24.331 Fundamentals of NSE

Lesson 10: Neutron Interactions with Matter II

General Nature of σ(E)

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See Section 7.4 in your text by Shultis & Faw (with lots of additional info)

Lesson 10 Objectives

- Describe qualitatively the energy dependence of most neutron cross sections.
- Compare the σ(E) behavior of typical absorption, elastic scattering, and inelastic scattering cross sections.
- Compare the σ(E) behavior for fission in fissile versus non-fissile nuclides.
- Use the JANIS program to obtain σ(E) plots and to get numerical values for various cross sections at a particular energy.
- Assuming 1/v behavior, show how one can obtain the absorption rate integrated over the thermal region.
- Describe the non-1/v factor and explain how it is computed and used.

Lesson 10 Objectives (cont.)

Describe the distribution function f(E) and explain how this is used to compute average cross sections and other average quantities.

Assuming a Maxwellian distribution function at low energies, compute the average energy, the most probable energy, and the energy associated with the most probable speed.

General Nature of σ(E)

Neutron cross sections are strong functions of the incident energy of the neutron involved in the reaction.

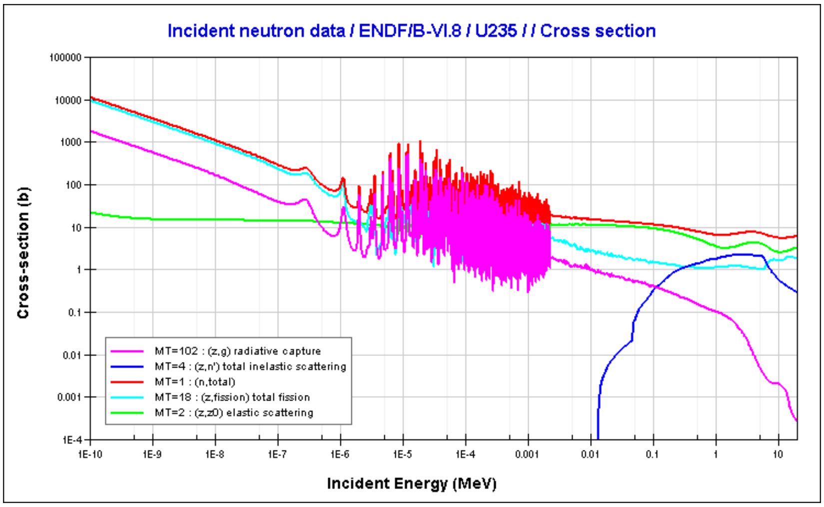
For most isotopes and reaction types, the energy spectrum can be broken into three ranges; low, resonance, and high energies.

The specific boundary between regions varies considerably, where, in general, the resonance region tends downward in energy as the mass number of the target becomes larger.

Thus, the boundaries between the three generic energy regions tend to shift with the mass number of the target nucleus.

The best way to illustrate these tendencies is through actual examples that show the behavior of σ(E) versus E for specific nuclides and reactions -- and several examples are given on the following slides (these figures were generated with the JANIS 3.0 program using ENDFB-VI data).

U235 Cross Sections



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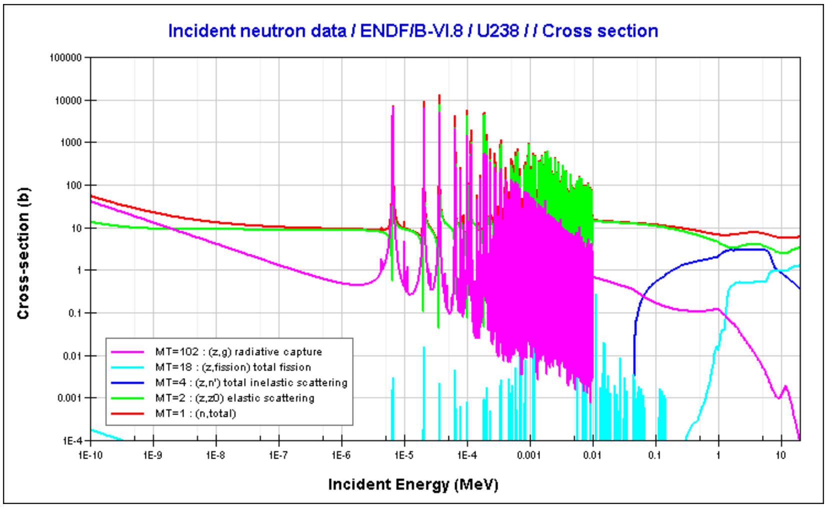
U238 Cross Sections



