Lab #4c -- Finite Difference (FD) Solutions to ODEs

Overview

This lab will highlight a few additional examples of solving ODEs (both IVPs and BVPs) using finite difference (FD) methods -- where you get to do all the work, with only occasional help from the lab instructor, as needed. The goal here is for you to take the discussions from class for the "Pendulum Problem" and the "Fin Heat Transfer Problem" and use your newly acquired insight to numerically solve the two ODEs given below on your own. In particular, there will be no initial formal lecture or overview by the instructor -- instead you should use the lab time to tackle the two problems outlined below using the two in-class examples as a guide for your work. After some quality time on your own, the lab instructor will be glad to assist, as needed, with any specific questions you may have. However, the intent here is for YOU to do these exercises, and not just follow along as the instructor explains how to do them…

Upon completion of this lab you should be more comfortable with the difference between IVPs and BVPs, and with the use of finite difference methods for solving problems of this type (and we have also thrown in the use of Matlab's ***ode45*** routine for solving IVPs as an extra bonus…). Although the ODEs given here are simply "math problems", the same techniques utilized here are used to solve the real application-specific ODEs that routinely occur in every field of science and engineering. Thus, this introduction to FD methods should prove useful in many real-world applications that you will face in some of your other classes and as a practicing engineer in your chosen field of interest.

Problem 1 -- Exact, FD, and ode45 Solutions for a simple 1^st^-order IVP

Consider the following 1st-order IVP:

$$x^2 y' + 2xy - x + 1 = 0 \qquad \text{with} \qquad y(1) = 0$$

a. Find the analytical solution to the given IVP -- this exact solution will be used to validate the numerical solutions generated in Parts b and c.

   Hint: This is a linear first-order ODE that can be written in the form $y' + p(x)y = q(x)$ with integrating factor $g(x) = e^{\int p(x)dx}$. Using the integrating factor method is probably the easiest approach for analytically solving this problem…

b. Now discretize the continuous ODE using a first-order forward FD derivative approximation for y'_i. Implement the resultant recurrence relation in Matlab and graphically compare your FD solution with the exact result from Part a on the same plot. For the comparisons, lets x vary over the domain $1 \le x \le 5$. Do the two solutions agree?

c. Now use Matlab's ***ode45*** routine to numerically solve this problem. Plot this numerical solution and your analytical solution to Part a on the same axes for $1 \le x \le 5$. Do these two solutions agree?

   Hint: Use the examples in your texts or in the Matlab ***help application*** for the proper syntax on how to use ***ode45***.

Problem 2 -- FD Solution for a simple 2nd-order BVP

Consider the following linear variable-coefficient BVP:

$$xy'' + 2y' + xy = 0 \qquad \text{with} \qquad y(\frac{\pi}{2}) = y_o = 1 \quad \text{and} \quad y(\pi) = y_f = 1$$

Develop a FD solution to this BVP and graphically compare to the exact solution over the domain $\pi/2 \le x \le \pi$ (the exact solution was derived via the Power Series Solution technique):

$$y(x) = \frac{\pi}{2x}\left(\sin x - 2\cos x\right)$$

Do the two solutions compare well? Is your FD solution very sensitive to the number of unknowns, N, used in the problem?