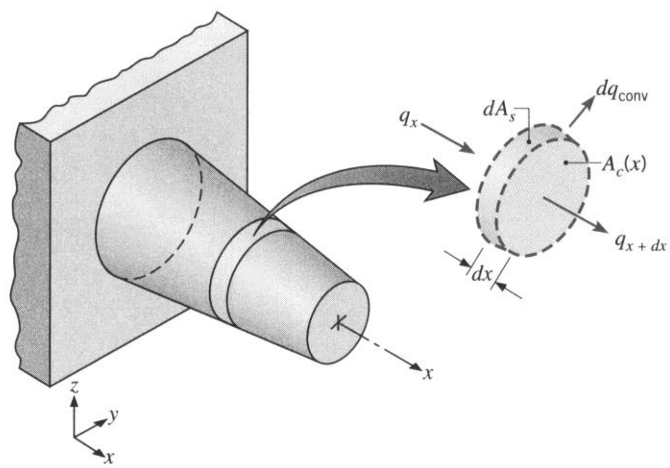
Heat Transfer in a Rectangular Fin

Fourier's law of heat conduction tells us that the rate of energy transfer is proportional to the heat transfer area. Because of this, extended surfaces are used in many applications to help enhance the energy transfer process. In general, the study of fin heat transfer is rather complicated because some of the geometries can become quite complex. In these cases, finite difference or finite element modeling is used to discretize the geometry of interest. Energy balances on each nodal volume are performed, which leads to a system of simultaneous algebraic equations. Solving these equations gives a discrete approximation to the temperature profile in the system. This process is relatively straightforward and many computer codes have been developed to handle this type of problem. However, the detailed study of finite difference (FD) or finite element (FE) methods for the solution of boundary value problems (BVPs) is beyond the scope of this course (although a brief introduction to the FD method is treated in one of the case studies in Lesson 4: Numerical Errors).

Fortunately, however, there are several practical cases that involve simple 1-D geometries that can be solved analytically. The usual assumption in these problems is that heat conduction along the fin is primarily in one direction, and a simple energy balance on a differential element leads to the defining ODE -- that is, the differential energy balance for the system. The solution of this ODE, along with specific boundary conditions, gives the 1-D temperature profile in the system of interest. Knowing the temperature profile, one can then determine the total energy transfer, the fin efficiency, and several other quantities of interest.

To illustrate this process, consider the sketch of a typical extended surface (obtained from Ref. 1). For steady state, a simple energy balance gives

energy flow rate in = energy flow rate out



or, using the notation from the diagram,

$$q_x = q_{x+dx} + dq_{conv} \qquad (1)$$

For 1-D problems, Fourier's law of conduction can be written as

$$q_x = -kA_c \frac{dT}{dx} \qquad (2)$$

and Newton's Law of Cooling for convection heat transfer between a solid surface and a fluid gives

$$dq_{conv} = h\, dA_s (T - T_\infty) \qquad (3)$$

where A_c = cross sectional area for conduction heat transfer

dA_s = surface area of differential element for convection heat transfer

Using a first-order Taylor series representation for q_{x+dx} within the 1-D steady state balance equation, we have

$$q_x = q_x + \frac{dq_x}{dx}dx + dq_{conv}$$

or

$$\frac{dq_x}{dx}dx + dq_{conv} = 0$$

Now, with the above expressions for q_x and dq_{conv}, we have

$$\frac{d}{dx}\left(-kA_c\frac{dT}{dx}\right) + h\frac{dA_s}{dx}\left(T - T_\infty\right) = 0$$