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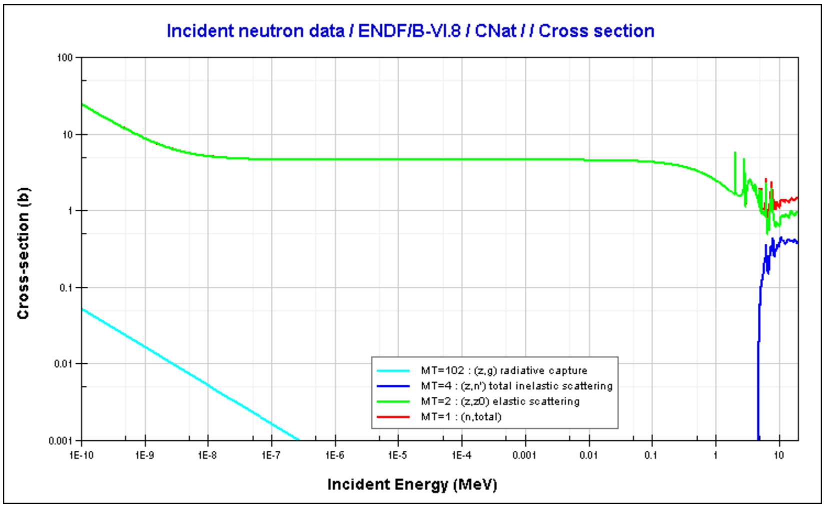
Natural Carbon Cross Sections



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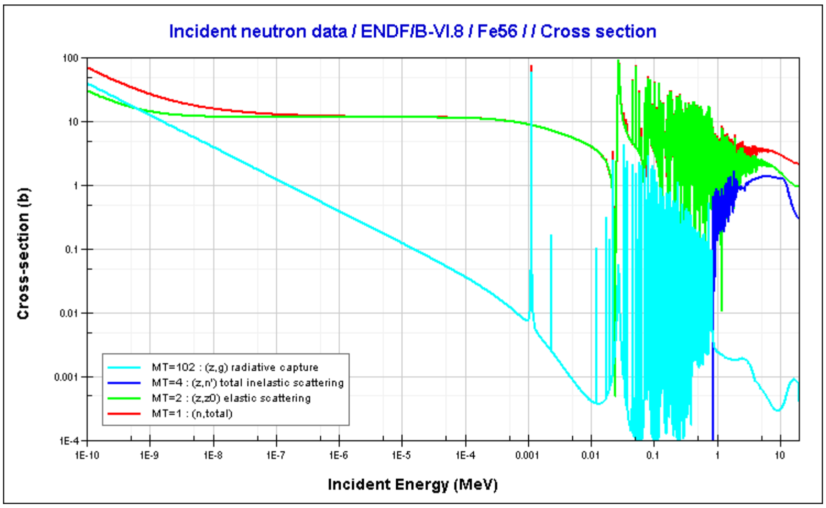
Fe56 Cross Sections



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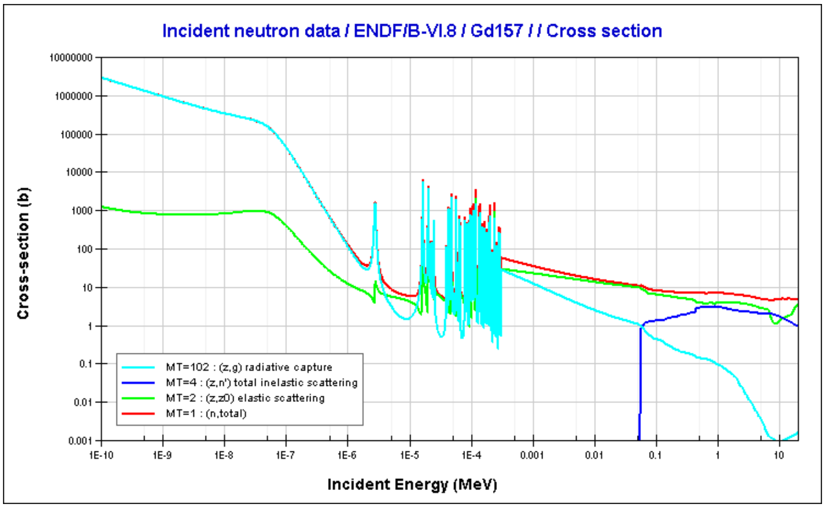
Gd157 Cross Sections



