Applied Engineering Problem Solving (CHEN.3170)

Optional Extra Credit Project -- Fall 2017

Solution of IVPs in Matlab -- The Pendulum Problem Revisited (again!!!)

Project Overview

We have studied the simple pendulum problem in some detail already in this class, and this exercise will hopefully add some additional insight into solving problems of this type -- that is, Initial Value Problems (IVPs) -- in Matlab. In Lesson #1 we derived the pendulum equations of interest in this study and they are repeated here for quick reference:

Nonlinear Model: $$\theta'' + \frac{c}{m}\theta' + \frac{g}{L}\sin\theta = 0 \qquad (1)$$

Linear Model: $$\theta'' + \frac{c}{m}\theta' + \frac{g}{L}\theta = 0 \qquad (2)$$

Specific Model Data: m = 1 kg, c = 2 kg/s, L = 1 m, and g = 10 m/s^2

Initial Conditions:

Case 1: $\theta(0) = \pi/6$ radians and $\theta'(0) = 0$ rad/s

Case 2: $\theta(0) = 2\pi/3$ radians and $\theta'(0) = 0$ rad/s

For this study we are interested in several different comparisons, including comparing the linear vs. nonlinear model behavior for the two different initial starting points (initial angle for Case 1 at 30^o and Case 2 at 120^o) as well as the implementation and comparison of different solution techniques, as described below.

In particular, we will start with the linear model since the IVP here has a simple analytical solution and we can compare the different numerical methods to a known exact solution. Then, once confidence with the numerical techniques has been achieved, we will solve the nonlinear pendulum using the same numerical schemes.

The three methods to be used are:

Method 1: analytical solution (see Lesson 1 Lecture Notes)

Method 2: numerical solution using the finite difference (FD) approach developed in Lesson 4

Method 3: using Matlab's built-in **ode45** routine

For the **Linear Model,** Methods 1 and 2 have already been discussed and demonstrated, so the only new approach here involves the use of **ode45**, which is discussed and illustrated in Chapter 23 of your numerical methods text by Chapra. The **ode45** routine is very easy to use, but it requires that the IVP be written as a set of coupled 1st order equations. In general, any nth order ODE can be converted to n 1st order ODEs and put into vector form. This is best illustrated via example, as follows:

Given $$\frac{d^3}{dt^3}y(t) + a_1\frac{d^2}{dt^2}y(t) + a_2\frac{d}{dt}y(t) + a_3 y(t) = u(t)$$

we can convert this 3rd order ODE into a system of three 1st order equations (often referred to as the state-space equations), by defining three new variables (called the state variables), as follows,

$$x_1 = y \qquad x_2 = y' = \frac{d}{dt}x_1 \qquad x_3 = y'' = \frac{d}{dt}x_2$$

or, in general,

$$x_j(t) = \frac{d}{dt}x_{j-1}(t) \qquad \text{with } x_1 = y$$

Now, from the defining equation with $y''' = \frac{d}{dt}x_3$, we have

$$\frac{d}{dt}x_3 = -a_1x_3 - a_2x_2 - a_3x_1 + u(t)$$

Now, the three 1st order ODEs can be put into vector form, giving

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ -a_1x_3 - a_2x_2 - a_3x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

which can be written for the general case of n 1st order equations using a simple vector notation as

$$\frac{d}{dt}\underline{x}(t) = \underline{f}(\underline{x}, t) + \underline{g}(t) \qquad \text{with} \quad \underline{x}(0) = \underline{x}_o \tag{3}$$

Note: The above simple procedure works in nearly every case -- except for a few special cases outside our interest at this time.

Equation (3) is the general form of an IVP that is expected for use in the **ode45** routine and, as noted above, your text by Chapra does a good job illustrating how to use several of the functions within the existing suite of Matlab's ODE solvers. This exercise will evaluate your ability to use the brief discussion from above and the examples from Chapra as applied to the pendulum dynamics problem described above. In particular, for the linear model, you will need to implement all three methods using both initial condition cases and show that they give essentially the same result -- giving confidence in the use of both Methods 2 and 3. Note that, for the **ode45** case, both $\theta(t)$ and $\theta'(t)$ will be available, but comparison of the angular position of the pendulum for all three methods will be sufficient to show that they give equivalent results [so only state 1 will be of interest here -- that is, $x_1(t) = \theta(t)$].

Now, for the **Nonlinear Model**, since the analytical solution is not straightforward, we will only apply the two numerical methods, **Method 2** and **Method 3**, and then do a whole series of inter-comparisons to compare the linear and nonlinear models and the various methods used in this study.

Specific Project Tasks

a. Solve the **linear pendulum problem** using all three methods for the two different initial conditions given above. Plot all three solutions on the same graph, with a different graph for Case 1 and Case 2. Do all three methods give the same result? Are the Case 1 and Case 2 behaviors as expected? Explain...

b. Solve the **nonlinear pendulum dynamics problem** using only the two numerical approaches (Methods 2 and 3) for both sets of ICs. Plot these solutions in a similar fashion as for Part a. Do these two techniques give the same result?

c. Do a **detailed comparison of the behavior of the linear and nonlinear pendulum models** for both sets of ICs. Do the linear and nonlinear pendulum models behave as expected? For example, is the same behavior as observed for Cases 1 and 2 in Part a repeated for the nonlinear model? Compare and contrast all the results and be sure to clearly explain any expected and unexpected behavior...

Documentation

This Extra Credit Project should be treated as a **Formal Project**, which means that a complete professional report documenting your work is expected. In general, formal project reports should have a brief **Introduction** that overviews the problem of interest, a **Procedure** section that discusses the solution procedures and any required derivations or manipulations (long derivations can be neatly hand-written and included in an Appendix), and a **Results/Discussion** section that summarizes and discusses the key results in tabular or graphical form, as appropriate. The section names do not need to be "Introduction", "Procedure", etc., but the document organization should reflect this general pattern. Note that, in this particular case, some of these sections can be quite brief, since the development of the problem has already been treated in some detail in previous sections of the course Lecture Notes (a formal reference may be appropriate here). Nevertheless, you should put together a complete document that can be used by an outside reviewer to learn about the particular subject under investigation in this project -- so enough information will be needed in each section to be informative by itself.

Also note that any figures or tabular data included in your report should be integrated directly into your reports (not as single page attachments), with the Matlab code listings included as a separate appendix. Each figure or table should have an informative title and, in your discussions, you should refer to a specific figure or table explicitly via its number. In completing your report, also be sure to thoroughly explain the procedure used in your analysis (including the development of the FD equations used in your Matlab code) and to fully discuss the results obtained -- and also to pay attention to grammar, format, style, etc. Again, since this is a rather straight-forward assignment, all this can be fairly brief (the discussion of the results will probably be the largest component of your report). Just remember that I am looking for a professional report here, even if it is not very lengthy!!!

This Extra Credit HW Assignment will be worth up to **25 extra points** towards your HW grade. Team efforts with two students per team (even if you have not formed a formal HW team) are encouraged for this project. Only one project report and Matlab code is needed for each team. Each student should sign and date their team's report to acknowledge their participation and contribution to the team project.

Note that a little extra credit can go a long ways in improving your HW grade, and the experience gained with the FD method and the **ode45** routine should be very useful in some of your subsequent classes -- so this is indeed a worthwhile exercise. Good Luck and have fun...