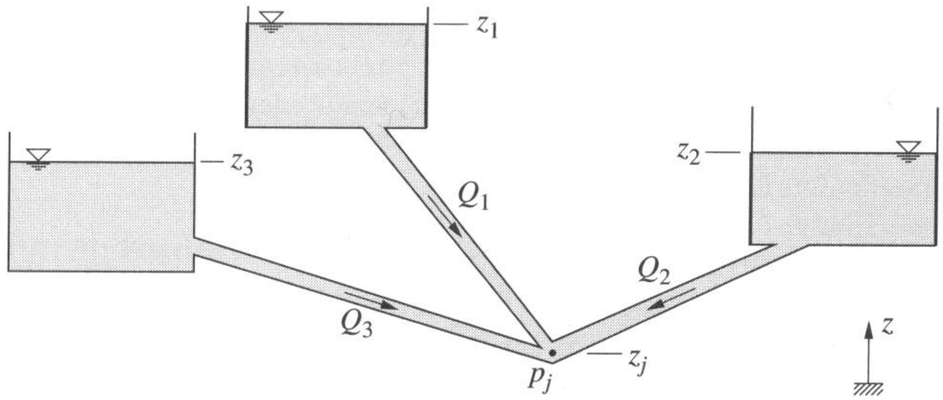
Applied Engineering Problem Solving (CHEN.3170)

Optional Extra Credit Project #2 -- Fall 2017

Solving Nonlinear Equations -- The Classical Three-Reservoir Problem

Project Overview

Consider the classical three-reservoir water distribution problem sketched below.



Three pipes connect the three reservoirs as shown and these meet at junction point j. The elevation at the surface of each reservoir is denoted as z_i, the length of each pipe is L_i, and the volumetric flow rate in line i is Q_i. At the junction of the three pipes, the continuity equation requires that

$$\sum_i Q_i = 0 \tag{1}$$

where Q_i is related to the average velocity in line i by

$$v_i = \frac{Q_i}{A_i} \tag{2}$$

where A_i is the flow area within the pipe.

Also, if we write the general energy equation from the free surface of reservoir i to point j, we have

$$\frac{P_{surface}}{\gamma} + \frac{v_{surface}^2}{2g} + z_i - h_{L_i} = \frac{P_j}{\gamma} + \frac{v_i^2}{2g} + z_j \tag{3}$$

where v_i refers to the average velocity in line i as given by eqn. (2). With $P_{surface} = 0$ and $v_{surface} = 0$ because of the free surface condition associated with a large reservoir, and $z_j = 0$ since we can choose this as the reference elevation, we have

$$\frac{P_j}{\gamma} + \frac{v_i^2}{2g} + h_{L_i} = z_i \tag{4}$$

Now, with eqn. (2) and Darcy's equation for the head loss term, h_L, with

$$h_L = f\frac{L}{D}\frac{v^2}{2g} = f\frac{L}{D}\frac{1}{2gA^2}Q^2 \tag{5}$$

we have

$$\frac{P_j}{\gamma} + \left(1 + f_i\frac{L_i}{D_i}\right)\frac{1}{2gA_i^2}Q_i^2 = z_i \tag{6}$$

To simplify, we can define the coefficient of the Q_i^2 term as a kind of resistance coefficient, K_i,

$$K_i = \left(1 + f_i\frac{L_i}{D_i}\right)\frac{1}{2gA_i^2} \tag{7}$$

and rewrite eqn. (6) as

$$\frac{P_j}{\gamma} + K_iQ_i^2 = z_i \tag{8}$$

However, it is important to note that K_i is not a constant. In fact, since the friction factor, f, is a complicated function of the Reynolds number, which, in turn, is a function of the fluid velocity, the friction coefficient is indeed a relatively complicated function of the flow rate, Q_i. We can summarize this relationship with the following equations (see any good Fluid Mechanics text):

Reynolds number: $$Re = \frac{\rho v D}{\mu} = \left(\frac{\rho D}{\mu A}\right)Q \tag{9}$$

friction factor: $$f = \frac{64}{Re} \quad \text{for } Re \le 2000 \quad \text{(laminar flow region)} \tag{10}$$

$$f = \frac{0.25}{\left[\log\left(\frac{\varepsilon/D}{3.7} + \frac{5.74}{Re^{0.9}}\right)\right]^2} \quad \text{for } Re > 2000 \quad \text{(turbulent flow)} \tag{11}$$

where we have extended the Swamee-Jain turbulent flow friction factor correlation down to the transition flow regime ($2000 < Re < 4000$).

Now, writing eqns. (1) and (8) specifically for the three-reservoir problem gives four equations with four unknowns, or

$$Q_1 + Q_2 + Q_3 = 0 \tag{12}$$

$$\frac{P_j}{\gamma} + K_1Q_1^2 = z_1 \tag{13}$$

$$\frac{P_j}{\gamma} + K_2Q_2^2 = z_2 \tag{14}$$

$$\frac{P_j}{\gamma} + K_3 Q_3^2 = z_3 \tag{15}$$

Clearly, this system of equations is nonlinear and, in general, they are not particularly easy to solve.