or

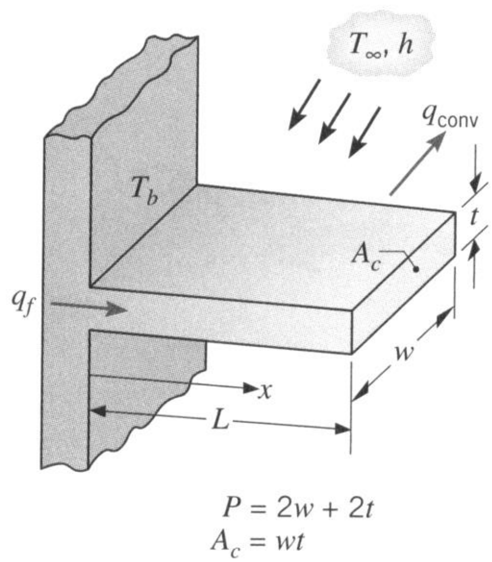
$$\frac{d}{dx}\left(kA_c\frac{dT}{dx}\right) - h\frac{dA_s}{dx}\left(T - T_\infty\right) = 0 \qquad (4)$$

This 2nd order ODE represents the 1-D steady state differential energy balance for an extended surface. In general, it is valid for any 1-D geometry, even ones with variable area (i.e. A_c and A_s vary with x). Note that the heat transfer coefficient, h, is assumed to be constant over the surface in the current development.

Now, since we want to keep things relatively simple to illustrate the basics, let's assume a simple rectangular fin arrangement as shown in the sketch (again taken from Ref. 1). Here we see that, in the special case of a rectangular fin, the following conditions apply

$$A_c = \text{constant} = wt \qquad \text{and} \qquad dA_s = Pdx$$

where P is the perimeter (P = 2w+2t).



Now, for the case of constant thermal conductivity, k, eqn. (4) reduces to

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

or

$$\frac{d^2T}{dx^2} - m^2(T - T_\infty) = 0 \qquad \text{with} \qquad m^2 = \frac{hP}{kA_c} \qquad (5)$$

where m^2 is simply a constant that simplifies writing the ODE. This equation is often studied in some detail in introductory heat transfer texts since it can give lots of insight into the subject of fin heat transfer.

To obtain a unique solution to eqn. (5), we need to specify two unique boundary conditions (BCs) for the problem. One common situation is for a fixed base temperature, T_b, and a convective tip condition. Mathematically, these specific BCs can be written as

$$T(0) = T_b \qquad \text{and} \qquad -k\left.\frac{dT}{dx}\right|_{x=L} = h\,(T - T_\infty)\big|_{x=L} \tag{6}$$

Equations (5) and (6) define a specific BVP. Since this problem has constant coefficients, its solution is particularly straightforward, as follows:

First, we let $\theta = T - T_\infty$ to give a homogeneous equation, where

$$\theta = T - T_\infty \qquad \frac{d\theta}{dx} = \frac{dT}{dx} \qquad \frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2}$$

Therefore, eqns. (5) and (6) become

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{7}$$

with

$$\theta(0) = T_b - T_\infty = \theta_b \qquad \text{and} \qquad -k\left.\frac{d\theta}{dx}\right|_{x=L} = h\left(T(L) - T_\infty\right) = h\theta\big|_{x=L} = h\theta_L \tag{8}$$

Now, since the ODE has constant coefficients and is homogeneous, we assume a solution of the form $\theta(x) = e^{rx}$, which leads to the characteristic equation

$$r^2 - m^2 = 0$$

with roots $r_{1,2} = \pm m$.

Thus, the general solution to eqn. (7) becomes

$$\theta(x) = A_1 e^{mx} + A_2 e^{-mx} \tag{9a}$$

or

$$\theta(x) = C_1 \sinh mx + C_2 \cosh mx \tag{9b}$$

where the hyperbolic sine and cosine functions are defined as follows:

$$\sinh mx = \frac{e^{mx} - e^{-mx}}{2} \qquad \text{and} \qquad \cosh mx = \frac{e^{mx} + e^{-mx}}{2}$$

For problems with a finite length, it is usually more convenient to use the second form. Thus, applying the BCs to eqn. (9b) gives

