Demo #1

Solve $Bx = y$ using row operations

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

1 – form augmented matrix $\tilde{A} = [A \; b]$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & -3 & 2 & -2 \end{array}\right]$$

2. – since the a_{31} element is already zero, no operations needed here

3. – multiply row 1 (the pivot eqn) by $-\frac{1}{2}$ and add to row 2

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & \frac{1}{2} & -2 & -\frac{1}{2} \\ 0 & -3 & 2 & -2 \end{array}\right]$$

done using row 1 to eliminate the elements below the pivot element, a_{11}

4. – now use pivot eqn (ie row 2) to eliminate the elements below the pivot element (element 2,2)
In particular, multiply row 2 by 6 and add to row 3

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & \frac{1}{2} & -2 & -1/2 \\ 0 & 0 & -10 & -5 \end{array}\right]$$

now in row echelon form

5. – use back substitution to get solution vector, x

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{1}{2} & -2 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ -5 \end{bmatrix}$$

from eqn 3 $\quad -10x_3 = -5 \quad \Rightarrow x_3 = \frac{1}{2}$

from eqn 2 $\quad \frac{1}{2}x_2 - 2\left(\frac{1}{2}\right) = -\frac{1}{2} \quad \Rightarrow \frac{1}{2}x_2 = \frac{1}{2} \quad \therefore x_2 = 1$

from eqn 1 $\quad 2x_1 - 1(1) + 0\left(\frac{1}{2}\right) = 1 \quad \Rightarrow 2x_1 = 2 \quad \therefore x_1 = 1$

$$\therefore \underline{x} = \begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix}$$

same answer as before

OK

Demo #2

Find B^{-1} using row operations

$$BX = I \quad \text{where } X = B^{-1}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & -3 & 2 \end{bmatrix} B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. - form augmented matrix

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & -3 & 2 & 0 & 0 & 1 \end{array}\right]$$

2. multiply row 1 by $\frac{1}{2}$ - normalize pivot eqn

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & -3 & 2 & 0 & 0 & 1 \end{array}\right]$$

3. add $-1 *$ row 1 to row 2

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -2 & -\frac{1}{2} & 1 & 0 \\ 0 & -3 & 2 & 0 & 0 & 1 \end{array}\right]$$

4. multiply row 2 by 2 - normalize pivot eqn

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -4 & -1 & 2 & 0 \\ 0 & -3 & 2 & 0 & 0 & 1 \end{array}\right]$$

5. add $3 *$ row 2 to row 3

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -4 & -1 & 2 & 0 \\ 0 & 0 & -10 & -3 & 6 & 1 \end{array}\right]$$

all zeros

now let's continue to make all these zero as well

6. multiply row 3 by $-\frac{1}{10}$ - normalize pivot eqn

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -4 & -1 & 2 & 0 \\ 0 & 0 & 1 & \frac{3}{10} & -\frac{6}{10} & -\frac{1}{10} \end{array}\right]$$

Find B^{-1} via Gauss Jordan Method (cont.)

7. multiply row 3 by 4 and add to row 2

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{10} & -\frac{4}{10} & -\frac{4}{10} \\ 0 & 0 & 1 & \frac{3}{10} & -\frac{6}{10} & -\frac{1}{10} \end{array}\right]$$

$\frac{5}{10} + \frac{1}{10} = \frac{6}{10}$

8. multiply row 2 by $\frac{1}{2}$ and add to row 1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{6}{10} & -\frac{2}{10} & -\frac{2}{10} \\ 0 & 1 & 0 & \frac{2}{10} & -\frac{4}{10} & -\frac{4}{10} \\ 0 & 0 & 1 & \frac{3}{10} & -\frac{6}{10} & -\frac{1}{10} \end{array}\right]$$

notice that this is the identity matrix

$$IX = X = B^{-1} =$$

$$\therefore \boxed{B^{-1} = \frac{1}{10}\begin{bmatrix} 6 & -2 & -2 \\ 2 & -4 & -4 \\ 3 & -6 & -1 \end{bmatrix}}$$

This is the same answer as determined with $B^{-1} = \frac{C^{+}}{\det B}$

*

- Using row operations is a little tedious but very systematic (if you are very careful with the arithmetic).
- Variations of this technique are used for computer evaluation of large systems of eqns. (will be discussed later in Lesson #6...)