Scattering Cross Sections

Elastic scattering:

Effectively a classical two-body collision process, where momentum and energy are transferred in the collision.

This is especially true in the potential scattering region where $\sigma_s \approx 4\pi R^2$ with R as the nuclear radius (no compound nucleus formation).

In the resonance region and above, a compound nucleus is formed but, upon ejection of the scattered neutron, the target is left in its ground state.

We will address elastic scattering in detail in a later Lesson…

Scattering Cross Sections (cont.)

Inelastic scattering:

Energy loss is through the excitation of the target nucleus, with the subsequent emission of γ-rays as the nucleus decays to its ground state.

The neutron's kinetic energy must be greater than the energy of the first excited state of the target -- thus, inelastic scattering is a threshold reaction.

The threshold energy decreases with increasing A -- but this is still a relatively high energy reaction in all nuclides.

Inelastic scattering is certainly important (especially in fast reactors and for shielding applications), but elastic scattering is usually the dominant reaction at most energies for many applications.

Absorption Cross Sections

Recall that $\sigma_a \approx \sigma_c + \sigma_f$. Therefore, absorption and radiative capture are nearly identical (for most non-fissionable nuclides).

Fission for non-fissile isotopes is a threshold reaction (e.g. U238).

The general behavior of the capture and fission processes was illustrated in the previous figures:

- **The low, resonance, and high energy regions are quite distinct.**
- **The wild resonance variations and the smoothly decreasing function at high energy are similar in nature to the elastic scattering cross section.**
- **In contrast, however, the low energy component has an approximate 1/v behavior at energies below the resonance region (as compared to the constant potential scattering region that occurs for elastic scatter).**

1/v Behavior

The 1/v behavior for capture and fission at low energies is very important and it justifies further analysis here.

In fact, we can make the following bold statement:

"For 1/v-absorbers, the thermal absorption rate is independent of the energy distribution of the neutrons and it is determined by the cross section at an arbitrary energy (usually E_o = 0.0253eV)."

If 1/v behavior is valid then

$$\sigma_a(E) = \frac{C_1}{v(E)} = \sigma_a(E_o)\frac{v_o}{v(E)} \quad \text{or} \quad \sigma_a(E) = \frac{C_2}{\sqrt{E}} = \sigma_a(E_o)\left(\frac{E_o}{E}\right)^{1/2}$$

Also recall that the absorption rate (denoted as F_a) can be written as

$$F_a = \int N\sigma_a(E)\phi(E)dE = \int N\sigma_a(E)n(E)v(E)dE = \int \Sigma_a(E)n(E)v(E)dE$$

1/v Behavior (cont.)

But, from above, we have

$$\Sigma_a(E) = \Sigma_a(E_o)\frac{v_o}{v(E)}$$

Therefore,

$$F_a = \Sigma_a(E_o)v_o\int n(E)dE = \Sigma_a(E_o)v_o n_{tot}$$

where n_{tot} is the total (energy integrated) neutron density with units of neutrons/cm^3.

Since the neutron density times the neutron speed is denoted as the neutron flux, $\phi(E) = n(E)v(E)$, we can define $\phi_o = n_{tot}v_o$ as the 2200 m/s flux, and the above expression for F_a becomes

$$F_a = \Sigma_a(E_o)\phi_o$$

$\phi_o = n_{tot}v_o$ is simply a convenient measure of the total neutron level in a thermal system

1/v Behavior (cont.)

This last relationship supports the above statement that the absorption rate is independent of the energy distribution and can be characterized at a single energy E_o.

