Lab #2a (Part 2) -- Introduction to Linear Algebra and Common Array and Matrix Operations in Matlab

Overview

This lab will essentially be a continuation from last week's lab work -- since we did not finish everything that was initially planned. In practice, there are two main techniques for performing hand calculations for finding the inverse matrix and for doing a number of linear algebra operations, but only one of these was demonstrated last week. In particular, last week's lab focused on computing the matrix inverse using the formula $A^{-1} = C^T / \det A$ where C is the cofactor matrix associated with matrix A. This approach allowed the introduction of a lot of standard linear algebra terminology, including determinants, cofactors, minors, matrix inverses, etc.. This week we will focus on the second approach -- using **row operations** -- to find det A, to solve a system of equations, and to determine the matrix inverse. In addition, a brief demonstration of how to compute the **eigenvalues and eigenvectors of a matrix** will be performed if time permits.

By the end of this lab you should have sufficient background to tackle all the tasks requested as part of HW2a, and be quite comfortable with a variety of basic linear algebra operations…

Note: For this exercise, we will use the same matrices that was given in last week's lab handout:

$$A = \begin{bmatrix} -1 & 1 & -3 \\ 3 & 0 & -1 \\ 5 & -2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & -3 & 2 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad y = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

Row Operations:

One of the most direct methods for solving systems of equations involves a sequence of **elementary row operations**. These operations represent legal algebraic manipulations that do not alter the basic equality associated with the original equations. ***The purpose of the row operations is to take the original equations and put them into a form that is easier to solve than the original equations.*** There are three row operations that are used to systematically simplify the original system of equations:

1. Interchange two rows
2. Multiply a row by a constant
3. Add a constant times one row to another row

The most well-known method that implements these row operations, called the ***Gauss Elimination Method***, takes the original system and converts the matrix into upper triangular form (often called ***row echelon form***). In this form, back substitution is used to evaluate the unknown solution vector **x**, since there is only one unknown per equation if evaluated sequentially starting with equation n and working backwards to the first equation. The required transformation (i.e. the ***elimination step*** as it is often called) can be represented symbolically using an augmented matrix notation, where $\tilde{\mathbf{A}} = \begin{bmatrix}\mathbf{A} & \mathbf{b}\end{bmatrix}$ is the augmented matrix. For example, a 3×3 system would be transformed as follows:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \Rightarrow \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

where the * notation implies a general nonzero entry and the last column in the original matrix contains the right hand side **b** vector. Of course, after transformation, the entries in the resultant matrix are different from the original case. However, this new system, with the n×n part of the augmented matrix in upper triangular form, is an equivalent representation of the original equation. Once in this form, one can easily use back substitution to solve for the unknown **x** vector.

Demo #1: Solve the following matrix equation for the solution vector x: Bx = y

The lab instructor will walk you through the basic steps to solve this problem. You should follow along with your own hand calculations…

The Matrix Inverse (via row operations):

Given the general matrix equation, **AX=B**, what happens if **B = I**, where **I** is the identity matrix? Remembering that the definition of the inverse matrix gives $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, we see that if **B = I**, then **X** must be $\mathbf{A}^{-1}$. This suggests that we can solve for $\mathbf{A}^{-1}$ by solving the equation **AX = I** using the standard row operations noted above (this is referred to as the ***Gauss-Jordan Method***).

Demo #2: Solve for $\mathbf{B}^{-1}$ using the Gauss-Jordan Method.

The lab instructor will walk you through the basic steps to solve this problem. Again, you should follow along with your own hand calculations…

Determinants:

The determinant of a matrix may or may not be altered under certain variations to the original matrix (i.e. by performing legal row operations). Since determinants are often computed using row operations, several important relations are noted as follows:

1. The det **A** is not altered if the rows are written as columns in the same order. Therefore,

   det $\mathbf{A}$ = det $\mathbf{A}^T$

2. If any two rows or columns are interchanged, the value of det **A** is multiplied by -1.
3. The value of det **A** is not altered if the elements of one row are altered by adding any constant multiple of another row to them.
4. The determinant is multiplied by a constant α if any row is multiplied by α.
5. The determinant of a diagonal matrix is simply the product of the diagonal elements. This is also true for triangular matrices. This observation, along with the above statements, establishes a method for computer calculation of the det **A** using row operations by transforming the original matrix into upper triangular form and then taking the product of the diagonal elements -- being careful to account for row interchanges and normalization steps

6. For square matrices,

   det (**AB**) = det (**BA**) = det **A** det **B**

Demo#3: Illustrate the above statements with a simple 2×2 matrix.

Demo#4: Find the determinant of matrix B using row operations.

The lab instructor will walk you through these demonstrations. Again, you should follow along with your own hand calculations…

Eigenvalues and Eigenvectors:

The classical discrete eigenvalue problem can be written in matrix form as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

which is a homogeneous system of equations. The eigenvalue, λ, is a scalar so the identity matrix is needed here so that both terms inside the parentheses are matrices of the same size. However, for a non-trivial solution to a homogeneous equation, the determinant of the coefficient matrix must be zero, so we have

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

which is referred to as the ***characteristic equation***. This gives rise to an n^{th} order polynomial in λ which has n roots -- the n eigenvalues of a square matrix of order n.

The eigenvector $\mathbf{x}_i$ associated with the i^{th} eigenvalue, λ_i, is found by evaluating the homogeneous equation

$$(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{x}_i = \mathbf{0}$$

where the notation, $\mathbf{x}_i$, refers to the i^{th} eigenvector, not the i^{th} element of a given vector.

Demo#5: Find the eigenvalues and eigenvectors of the 2×2 matrix, $\mathbf{C} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$**.**

The lab instructor will walk you through the basic steps to solve this problem (and also show how to do this in Matlab, if time permits). Again, you should follow along with your own hand calculations…

Note: Make sure you understand how to do all the tasks (i.e. Demos) performed here, since there is a good chance that many of these same manipulations will be required in subsequent HWs and Quizzes/Exams…