Applied Engineering Problem Solving

Lesson #4: Illustrative Example -- Intro to FD Methods for Solution of ODEs

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FD Methods for Solution of ODEs

In Lesson #4, the Taylor series was used to derive a set of FD approximations for the discrete derivative of a function at point x_i.

These FD approximations are quite useful, and here we illustrate their use for the numerical solution of ODEs (IVPs and BVPs).

There are many variations of this basic theme that gives rise to a number of specific methods -- we only focus on one option.

Here we will apply the basic FD method to two relatively simple problems, an IVP and a BVP, as follows:

- Case 1: Pendulum Dynamics via the FD Method (see the Lesson 1 Lecture Notes and the pendulum_dynamics.pdf file for background).
- Case 2: Heat Transfer in a Rectangular Fin via the FD Method (see the rect1d_fin_1.pdf file that was studied after Lesson 3 for the development of the pertinent equations).

Pendulum Dynamics – Analytical

The continuous linearized pendulum model is given by

$$\theta'' + \frac{c}{m}\theta' + \frac{g}{L}\theta = 0 \qquad \text{with} \quad \theta(0) = \theta_o \quad \& \quad \theta'(0) = \omega_o$$

The analytical solution to this 2nd order linear IVP is

$$\theta(t) = e^{\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right)$$

with $\alpha = -\frac{c}{2m}$ and $\beta = \sqrt{\frac{g}{L} - \left(\frac{c}{2m}\right)^2}$

and $c_1 = \theta_o$ and $c_2 = \frac{\omega_o - \alpha c_1}{\beta}$

analytical solution

see the Lesson 1 Lecture Notes and the pendulum_dynamics.pdf file for more details

Pendulum Dynamics – FD Solution

Now let's solve the pendulum dynamics problem using a simple Finite Difference scheme -- as an illustration of how to use the FD method for IVPs.

For this method, we start by discretizing the time variable, or $t \rightarrow t_i$, $t+\Delta t \rightarrow t_{i+1}$, etc., with i being a discrete time index (with $t_1 = t_o = 0$ for this problem).

Now, to discretize the continuous ODE, we simply evaluate every time dependent term in the given ODE at discrete time point t_i, or

$$\theta''\big|_{t_i} + \frac{c}{m}\theta'\big|_{t_i} + \frac{g}{L}\theta\big|_{t_i} = 0$$

Using central FD approximations for both derivatives evaluated at t_i, we have

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} + \frac{c}{m}\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta t} + \frac{g}{L}\theta_i = 0$$

2nd order estimates

Pendulum Dynamics – FD Solution

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Multiplying this recursive equation by Δt^2 gives

$$\theta_{i-1} - 2\theta_i + \theta_{i+1} + \frac{c}{m}\frac{\Delta t}{2}\left(\theta_{i+1} - \theta_{i-1}\right) + \frac{g}{L}\Delta t^2 \theta_i = 0$$

and collecting terms gives

$$\left(1 + \frac{c\Delta t}{2m}\right)\theta_{i+1} = \left(2 - \frac{g\Delta t^2}{L}\right)\theta_i + \left(\frac{c\Delta t}{2m} - 1\right)\theta_{i-1}$$

To simplify this a little, we can define some constants

$$a = \left(1 + \frac{c\Delta t}{2m}\right) \qquad b = \left(2 - \frac{g\Delta t^2}{L}\right) \qquad d = \left(\frac{c\Delta t}{2m} - 1\right)$$

and write the final recurrence relationship as

$$\theta_{i+1} = \frac{b}{a}\theta_i + \frac{d}{a}\theta_{i-1}$$

simple discrete recurrence relation

Pendulum Dynamics – FD Solution

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If we know θ_1 and θ_2, then the discrete equation can be used to estimate θ_3.

Knowing θ_2 and θ_3 then leads to θ_4, and so on -- this is why the discrete equation is said to be a recursive equation.

To simulate the dynamics of the linear pendulum, all we need is two starting positions, θ_1 and θ_2.

The first point, θ_1, is given directly as part of the initial conditions.