Since much of the available data is tabulated at 0.0253 eV, E_o is usually taken to be this value.

The neutron speed, $v_o = \sqrt{2E_o / m}$, which corresponds to E_o = 0.0253 eV is approximately 2200 m/s -- this is why we called ϕ_o the 2200 m/s flux).

In some cases, the assumption of 1/v behavior may not be exact, and this leads to the discussion of the "non-1/v factor" (see next slide)…

Non-1/v Factor

If one assumes that n(E) is given by the Maxwellian distribution (see below) characterized at some temperature T, then a correction factor or non-1/v factor can be defined as follows:

$$F_a(T) = \int \Sigma_a(E)n(E,T)v(E)dE = g_a(T)\Sigma_a(E_o)\phi_o$$

or

$$g_a(T) = \frac{F_a(T)}{\Sigma_a(E_o)\phi_o}$$

where $g_a(T)$ is called the non-1/v factor for absorption.

In practice, values of $g_a(T)$ are computed by numerically evaluating the first part of the above equation with n(E,T) given by the Maxwellian distribution, and also by calculating $\Sigma_a(E_o)\phi_o$ (which assumes 1/v behavior) -- the ratio of the two quantities is $g_a(T)$ as given above.

These non-1/v factors are then tabulated for later use…

Distribution Functions

Up to this point, we have noted several times, without any real justification, that a neutron energy of 0.0253 eV -- which corresponds to a neutron speed of 2200 m/s -- takes on special importance when dealing with thermal systems.

In addition, in the above slides, we have referred to the so-called thermal Maxwellian distribution without much explanation at all.

In fact, these two subjects are directly related, and the next few slides will try to shed some light on these concepts as well as on a more general picture of the relationship between neutron flux and neutron density.

In general, the neutron density is usually a function of time, three spatial coordinates, and three velocity coordinates.

This seven-dimensional phase space is usually too complicated to work with for normal everyday applications.

Distribution Functions

Thus, as part of a practical model reduction process, we seek to simplify the full functional dependence of the angular neutron flux:

$$\phi(\vec{r},\vec{v},t) = n(\vec{r},\vec{v},t)\,v(E)$$

To simplify things, let's write the neutron velocity vector, $\vec{v}$, in terms of a magnitude or speed, v, and a unit vector, $\hat{\Omega}$, that points in the direction of neutron travel, or $\vec{v} = v\hat{\Omega}$.

Also, since the neutron energy and speed are directly related via $E = ½ mv^2$, we can interchange the dependence on v or E as desired.

Thus, the angular flux is usually written as

$$\boxed{\phi(\vec{r},E,\hat{\Omega},t) = n(\vec{r},E,\hat{\Omega},t)\,v(E)}$$ angular neutron flux

Distribution Functions (cont.)

However, in most applications, we are interested in only the space and energy distribution of the scalar neutron flux, $\phi(\vec{r},E)$.

To reduce $\phi(\vec{r},E,\hat{\Omega},t)$ to this level, we assume steady state behavior and integrate over all angles (i.e. the scalar flux is simply the angular flux integrated over all angles).

Now, the scalar flux, $\phi(\vec{r},E)$, can be written in terms of the scalar neutron density, $n(\vec{r},E)$, which has units of neutrons per unit volume per unit energy, and the neutron speed, $v(E)$, written in units of distance per unit time.

Doing this gives,

$$\boxed{\phi(\vec{r},E) = n(\vec{r},E)\,v(E)}$$ scalar neutron flux

Distribution Functions (cont.)

We can define the scalar neutron density precisely in differential terms as,

$$n(\vec{r},E)d\vec{r}dE = \begin{array}{c}\text{number of neutrons in } d\vec{r} \text{ around } \vec{r}\\ \text{and in } dE \text{ around } E\end{array}$$

If we can make a case for separability of space and energy, then

$$n(\vec{r},E) = n(\vec{r})\,f(E)$$

where

n([redacted]

$$f(E)\,dE = \begin{array}{c}\text{probability of finding a particle}\\ \text{in energy interval } dE \text{ around } E\end{array}$$

It is this last term, f(E)dE, that is an example of a probability distribution function.

Distribution Functions (cont.)

