matrix Operations

#2

Given the following matrices!

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ -1 & 3 & 2 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Perform the following operations:

$$B^T B = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & -3 & 1 \\ -3 & 9 & 3 \\ 1 & 3 & 5 \end{bmatrix}} \quad \underline{\text{ans}}$$

$$(3\times 2)(2\times 3) \Rightarrow (3\times 3)$$

$$ABC = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$$

$$(2\times 3)(3\times 3) \Rightarrow (2\times 3)$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 6 & 6 \\ -2 & 0 & 0 \end{bmatrix}$$

$$(2\times 2)(2\times 3) \Rightarrow (2\times 3)$$

$$= \boxed{\begin{bmatrix} -2 & 0 & 0 \\ 3 & 6 & 6 \end{bmatrix}} \quad \underline{\text{ans}}$$

$$x^T A x = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \boxed{-4} \quad \underline{\text{ans}}$$

$$x x^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}} \quad \underline{\text{ans}}$$

$$(2\times 1)(1\times 2) \Rightarrow (2\times 2)$$

$$A^2 = A * A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}} \quad \underline{\text{ans}}$$

$$A .* A = \begin{bmatrix} 0^2 & 1^2 \\ 1^2 & -2^2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}} \quad \underline{\text{ans}}$$

```
>> matrix_ops_2
Matrices for the various exercises
A =
     0     1
     1    -2
B =
     1     0     2
    -1     3     1
C =
     1     0     2
     0    -1     0
    -1     3     2
x =
     1
     2
Find B^T*B
ans =
     2    -3     1
    -3     9     3
     1     3     5
Find A*B*C
ans =
    -2     0     0
     3     6     6
Find x^T*A*x
ans =
    -4
Find x*x^T
ans =
     1     2
     2     4
Find A*A
ans =
     1    -2
    -2     5
Find A.*A
ans =
     0     1
     1     4
>>
```

```
%
%   MATRIX_OPS_2.M  MATLAB file to verify some hand calculations
%
%   This file simply does a number of matrix multiplication tasks to verify
%   some hand calculations that were performed.  It should validate that you
%   understand how to do the hand computations as well as give some further
%   experience with doing matrix computations in Matlab (although this part
%   is pretty straightforward, since this is what Matlab does best!!!).
%
%   File prepared by J. R. White, UMass-Lowell (last update: Sept. 2017)
%

     clear all,   close all
     format compact
%
%   define matrices for the problems
     disp('Matrices for the various exercises ')
     A = [0 1; 1 -2],  B = [1 0 2; -1 3 1],  C = [1 0 2; 0 -1 0; -1 3 2],  x = [1 2]'
%
%   do the desired calculations
     disp('Find B^T*B');      B'*B
     disp('Find A*B*C');      A*B*C
     disp('Find x^T*A*x');    x'*A*x
     disp('Find x*x^T');      x*x'
     disp('Find A*A');        A*A
     disp('Find A.*A');       A.*A
%
%   end of program
```

(a) Calc det(A) where $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & -3 \\ -4 & 0 & 1 \end{bmatrix}$

→ via Laplace's Expansion (down row 2)

$$\det A = a_{12}c_{12} + a_{22}c_{22} + \cancel{a_{32}}^{\,0}c_{32}$$

$$= 2(-1)\begin{vmatrix} 2 & -3 \\ -4 & 1 \end{vmatrix} - 1(+1)\begin{vmatrix} 3 & -1 \\ -4 & 1 \end{vmatrix}$$

$$= -2(-10) - 1(-1)$$

$$= \boxed{21} \text{ ans}$$

→ via row operations

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & -3 \\ -4 & 0 & 1 \end{bmatrix}$$

multiply row 1 by $-2/3$ and add to row 2

$-\frac{4}{3} - 1 = -\frac{7}{3}$

$\frac{2}{3} - 3 = -\frac{7}{3}$

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & -7/3 & -\frac{7}{3} \\ -4 & 0 & 1 \end{bmatrix}$$

multiply row 1 by $4/3$ and add to row 3

$-\frac{4}{3} + 1 = -\frac{1}{3}$

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & -7/3 & -7/3 \\ 0 & \frac{8}{3} & -\frac{1}{3} \end{bmatrix}$$

multiply row 2 by $\frac{8}{7}$ and add to row 3

$\left(-\frac{7}{3}\right)\frac{8}{7} - \frac{1}{3} = -\frac{9}{3} = -3$

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & -7/3 & 7/3 \\ 0 & 0 & -3 \end{bmatrix}$$