Demo #3

Affect of row operations on determinant of A

(1) $\det A = \det A^t$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\qquad A^t = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$\det A = 1$ $\qquad \det A^t = 1$ $\qquad$ ok

(2) multiply by -1 for each row interchange

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\det A = 1$ $\qquad \det B = -1 \Rightarrow -1 * \det A$ since one row interchange

(3) determinant not altered if one row is altered by adding any constant multiple of another row

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ ← 2 * row 1 + row 2

$\det A = 1$ $\qquad \det B = 1$ $\qquad$ unchanged ✓

(4) determinant is multiplied by constant if any row is multiplied by that constant.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ ← 2 * row 1

$\det A = 1$ $\qquad \det B = 2 \Rightarrow 2 * \det A$ $\qquad$ ok

(5) det of triangular matrix = prod of diagonals

↳ can be easily shown from Laplace's expansion

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = a_{11} C_{11} + \cancel{a_{21}}^{0} C_{21} + \cancel{a_{31}}^{0} C_{31}$$ ← expanding down col #1

$$= 1 \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = 1(2)(2)$$ = prod of diagonal elements

(6) $\det(AB) = \det(BA) = (\det A)(\det B)$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\qquad AB = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$\det A = 1$ $\qquad \det B = 1$ $\qquad \det(AB) = 3 - 2 = 1$ $\qquad$ ok

$BA = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $\qquad \det(BA) = 1$ $\qquad$ ok

Demo #4

Find Determinant of matrix $B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & -3 & 2 \end{bmatrix}$

→ from Demo #1, we have already done a set of row operations on [A y] to give

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & \frac{1}{2} & -2 & -\frac{1}{2} \\ 0 & 0 & -10 & -5 \end{array}\right]$$

$\underbrace{\hspace{3cm}}$ call this BB

this is the modified B matrix and det BB = prod of diagonals $= \boxed{-10}$

note that only Row Op #3 was used so that the determinant is unchanged

det B = det BB

→ Also recall from last week using Laplace's Expansion that we had

$$\det B = b_{11}c_{11} + b_{21}c_{21} + b_{31}c_{31}$$

expanding down col #1 of B

$$= 2\begin{vmatrix} 0 & -2 \\ -3 & 2 \end{vmatrix} + 1(-1)\begin{vmatrix} -1 & 0 \\ -3 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix}$$

$$= 2(-6) - (-2) + 0$$

$$= -10$$

OK

Demo # 5

(d) Calculate the eigenvalues and eigenvectors of C where $C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

→ eigenvalues

$$|C - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) = 0$$

$\therefore \lambda_1 = 1$ and $\lambda_2 = 3$ $\boxed{\lambda_{1,2} = 1, 3}$

ans

→ eigenvectors

$$(\underline{\underline{C}} - \lambda \underline{\underline{I}}) \underline{x} = 0$$

for $\lambda = \lambda_1 = 1$

$$\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 0x_1 + 0x_2 = 0 \\ 2x_1 + 2x_2 = 0 \end{matrix} \Rightarrow x_2 = -x_1 \qquad \therefore \boxed{\underline{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

ans

for $\lambda = \lambda_2 = 3$

$$\begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} -2x_1 + 0x_2 = 0 \\ 2x_1 + 0x_2 = 0 \end{matrix} \Rightarrow x_1 = 0 \qquad \therefore \boxed{\underline{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

ans

note that we desire non-trivial solns so $\underline{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is NOT a valid eigenvector although it satisfies the homo eqn.