2} \qquad (91)$$

Derivation of $\bar{\mu}_0$ = <cosθ > for P(μ) = 1/2

The average value of the cosine of the scattering angle in the lab system is given by

$$\bar{\mu}_0 = <\mu_0> = <\cos\theta_L> = \int_{-1}^{1} \cos\theta_L(\mu) P(\mu)\, d\mu = \frac{1}{2}\int_{-1}^{1} \mu_0(\mu)\, d\mu = \frac{1}{2}\int_{-1}^{1} \frac{(1+A\mu)}{\sqrt{A^2+2A\mu+1}}\, d\mu$$

From a table of integrals, one has

$$\int \frac{cx+d}{\sqrt{ax+b}}\, dx = \frac{2}{3a^2}\left(3ad - 2bc + acx\right)\sqrt{ax+b}$$

Applying this generic expression to the specific case of interest here gives

a = 2A b = A^2 + 1 c = A and d = 1

which leads to

$$\bar{\mu}_0 = \left\{\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{1}{4A^2}\right)\left[6A - 2A(A^2+1) + 2A^2\mu\right]\sqrt{2A\mu + A^2 + 1}\right\}\Bigg|_{-1}^{1}$$

$$= \left\{\frac{1}{12A^2}\left[-2A^3 + 4A + 2A^2\mu\right]\sqrt{2A\mu + A^2 + 1}\right\}\Bigg|_{-1}^{1}$$

$$= \left\{\frac{1}{6A}\left[-A^2 + A\mu + 2\right]\sqrt{A^2 + 2A\mu + 1}\right\}\Bigg|_{-1}^{1}$$

$$= \frac{1}{6A}\left[\left(-A^2 + A + 2\right)(A+1) - \left(-A^2 - A + 2\right)(A-1)\right]$$

$$= \frac{1}{6A}\left[\left(-A^3 + 3A + 2\right) - \left(-A^3 + 3A - 2\right)\right] = \frac{4}{6A} = \frac{2}{3A}$$

or simply

$$\bar{\mu}_0 = \frac{2}{3A}$$

which is the summary result given in eqn. (84).

As an example, for A = 1, $\bar{\mu}_0$ = 0.667 and this gives $\bar{\theta}_L \approx 48^o$. For A very large, $\bar{\mu}_0 \to 0$ and $\bar{\theta}_L \to 90°$. Thus, scattering in the lab system is somewhat forward peaked. The average scattering angle for light nuclei is in the 45° range and for very heavy nuclides scattering occurs around $\bar{\theta}_L \approx 90^o$ on the average.

Characterization of Moderators

The parameters discussed in the previous subsection on Elastic Scattering can now be used to help characterize the neutronic performance of neutron moderators. Recall that a moderator is a material whose primary function is to slow down neutrons from fast energies to thermal. As such, it should have a relatively high scattering cross section and, since we want the neutrons to make it to thermal and eventually cause fission, the absorption cross section of the moderator

should be as small as possible. In addition, we would like the average energy loss per collision with a moderator nucleus to be as high as possible. This will guarantee that the neutron will slow down quickly (in a minimum number of collisions). Several common non-neutronic constraints on any material for use in the reactor are also applicable. Some of these considerations are relative abundance, low cost, and chemical compatibility with other materials. All these factors, neutronic and non-neutronic, must be considered in choosing the best moderator material for a particular reactor design.

The neutronic behavior of moderators is of primary interest here. What is desired is a single quantity that can be used to measure performance. The slowing down power (which is the product of ξ and Σ_s) is a measure of the average increase in lethargy per distance traveled. It combines the desire to have a high Σ_s and a large energy decrement per collision, ξ. The moderating ratio (which is $\xi\Sigma_s/\Sigma_a$) introduces the fact that absorption properties are also important. Thus, the higher the moderating ratio, the better the material is as a medium to slow down neutrons.

In summarizing this topic, a short table which highlights the scattering properties of several nuclei is given below. In addition, the slowing down power and moderating ratio are also tabulated for several potential moderators. These data are given in the following tables:

Scattering Properties of Several Nuclei

Element	A	α	ξ	Collisions to Thermalize (2 MeV $\rightarrow$ 0.025 eV)
hydrogen	1	0	1.000	18
deuterium	2	0.111	0.725	25
beryllium	9	0.640	0.206	86
boron (natural)	10.8	0.690	0.171	106
carbon	12	0.716	0.158	114
uranium	238	0.983	0.0084	2170

Characterization of Common Moderators

Moderator	Slowing Down Power ($\xi\ \Sigma_s$)	Moderating Ratio ($\xi\ \Sigma_s/\Sigma_a$)
water	1.28	58
heavy water	0.18	21000
beryllium	0.16	130
boron (natural)	0.06	0.0006
carbon (reactor grade)	0.065	200

The main observations to be made here are:

1. Nuclei with low mass number have the best scattering properties.
2. However, the scattering cross section and absorption cross section play a key role in the overall characterization of a good moderator.