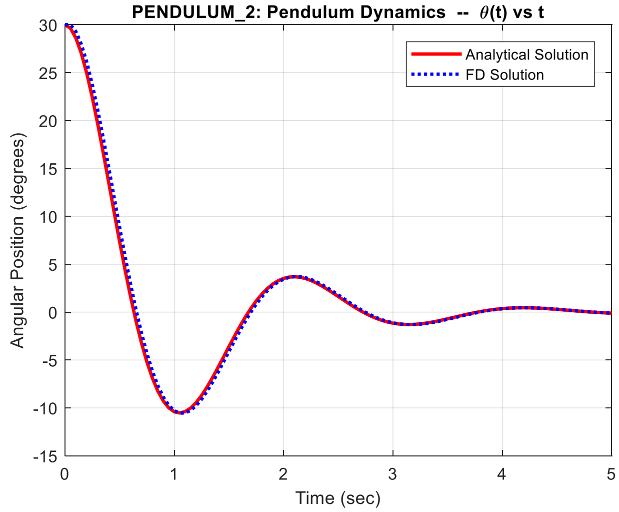
For the second point, θ_2, we can use the initial condition on $d\theta/dt$ and a forward FD approximation, as follows:

$$\left.\frac{d}{dt}\theta\right|_{t_1=0} = \omega_o \approx \frac{\theta_2 - \theta_1}{\Delta t} \qquad \text{or} \qquad \theta_2 = \theta_1 + \omega_o \Delta t$$

1st order estimate!!!

Let's implement this in Matlab (see pendulum_2.m)

Pendulum Dynamics – FD Solution



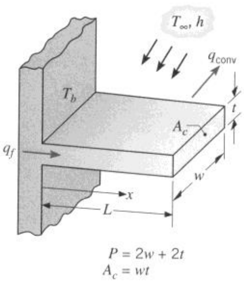
Fin Heat Transfer – Analytical

The governing continuous ODE for this problem is given by

$$\frac{d^2T}{dx^2} - m^2(T - T_\infty) = 0 \quad \text{with} \quad m^2 = \frac{hP}{kA_c}$$

where the specific BCs for this problem are

$$T(0) = T_b \quad \text{and} \quad -k\left.\frac{dT}{dx}\right|_{x=L} = h(T - T_\infty)\big|_{x=L}$$



The analytical solution to this BVP is

$$T(x) = T_\infty + \frac{\cosh m(L-x) + \dfrac{h}{mk}\sinh m(L-x)}{\cosh mL + \dfrac{h}{mk}\sinh mL}\left(T_b - T_\infty\right)$$

analytical solution

see the rect1d_fin_1.pdf file for more details

Fin Heat Transfer – FD Solution

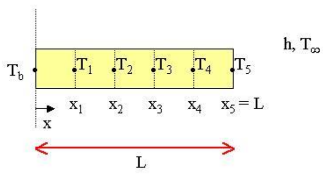
To develop a numerical solution using the FD method, we again start by discretizing the independent variable, x.

Here, we need to be careful to number only the nodal points where the temperature is to be determined.

For example, let's say N = number of unknowns = 5.

In this case, a side view of the fin geometry would give the sketch shown, and we can compute the discrete spatial increment, Δx, as

$$\Delta x = \frac{L-0}{N} = \frac{L}{N}$$



The vector that gives the location of the unknown temperatures to be computed can be written as x = [x_1 x_2 x_3 x_4 x_5] which, in Matlab, can be easily generated with the use of the colon operator, x = dx:dx:L, where dx = Δx.

Fin Heat Transfer – FD Solution (cont.)

With the nodal arrangement defined, we discretize the continuous ODE, or

$$\left.\frac{d^2T}{dx^2}\right|_{x_i} - m^2 (T - T_\infty)\big|_{x_i} = 0$$

Using a 2^{nd} order central approximation for the 2^{nd} derivative, we have

$$\frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2} - m^2 (T_i - T_\infty) = 0$$

or

$$T_{i-1} - \left(2 + m^2 \Delta x^2\right) T_i + T_{i+1} = -m^2 \Delta x^2 T_\infty$$

interior nodes only

Note that this expression is only valid for interior nodes (i = 2:N-1) -- since we used a central approximation for d^2T/dx^2.

We will always need to treat the end nodes as special cases!!!

Fin Heat Transfer – FD Solution (cont.)

For i = 1, the above equation can be used directly if we note that $T_{i-1} = T_0 = T_b$, the fin's base temperature.

Thus, for i = 1 (which is internal to the geometry), we have

$$-\left(2+m^2\Delta x^2\right)T_1 + T_2 = -m^2\Delta x^2 T_\infty - T_b$$

node 1

For i = N, we can not use a central approximation, since nothing is known to the right of node N -- that is, T_{N+1} is not defined.

Instead, we need to develop a backward approximation to the desired derivative at x = L.

To do this, let's write T_N'' as follows

$$T''\big|_{x_N} = T_N'' = \frac{d}{dx}\left(T'\right)\Big|_{x_N} \approx \frac{T_N' - T_{N-1}'}{\Delta x}$$

Fin Heat Transfer – FD Solution (cont.)