If the scalar neutron density is integrated over all space and energy, the result is simply the total number of neutrons in the volume of interest, or

$$\int n(\vec{r}) f(E) d\vec{r} dE = \int n(\vec{r}) d\vec{r} \int f(E) dE = \text{number of particles in volume}$$

On the Subject of Separability

To put the separability assumption into perspective, let's consider the case of a very large system.

As discussed previously, neutrons must eventually be absorbed or leak out of the system.

As the system becomes large, the leakage term decreases and, in the limit, the leakage goes to zero as the volume becomes infinite.

Distribution Functions (cont.)

When we develop the full neutron balance equation (as part of the Reactor Theory course next semester), it will become clear that it is the differential leakage term that accounts for the spatial redistribution of neutrons among different regions.

If a critical reactor system is homogeneous (i.e. with constant material properties) and infinite in size (relative to the distance traveled by the neutron), there is no leakage and thus the spatial distribution of the neutrons is simply a constant.

Thus, in an infinite homogeneous critical system, the neutron density becomes

$$n(\vec{r}, E) = n_{tot} f(E)$$

where n_{tot} is a constant (with units of neutrons per cm^3).

Therefore, in an infinite homogeneous critical reactor, the separability approximation used above becomes exact!!!

Distribution Functions (cont.)

In a real reactor of finite proportions, we are interested in both the space and energy distribution of the neutrons.

However, in determining this behavior, it is convenient to first approximate the energy behavior with the infinite reactor distribution function, f(E).

Then one uses this distribution function to formally break the continuous energy variable into a discrete multigroup formulation.

This is done by averaging all the variables over discrete energy ranges.

Recall that, when averaging any quantity over energy, one needs to weight that quantity by the probability of finding the particles in any given energy region.

Distribution Functions (cont.)

Thus, the average value of any quantity, g(E), over some energy interval, ΔE, is given by

$$\bar{g} = \langle g \rangle = \frac{\int_{\Delta E} g(E) f(E) dE}{\int_{\Delta E} f(E) dE}$$

where the distribution function, f(E), is the importance or weight function that properly accounts for the fact that there may be more particles at some energies than at others.

Now, in the context of reactor physics applications, we are interested in calculating average multigroup cross sections for use in solving the multigroup diffusion equation (to be derived in the Reactor Theory class).

Distribution Functions (cont.)

If we denote σ_g as the average value of $\sigma(E)$ over energy interval ΔE_g, then σ_g is usually given by

$$\sigma_g = \frac{\int_{\Delta E_g} \sigma(E)\phi(E)dE}{\int_{\Delta E_g} \phi(E)dE}$$

where in many applications, the weight function is simply the energy dependent neutron flux.

Recalling that,

$$\phi(E) = n(E)v(E) \approx n_{tot} f(E)v(E)$$

we see that knowing the distribution function, f(E), will allow determining the desired averages.

The Maxwellian Distribution

In a dilute gas, the energies of the gas atoms or molecules are distributed according to the well-known Maxwellian distribution function.

In a thermal reactor, the low energy neutrons can be thought of as a dilute neutron gas and, to a good approximation, the energy distribution of these neutrons is given by the same distribution function that describes real gas particles.

In particular, the Maxwellian distribution is given as

$$\boxed{f(E) = CE^{\frac{1}{2}} e^{-\frac{E}{kT}}}$$

where k is Boltzmann's constant (8.6173×10^{-5} eV/K) and T is the absolute temperature of the (neutron) gas.

The Maxwellian Distribution

For this to be a properly normalized distribution function, we require that the integral over all energy be unity, or

$$\int_0^\infty f(E)dE = C\int_0^\infty E^{\frac{1}{2}} e^{-\frac{E}{kT}} dE = 1$$

Performing the integration and solving for C gives: $\boxed{C = \frac{2\pi}{(\pi kT)^{\frac{3}{2}}}}$

Given that the energy distribution of the low energy neutrons is described by f(E), it is appropriate to ask, What is the average particle energy? or What is the most probable particle energy? .

From the above discussion, we see that the average energy, < E >, is given by

$$<E> = \frac{\int_0^\infty Ef(E)dE}{\int_0^\infty f(E)dE}$$

The Maxwellian Distribution (cont.)