The determinant here is the product of the diagonal elements

$$\cancel{3}\left(-\frac{7}{\cancel{3}}\right)(-3) = \boxed{21} \text{ ans}$$

OK

not requested explicitly — but needed for Part b

also calc det(B) where $B = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 1 & 3 \\ 2 & 4 & -9 \end{bmatrix}$

→ via Laplace's expansion (along row 1)

$$\det B = 2(+1)\begin{vmatrix} 1 & 3 \\ 4 & -9 \end{vmatrix} + 2(-1)\begin{vmatrix} 2 & 3 \\ 2 & -9 \end{vmatrix} - 1(+1)\begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix}$$

$$= 2(-21) - 2(-24) - 1(6)$$

$$= -42 + 48 - 6 = \boxed{0} \text{ ans}$$

→ via row operations

$$\begin{bmatrix} 2 & 2 & -1 \\ 2 & 1 & 3 \\ 2 & 4 & -9 \end{bmatrix}$$

multiply row 1 by −1 and add to row 2

$$\begin{bmatrix} 2 & 2 & -1 \\ 0 & -1 & 4 \\ 2 & 4 & -9 \end{bmatrix}$$

multiply row 1 by −1 and add to row 3

$$\begin{bmatrix} 2 & 2 & -1 \\ 0 & -1 & 4 \\ 0 & 2 & -8 \end{bmatrix}$$

multiply row 2 by 2 and add to row 3

$$\begin{bmatrix} 2 & 2 & -1 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

↑ the determinant is the product of the diagonal elements

$\therefore \det B = \boxed{0}$

OK

(b) Find the inverses of A and B

→ Inverse of A

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & -3 \\ -4 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{\det A} = \frac{(\text{cofactor of } A)^T}{\det A}$$

$$\text{cofactor matrix} = \begin{bmatrix} +(-1) & -(-10) & +(-4) \\ -2 & +(-1) & -8 \\ +(-7) & -(-7) & +(-7) \end{bmatrix}$$

$$\therefore \boxed{A^{-1} = \frac{1}{21}\begin{bmatrix} -1 & -2 & -7 \\ 10 & -1 & 7 \\ -4 & -8 & -7 \end{bmatrix}}$$

check

$$A^{-1}A = \frac{1}{21}\begin{bmatrix} -1 & -2 & -7 \\ 10 & -1 & 7 \\ -4 & -8 & -7 \end{bmatrix}\begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & -3 \\ -4 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{21}\begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix} = I$$

ok

→ Inverse of B

since $\det B = 0$ Then B^{-1} does not exist

i.e. <u>B is a singular matrix</u>

(c) Solve the matrix eqn for vector x: $Ax = y$

well, since we already know A^{-1}, let's calc

$$y = \begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix}$$

$$x = A^{-1} y$$

$$= \frac{1}{21} \begin{bmatrix} -1 & -2 & -7 \\ 10 & -1 & 7 \\ -4 & -8 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} -1 + 8 + 35 \\ 10 + 4 - 35 \\ -4 + 32 + 35 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 42 \\ -21 \\ 63 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}}$$

ans

(d) Find the eigenvalues and eigenvectors of $C = \begin{bmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{bmatrix}$

→ for the eigenvalues

$$|C - \lambda I| = \begin{vmatrix} -4-\lambda & 4 & 0 \\ 6 & -12-\lambda & 6 \\ 0 & 4 & -4-\lambda \end{vmatrix}$$

expand along row 1

$$= (-4-\lambda) \begin{vmatrix} -12-\lambda & 6 \\ 4 & -4-\lambda \end{vmatrix} + 4(-1) \begin{vmatrix} 6 & 6 \\ 0 & -4-\lambda \end{vmatrix}$$

$$= (-4-\lambda)\Big((-12-\lambda)(-4-\lambda) - 24 \Big) - 24(-4-\lambda)$$

$$= -(\lambda+4)\left[(\lambda+12)(\lambda+4) - 48 \right]$$

$$= -(\lambda+4)(\lambda)(\lambda+16) = 0$$

$$\begin{array}{r} \lambda+12 \\ \lambda+4 \\ \hline \lambda^2 + 16\lambda + 48 \\ -48 \\ \hline \lambda^2 + 16\lambda \end{array}$$