From the second table, all other things being equal, we see that heavy water is by far the best choice as a moderating medium. However, other factors such as cost, availability, compatibility, tritium production (important in heavy water), etc., are also very important. The final choice for a particular reactor design is made by considering a number of aspects and by making the best overall compromises.

Obtaining Broad Group Data for Reactor Calculations

The last subject to be treated in this section is related to the question, "How does one get few group data for use in core physics calculations?" This is not an easy question to answer and the discussion that follows is only intended as a rough overview or outline of the basic procedure. The details of each step of the process are usually treated in graduate level reactor physics and/or nuclear physics courses (and these details are well beyond the scope of this introductory course).

The starting point for any physics analysis is at the level of the basic neutron reaction data. We have discussed the general nature of $\sigma(E)$. There is an enormous amount of information contained here -- this can be appreciated by exploring the information available via the JANIS program (Java-based Nuclear Information Software). To quantitatively process these data for use in reactor physics applications is a large undertaking. The data have been compiled in various standard formats, such as the Evaluated Nuclear Data File (ENDF) format. The latest version of ENDF data is ENDF/B-VII, but many people are still using ENDF/B-VI data since each update takes years of effort to assemble and evaluate and to get into general use. The data contained include pointwise cross sections versus energy for all the partial cross sections (capture, fission, elastic and inelastic scattering, charged particles reactions, etc.), enough resonance region information to reconstruct many of the isolated resonances, detailed emission spectra for both prompt and delayed neutrons, and also uncertainty information reflecting an estimate of the errors/uncertainties in the original experimental data and nuclear models used to obtain the basic $\sigma(E)$ data.

The data contained in the ENDF/B files represent a considerable inventory of basic knowledge. However, the data files are too large and complicated to be accessed and processed on a daily basis, each time a new core physics computation is needed. Instead, the ENDF files are usually processed into "Pseudo Problem Independent Fine Group Libraries", which are more manageable data sets that can be accessed as necessary for routine reactor computations.

The basic procedure here involves averaging the pointwise data (essentially continuous energy data) over relatively small energy intervals to obtain a set of fine group data. Fine group libraries refer to data sets that may have up to a few hundred energy intervals and still retain sufficient information and energy resolution to address a variety of different problem types, but are significantly reduced in scope and complexity from the raw nuclear data in the ENDF files. The fine group averaged cross sections are computed using a weight function that describes the probability of finding neutrons within a certain energy range, or

$$\sigma_g = \frac{\int_{\Delta E_g} \sigma(E)\phi(E)dE}{\int_{\Delta E_g} \phi(E)dE} \qquad (92)$$

where $\phi(E)$ is this weight (or distribution) function and ΔE_g is the energy interval of interest.

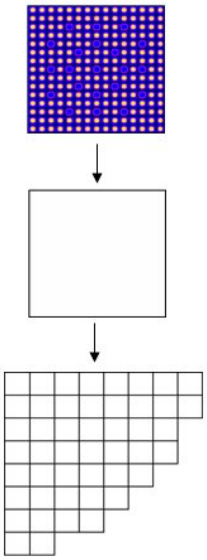
The neutron flux per unit energy, $\phi(E)$, that is used as an appropriate weight function is usually an assumed function for a particular class of problems. It is usually determined from simplified theoretical calculations. For example, for thermal reactor analyses, a typical weight function would be a Maxwellian at low energies, $1/(\Sigma_t E)$ spectrum at intermediate slowing down energies (this can be developed from a formal study of slowing down theory), and a $\chi(E)$ fission spectrum at high energies. Using this weight function would give a pseudo problem independent library for thermal reactor applications. Other weight functions appropriate to fast reactor work, shielding applications, or fusion analysis can be used to develop fine group libraries for these pseudo problem independent applications.

The next step is to collapse the fine group generic data to problem specific few group constants that can be used with multidimensional core physics codes. This processing step is functionally similar to the previous one (going from point data to fine group data) in that we again want to average the cross sections over energy. This time, however, we need to build in some details concerning the specific system of interest (such as geometry, material composition, operating temperatures, etc.). This is required since much of the energy detail will be lost through the averaging process. In addition, if the cross sections are to be averaged over some spatial region, the spectral and spatial variation of $\sigma(\vec{r},E)$ and the weight function $\phi(\vec{r},E)$ must be considered.