Since f(E) is already properly normalized, this becomes

$$<E> = C\int_0^\infty E^{\frac{3}{2}} e^{-\frac{E}{kT}} dE = \tfrac{3}{2}kT$$

Thus, the average energy in a thermal system operating at temperature T is given by

$$E_{ave} = <E> = \tfrac{3}{2}kT$$

For the most probable energy, one differentiates the function and sets the derivative to zero.

Setting df(E)/dE = 0 gives E_p = ½ kT as the most probable energy in a thermal system operating at temperature T.

As apparent from these last two results, the combination of terms, kT, appears frequently in reactor physics work.

The Maxwellian Distribution (cont.)

In fact, if one transforms the Maxwellian distribution function, f(E), to a velocity space distribution, f(v), and then finds the most probable speed (using a procedure similar to that described above), the result is

$$v_p = \sqrt{2kT/m}$$

And, since E = ½ mv², we see that the energy associated with the most probable speed is

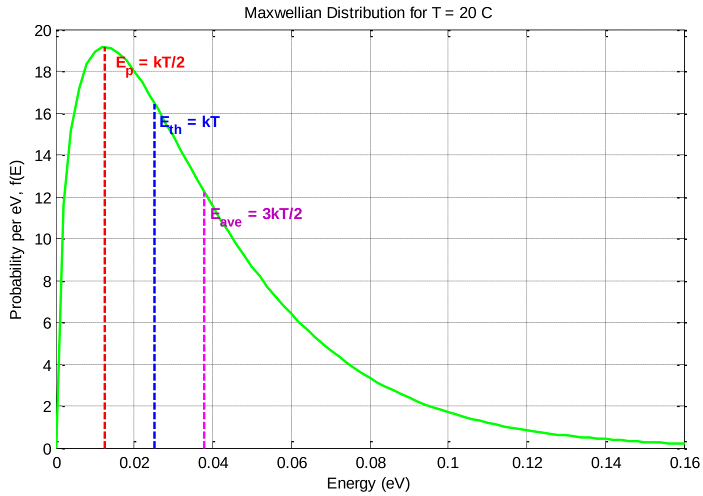
$$\boxed{E_{th} = kT}$$ **thermal energy**

which is usually associated with the term "thermal energy".

At room temperature (T = 20 °C or 293 °K), E_{th} = kT = 0.0253 eV and $v_p = v_{th}$ = 2200 m/s -- and this analysis finally shows where we obtained the E_o = 0.0253 eV value .

The values, E_p, E_{th}, and E_{ave}, are shown on a typical Maxwellian profile on the next slide…

The Maxwellian Distribution (cont.)



The Gamma Function

One final note before leaving our discussion of the Maxwellian distribution concerns the procedure for actually performing the integrals that are often needed.

There are several ways to tackle this mathematical problem; integration by parts, use of integral tables, etc.

However, one specific technique that is frequently used for integrals involving the exponential function is the use of the gamma function, $\Gamma(n)$.

Some properties of $\Gamma(n)$ that are useful for the specific task of evaluating the above integrals are (see the appropriate literature for additional details):

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \quad \text{with} \quad \begin{cases} \Gamma(n+1) = n\Gamma(n) & \\ \Gamma(n) = (n-1)! & \text{for positive integers} \\ \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} & \end{cases}$$

Lesson 10 Summary

In this Lesson we have briefly discussed the following subjects:

The qualitative energy dependence of most neutron cross sections.

The behavior of $\sigma(E)$ for typical absorption, elastic scattering, and inelastic scattering cross sections.

The behavior of $\sigma(E)$ for fission in fissile versus non-fissile nuclides.

The use of the JANIS program to obtain $\sigma(E)$ plots and to get numerical values for various cross sections at a particular energy.

Assuming 1/v behavior, how one can obtain the integrated absorption rate.

The non-1/v factor and how it is computed and used.

Lesson 10 Summary

The distribution function f(E) and how this is used to compute average cross sections and other average quantities that are functions of energy.

Assuming a Maxwellian distribution function at low energies, how to compute the average energy, the most probable energy, and the energy associated with the most probable speed.