$$\therefore \boxed{\lambda_1 = 0 \qquad \lambda_2 = -4 \qquad \lambda_3 = -16}$$

eigenvalues

→ for the eigenvectors

for $\lambda_1 = 0$

$(\underline{\underline{C}} - \lambda \underline{\underline{I}}) \underline{X}_1 = \underline{0}$

$$\begin{bmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$-4X_1 + 4X_2 = 0 \qquad X_1 = X_2$

$6X_1 - 12X_2 + 6X_3 = 0 \qquad$ ok

$4X_2 - 4X_3 = 0 \qquad X_2 = X_3$

$$\therefore \underline{X}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

for $\lambda_2 = -4$

$(\underline{\underline{C}} - \lambda \underline{\underline{I}}) \underline{X}_2 = \underline{0}$

$$\begin{bmatrix} 0 & 4 & 0 \\ 6 & -8 & 6 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$4X_2 = 0 \qquad X_2 = 0$

$6X_1 - 8X_2 + 6X_3 = 0$

$4X_2 = 0 \qquad$ ok $\qquad \rightarrow X_1 = -X_3$

$$\therefore \underline{X}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda_3 = -16$

$(\underline{\underline{C}} - \lambda \underline{\underline{I}}) \underline{X}_3 = \underline{0}$

$$\begin{bmatrix} 12 & 4 & 0 \\ 6 & 4 & 6 \\ 0 & 4 & 12 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$12X_1 + 4X_2 = 0 \qquad X_2 = -3X_1$

$6X_1 + 4X_2 + 6X_3 = 0 \qquad$ ok

$4X_2 + 12X_3 = 0 \qquad X_3 = \frac{-X_2}{3}$

or $X_3 = X_1$

$$\therefore \underline{X}_3 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

```
>> linear_algebra_5
Matrices for Problems 2a - 2d
A =
     3     2    -1
     2    -1    -3
    -4     0     1
B =
     2     2    -1
     2     1     3
     2     4    -9
C =
    -4     4     0
     6   -12     6
     0     4    -4
y =
     1
    -4
    -5
Find det A
ans =
    21
Find det B
ans =
     0
Find inverses of A and B
AI =
   -0.0476   -0.0952   -0.3333
    0.4762   -0.0476    0.3333
   -0.1905   -0.3810   -0.3333
Warning: Matrix is singular to working precision.
  In linear_algebra_5 at 24
BI =
   Inf   Inf   Inf
   Inf   Inf   Inf
   Inf   Inf   Inf
Solve A*x = y
x =
     2
    -1
     3
Find eigenvalues & eigenvector of C
evec =
   -0.3015   -0.7071    0.5774
    0.9045   -0.0000    0.5774
   -0.3015    0.7071    0.5774
eval =
  -16.0000         0         0
         0   -4.0000         0
         0         0   -0.0000
>>
```

```
%
%   LINEAR_ALGEBRA_5.M  MATLAB matrix tasks as part of Prob. #2 in HW2
%
%   This file just does some simple linear algebra manipulations within Matlab.
%   The goal here is to compare with some hand calculations to confirm our
%   general understanding.
%
%   File prepared by J. R. White, UMass-Lowell (last update: Sept. 2017)
%

    clear all,   close all
    format compact
%
%   define matrices for problems
    disp('Matrices for Problems 2a - 2d')
    A = [3 2 -1;2 -1 -3;-4 0 1],  B = [2 2 -1;2 1 3;2 4 -9]
    C = [-4 4 0;6 -12 6;0 4 -4],  y = [1 -4 -5]'
%
%   *** Problem 2a ***
    disp('Find det A');          det(A)
    disp('Find det B');          det(B)
%
%   *** Problem 2b ***
    disp('Find inverses of A and B'),  AI = inv(A),  BI = inv(B)
%
%   *** Problem 2c ***
    disp('Solve A*x = y');       x = A\y
%
%   *** Problem 2d ***
    disp('Find eigenvalues & eigenvector of C'),  [evec,eval] = eig(C)
%
%   end of program
```