## Lab #4b -- Taylor Series, FD Derivatives, Error Analysis, Etc…

### Overview

As we have discussed in class, Taylor series (TS) play an important role in most numerical methods, and certainly they are key to understanding the concept of the "truncation error" that is inherent in many numerical solution algorithms. This lab will try to enhance your understanding of Taylor series with an example that involves writing and evaluating the function f(x) = sinh x as a TS expansion about the point $x_o = 0$, and it will also emphasize the inherent truncation error associated with the finite difference (FD) derivatives that are used extensively in many numerical solution methods.

Upon completion of this lab you should be more comfortable with Taylor series (TS), with the development and use of several low-order FD derivatives, and with how the truncation error in many numerical schemes is related to the step size and to the number of terms retained within the TS approximation. These topics should give you a good foundation for many of the subjects to be discussed in the remainder of the semester and, certainly, they should prove to be useful when completing your next formal HW assignment (and for several additional future HWs) -- so make sure you get a good handle on these topics before leaving the lab session…

### Problem 1 -- On Formulating and Evaluating Infinite Power Series -- An Example

One of the formal "Illustrative Examples" for Lesson 4 involves working with the TS approximation for the hyperbolic sine function, f(x) = sinh x (see **infinite_series.pdf** and **sinh_series.m** on the course website). As the first exercise for today's lab, the instructor will walk you through the development of the TS approximation for this function, he will then discuss some of the issues associated with evaluating the infinite power series that results, and finally, he will describe and show a Matlab file that implements a general algorithm that can be used to efficiently evaluate power series of this type. There are several subtle points in this example, so please study this illustrative example in some detail **before** the lab so that you can have the lab instructor resolve any questions that may arise…

### Problem 2 -- Working with Taylor Series -- Derivative Approximations

A Taylor series expansion for the functions f(x+h) and f(x−h) can be written in terms of the function f(x) and all its derivatives evaluated at the point x, where $h = |\Delta x|$. Using a discrete notation (i.e. $f_i$, $f_{i+1}$, etc.), for convenience, these are given as follows:

$$f_{i+1} = f_i + f_i' h + \frac{f_i'' h^2}{2!} + \frac{f_i''' h^3}{3!} + \frac{f_i'''' h^4}{4!} + \cdots \qquad \text{(Forward Taylor Series)}$$

and

$$f_{i-1} = f_i - f_i' h + \frac{f_i'' h^2}{2!} - \frac{f_i''' h^3}{3!} + \frac{f_i'''' h^4}{4!} - \cdots \qquad \text{(Backward Taylor Series)}$$

a. Now, using the expressions from above, **derive a central difference approximation to df/dx** (i.e. the **first derivative**) at the point $x_i$, and **obtain an estimate of the order of error**. This should be a formal development!

b. An object falling through a viscous medium, where the friction force is proportional to the velocity, can be modeled with the following ODE:

$$m\frac{dv}{dt} = mg - kv \qquad \text{or} \qquad \frac{dv}{dt} = g - cv \qquad \text{where } c = \frac{k}{m}$$

In this equation, c is the drag coefficient per unit mass (with units of $s^{-1}$, for example) and $g = 9.8\ m/s^2$ is the gravitational acceleration.

In a particular situation the position of an object versus time was measured over a period of 4 seconds at 1 second intervals as noted in the table below:

**Table I Experimental Data – Position vs Time**

| **i** | **$t_i$ (sec)** | **$y_i$ (m)** |
|---|---|---|
| 1 | 0 | 0.00 |
| 2 | 1 | 4.18 |
| 3 | 2 | 14.4 |
| 4 | 3 | 28.4 |
| 5 | 4 | 44.5 |

Noting that velocity is the rate of change of position, v = dy/dt, use these measured data to estimate the drag coefficient, c, in the above mathematical model. Explain your solution logic and support your results with the appropriate computations.

**Hint:** Evaluate the ODE at time point i = 3 and solve for c.

**Problem 3 -- Taylor Series Derivative Approximations -- Error Analysis**

Given the function $f(x) = 2x^4 + 5x^3 - x + 1$, write a Matlab code to do the following:

a. Compute the exact derivative of f(x) at x = 1.5.

b. Estimate the first derivative of f(x) at x = 1.5 using a **backward** Taylor series approximation with the following step sizes:

$$h = [10^{-4}\ \ 10^{-3}\ \ 10^{-2}\ \ 10^{-1}\ \ 10^{0}]$$

Using the exact result from Part a, make a plot of the magnitude of the relative error in $f'(x)\big|_{x=1.5}$ versus step size -- use a log-log plot. From the plot, determine the value of n in the error expression, $\varepsilon = \alpha h^n = O(h^n)$. Explain!!!

c. Repeat Part b using a **central** Taylor series approximation for $f'(x)\big|_{x=1.5}$. What is the "order of error" in this case? Plot the error vs. step size for this case in the same plot as above. Explain your results!!!

**Note:** You may want to use Matlab's ***polyval*** function to evaluate the polynomials for this problem (see ***help polyval***).