Now, we can write a central approximation for the 1st derivative at point N-1, or

$$T_{N-1}' = \frac{T_N - T_{N-2}}{2\Delta x}$$

and, for T_N', we can directly use the given BC at x = L,

$$-kT_N' = h(T_N - T_\infty)$$

Substitution of these expressions into the backward approximation for T_N'' gives

$$T_N'' = \frac{-\frac{h}{k}\left(T_N - T_\infty\right) - \frac{T_N - T_{N-2}}{2\Delta x}}{\Delta x} = -\left(\frac{h}{k\Delta x} + \frac{1}{2\Delta x^2}\right)T_N + \frac{h}{k\Delta x}T_\infty + \frac{1}{2\Delta x^2}T_{N-2}$$

Fin Heat Transfer – FD Solution (cont.)

Finally, putting this expression into the discrete balance equation for i = N gives a proper equation for the last node in the fin's discrete geometry representation,

$$-\left(\frac{h}{k\Delta x}+\frac{1}{2\Delta x^2}\right)T_N+\frac{h}{k\Delta x}T_\infty+\frac{1}{2\Delta x^2}T_{N-2}-m^2\left(T_N-T_\infty\right)=0$$

or

$$T_{N-2}-\left(2m^2\Delta x^2+1+\frac{2h\Delta x}{k}\right)T_N=-\left(\frac{2h\Delta x}{k}+2m^2\Delta x^2\right)T_\infty$$

node N

Together, the three highlighted equations give a system of N equations with N unknowns -- the unknown temperature at each discrete x_i location.

These coupled equations can be written in matrix form, AT = b, and easily solved in Matlab for the desired temperature vector, T, using the backslash operator, or $T = A\backslash b$

desired solution

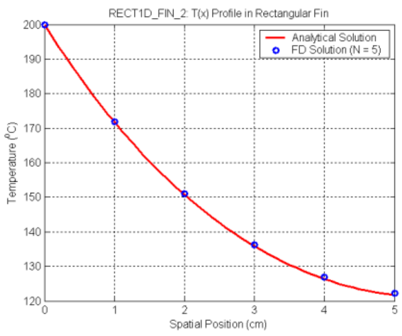
Fin Heat Transfer – FD Solution (cont.)

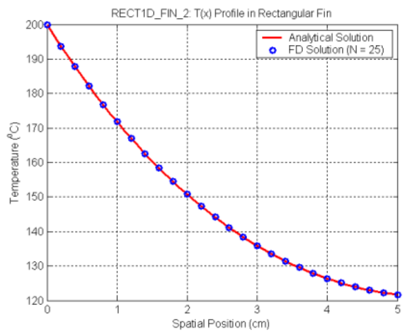
Solution Algorithm

1. set problem parameters
2. compute analytical solution (for comparison purposes)
3. set up the coefficient matrices, A and b,
4. solve the resultant system of equations
5. plot both the analytical and FD solutions
6. perform mesh sensitivity studies, as desired…

Let's implement this in Matlab (see rect1d_fin_2.m)

Fin Heat Transfer – FD Solution (cont.)





Often want to do a mesh sensitivity study to make sure that the numerical solution is converged…

FD Solution -- IVPs vs. BVPs

The two examples given here highlight the difference between initial value problems (IVPs) and boundary value problems (BVPs).

The IVP leads to a simple recurrence relation because enough initial condition information is available to compute the dependent variable at node i+1, y_{i+1}, in terms of known values at two previous nodes, y_i and y_{i-1} (for a 2nd order system).

This can be represented mathematically as

$$y_{i+1} = f(y_i, y_{i-1})$$

2nd order IVP

Since two initial conditions are needed for a 2nd order IVP, we have enough information to compute y_1 and y_2 to start the recursive expression given above.

IVPs give simple recurrence relationships…

FD Solution -- IVPs vs. BVPs (cont.)

However, for a 2nd order BVP, there is only one condition given at each end point -- which means that we do not have sufficient information to get the recursive algorithm started.

Thus, for BVPs, the 2nd order difference equation is usually written in the following form,

$$f(y_{i-1}, y_i, y_{i+1}) = 0$$

2nd order BVP

and, since there are N equations of this type (one for each node in the system), the result is a system of N equations with N unknowns -- which must be solved simultaneously.

BVPs give a system of coupled algebraic equations…

FD Solution -- IVPs vs. BVPs (cont.)

Summary:

IVPs give simple recurrence relationships…

$$y_{i+1} = f(y_i, y_{i-1})$$

2nd order IVP

BVPs give a system of coupled algebraic equations…

$$f(y_{i-1}, y_i, y_{i+1}) = 0 \qquad \text{for} \quad i = 1, 2, \cdots N$$

2nd order BVP

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