Before we actually attempt a solution however, we need to address one more level of complication. As you know, the energy equation for incompressible flow [see eqn. (3)] must be written in the direction of flow. In the above development, we assumed a direction -- from the reservoir surface to the junction point -- but, in fact, for a steady flow situation, at least one of the flow paths must be reversed. For example, if we assume that Q_1 and Q_2 are positive (i.e. in the assumed direction) and Q_3 is negative (i.e. opposite to the assumed direction), then the energy equation for line 3 would be

$$\frac{P_j}{\gamma} + \frac{v_3^2}{2g} + z_j - h_{L_3} = \frac{P_{surface}}{\gamma} + \frac{v_{surface}^2}{2g} + z_3$$

or, with the same simplifications as above, we have

$$\frac{P_j}{\gamma} + \left(1 - f_3 \frac{L_3}{D_3}\right)\frac{1}{2gA_3^2} Q_3^2 = z_3$$

The only difference between this expression and eqns. (6) and (7) with i = 3 is the sign of the head loss term. Thus, we can modify eqns. (6) and (7) to generalize the situation to be independent of the flow direction by using the **sign(Q)** function, or

$$K_i = \left(1 + \mathrm{sign}(Q_i)\, f_i \frac{L_i}{D_i}\right)\frac{1}{2gA_i^2} \tag{16}$$

Thus, eqns. (9) - (16) completely define this nonlinear flow situation.

Note: Since Q and v can take on negative values (just indicates that the flow direction is opposite to the assumed direction), be sure to use abs(Q) and/or abs(v) when computing the Reynolds number or other quantities that are dependent on the magnitude of the flow rate.

Solution Schemes -- Linearized Iteration Method and Matlab's fsolve Function

As discussed in class, one relatively easy way to solve "mildly" nonlinear problems is via a linearized iteration method. In this method, we simply linearize the nonlinear terms by making the coefficient matrix a function of the solution vector, $\mathbf{A} = \mathbf{A}(\mathbf{x})$. However, with a given guess for $\mathbf{x}$, $\mathbf{A}(\mathbf{x})$ is known -- call this $\mathbf{A}^p$ for the p^{th} iteration -- and then we have a set of simple linear equations,

$$\mathbf{A}^p \mathbf{x}^{p+1} = \mathbf{b} \tag{17}$$

which we can solve for the $(p+1)^{th}$ estimate of the desired solution vector $\mathbf{x}$. This process is continued until convergence or the maximum number of iterations is reached.

Note that one often simply lets the next estimate of the solution be the current best guess, $\mathbf{x}_{old} = \mathbf{x}_{new}$ (where $\mathbf{x}_{old}$ is the guess used for the next iteration and $\mathbf{x}_{new}$ is the current best estimate of the solution $\mathbf{x}^{p+1}$). However, in some cases, the computed solution tends to oscillate around the real solution if this approach is used. When this happens, we often use the next guess as the average of the previous two solution estimates, $\mathbf{x}_{old} = (\mathbf{x}_{new} + \mathbf{x}_{old})/2$ -- where the $\mathbf{x}_{old}$ on the left hand side (LHS) is the new guess for the next iteration and the $\mathbf{x}_{old}$ on the RHS is the old guess used to generate $\mathbf{x}_{new}$. This approach usually eliminates the oscillatory behavior that can occur.

For the current problem, we can write the linearized form of the equations as

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1/\gamma & K_1Q_1 & 0 & 0 \\ 1/\gamma & 0 & K_2Q_2 & 0 \\ 1/\gamma & 0 & 0 & K_3Q_3 \end{bmatrix} \begin{bmatrix} P_j \\ Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{18}$$

where we see that the 4x4 **A**-matrix is indeed a function of the solution $\mathbf{x}^T = [P_j \; Q_1 \; Q_2 \; Q_3]$.

In the Lecture Notes we also illustrated the use of Matlab's built-in **fsolve** function to solve unconstrained nonlinear equations. The goal here is to find the solution vector **x** to a system of coupled nonlinear equations written in the form $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ -- that is, what is **x** such that $\mathbf{f}(\mathbf{x}) = \mathbf{0}$? If we visualize eqn. (18) as $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \mathbf{A}(\mathbf{x})$, then we can simply write $\mathbf{f}(\mathbf{x}) = \mathbf{Ax} - \mathbf{b}$ and search for the vector **x** that makes this vector function vanish. This is what the **fsolve** routine is designed to do...

Problem Description

Your job for this project is to solve this system of equations using both the linearized iteration scheme and Matlab's **fsolve** routine for a variety of conditions and to interpret the results within the context of this fluid flow application. In particular, you should break the project into two major tasks, as follows:

Part A: With the above background and the following data for the three-reservoir problem:

1. 20 ^{o}C water is the working fluid.
2. All pipes are 3" Sch 40 steel pipes.
3. The three pipe lengths are: $L_1 = 75$ m $L_2 = 50$ m $L_3 = 150$ m
4. The reservoir heights relative to the junction level are: $z_1 = 60$ m $z_2 = 40$ m $z_3 = 20$ m

find the three volumetric flow rates (m^3/s) and the junction pressure (kPa) for the specific situation given here using both of the methods outlined in the above paragraphs (this is done as a check on the base methods). In addition, you should print out a table of intermediate results and also verify that the energy equation is in balance for each pipeline (**do this for both methods**). Is the continuity equation also satisfied? Discuss and explain your results in detail. Does everything make sense?

Part B: Now that you have a good understanding of the basic system, select one of the methods (either the linearized iteration method or the use of **fsolve**) to perform a parametric study of how things change as the height of Reservoir #3 varies. In particular, let the height of Reservoir #3 vary from 20 m to 140 m and plot the three flow rates and the junction pressure versus z_3. Do the results make sense? Explain and interpret your solutions within the context of the physical problem. Did you observe any interesting or unexpected behavior? Explain in detail…

Documentation

You are expected to treat this extra credit assignment as a **Formal Project**, and to produce a short, but formal report on your work (refer to the Project #1 write-up for an overview of what is expected). Again, team efforts with two students per team are encouraged for this project, with each two-person team submitting a single report. This assignment will be worth up to **25 extra points** towards your HW grade, so it can contribute significantly to your composite HW score. Good Luck -- this should be a lot of fun!!!