The equation to be used for collapsing the fine group data for some spatial region z is of the form

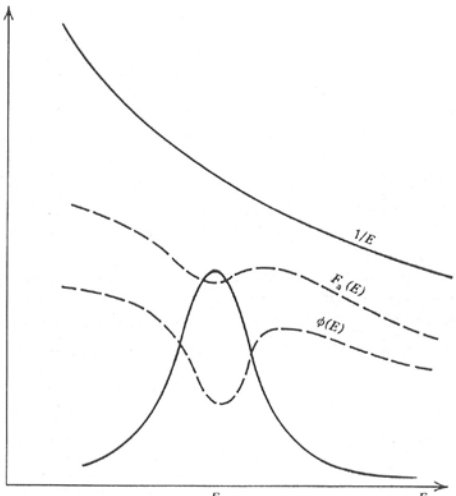
$$\sigma_{gz} = \frac{\frac{1}{N_z}\sum_{g'\in g}\int_{v_z} N(\vec{r})\sigma_{g'}(\vec{r})\phi_{g'}(\vec{r})d\vec{r}}{\sum_{g'\in g}\int_{v_z}\phi_{g'}(\vec{r})d\vec{r}} \qquad (93)$$

where g' refers to the fine group number and g is a broad group index. To evaluate eqn. (93), one needs to know the geometry (denoted by the spatial variable $\vec{r}$ and domain of interest v_z), the material composition (denoted by $N(\vec{r})$ and its average value over v_z as N_z), and the spatial variation of the fine group weight function, $\phi_{g'}(\vec{r})$. This typically requires that a fine group 1-D or 2-D model be employed to solve for $\phi_{g'}(\vec{r})$.



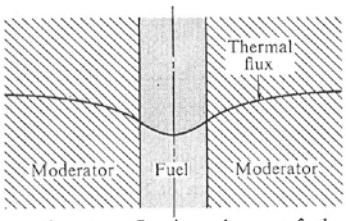
In doing this, some representative portion of the overall heterogeneous geometry is modeled in one-dimensional or two-dimensional geometry. This unit cell or unit assembly calculation is designed to be simple enough so that $\phi_{g'}(\vec{r})$ can be computed, but accurate enough so that the resultant average cross sections are indicative of a full multidimensional heterogeneous geometry fine group analysis. The broad group cell or assembly averaged cross sections can then be used in full or partial core 2-D and 3-D computer models that only incorporate homogeneous regions within the models.

The key goal, of course, is to somehow incorporate all the detailed information that is available in the heterogeneous geometry and fine-group energy resolution into the broad group assembly averaged cross sections. This is not an easy task, and there is still ongoing research on how best to accomplish this task. The two key issues involved here relate to the concepts of space and energy self shielding. These two concepts are illustrated nicely in the two sketches: one showing the flux depression that can occur in the neighborhood of a resonance, and the other showing the depression in the thermal flux that often occurs in the vicinity of a fuel rod (or other absorber material). These indicate the fine detail that must be accounted for in the space and energy dependent weight function, $\phi_{g'}(\vec{r})$, when collapsing to the problem-dependent broad group level.



Flux depression in the neighborhood of a resonance.

Clearly some details have been left out of the above discussion. In addition, it should be noted that there is no single path to follow, and it is somewhat of an art to perform these kinds of computations. The definition of the unit cell, the choice of the initial fine group and resultant broad group energy structures, and the selection of the calculational methods to compute $\phi_{g'}(\vec{r})$ are decisions that need to be made. Only some experience (and good fortune) will get you through the torturous path of generating broad group cross sections.



Thermal neutron flux in and near a fuel rod.

The above discussion is intended only as an introduction to this subject. We will be discussing and using broad group cross section data (typically 1 to 4 energy groups) in subsequent discussions and assignments. These data sets don't appear by magic, but are developed by careful processing of the basic nuclear data. There are three levels of data organization and two separate processing methodologies used to process the data to the next lower level. This interaction can be summarized as follows:

ENDF/B (continuous energy)	] assume/compute generic $\phi(E)$ to collapse from $E \rightarrow g'$
Pseudo Problem Independent (fine group)	specify geometry and material configuration and calculate $\phi_{g'}(\vec{r})$ to collapse from $g' \rightarrow g$ averaged over some spatial domain
Problem Dependent (few group)	

The broad-group data are then used in 2-D and 3-D full or partial core physics calculations to determine the global flux and power distributions. Our emphasis in the next section of this course is on the calculation and understanding of core performance. Our focus will be on the spatial distribution of neutrons in moderating systems and in critical core configurations. These discussions will highlight the use of few group theory (primarily 1 and 2 groups) for illustrating and understanding the basic concepts of interest. Thus, we will rely heavily on the availability of broad group data for these analyses. The whole point of this last subsection is to make you aware, in a general way, of how these broad group data are generated…