BC #1: $\quad \theta(0) = C_1(0) + C_2(1) = \theta_b \qquad \text{or} \qquad C_2 = \theta_b$ (10)

BC #2: $\quad -k\left.\dfrac{d\theta}{dx}\right|_{x=L} = h\theta\big|_{x=L}$

$$-k\left(C_1 m\cosh mx + C_2 m\sinh mx\right)\big|_{x=L} = h\left(C_1 \sinh mx + C_2 \cosh mx\right)\big|_{x=L}$$

$$-k\left(C_1 m\cosh mL + C_2 m\sinh mL\right) = h\left(C_1 \sinh mL + C_2 \cosh mL\right)$$

and, with the use of eqn. (10), we can solve this for C_1, giving

$$C_1 = -\theta_b \frac{\left(\sinh mL + \frac{h}{mk}\cosh mL\right)}{\cosh mL + \frac{h}{mk}\sinh mL} \tag{11}$$

Putting the constants C_1 and C_2 back into the general solution gives the unique solution,

$$\frac{\theta(x)}{\theta_b} = \cosh mx - \frac{\left(\sinh mL + \frac{h}{mk}\cosh mL\right)}{\cosh mL + \frac{h}{mk}\sinh mL}\sinh mx$$

We can simplify this a bit by putting all terms on the RHS over the same denominator,

$$\frac{\theta(x)}{\theta_b} = \frac{\left(\cosh mL + \frac{h}{mk}\sinh mL\right)\cosh mx - \left(\sinh mL + \frac{h}{mk}\cosh mL\right)\sinh mx}{\cosh mL + \frac{h}{mk}\sinh mL}$$

Now, from the following identities for the hyperbolic functions,

$$\sinh(u - v) = \sinh u \cosh v - \cosh u \sinh v$$

and

$$\cosh(u - v) = \cosh u \cosh v - \sinh u \sinh v$$

we have, upon substitution of $u = mL$ and $v = mx$, our final expression for the desired temperature profile,

$$\frac{\theta(x)}{\theta_b} = \frac{\cosh m(L - x) + \frac{h}{mk}\sinh m(L - x)}{\cosh mL + \frac{h}{mk}\sinh mL} \tag{12}$$

Once one has the desired temperature profile, a variety of analyses can be performed. Here we will compute the total heat transfer, q_f, the fin effectiveness, ε_f, and the fin efficiency, η_f, which are defined as follows:

total heat transfer, q_f:

$$q_f = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = -kA_c \left.\frac{d\theta}{dx}\right|_{x=0} = \left.\frac{\theta_b k A_c m\left[\sinh m(L - x) + \frac{h}{mk}\cosh m(L - x)\right]}{\cosh mL + \frac{h}{mk}\sinh mL}\right|_{x=0}$$

or

$$q_f = \frac{M\left[\sinh mL + \frac{h}{mk}\cosh mL\right]}{\cosh mL + \frac{h}{mk}\sinh mL} \quad (13)$$

where $M = \theta_b\sqrt{kA_c hP}$, since $kA_c m = kA_c\sqrt{\frac{hP}{kA_c}} = \sqrt{kA_c hP}$

fin effectiveness, ε_f:

$$\varepsilon_f = \frac{\text{energy transfer with fin}}{\text{energy transfer without fin}} = \frac{q_f}{hA_c\theta_b} \quad (14)$$

fin efficiency, η_f:

$$\eta_f = \frac{\text{energy transfer with fin}}{\text{max energy that could be transferred with fin}} = \frac{q_f}{q_{max}} = \frac{q_f}{hA_f\theta_b} \quad (15)$$

where A_f is the total heat transfer area of fin, $A_f = PL + A_c$.

As a specific application of the above development, let's consider a rectangular fin with the following properties:

L = 5 cm = .05 m t = 1.0 cm = 0.01 m w = 1 m (unit width)

T_b = 200 C T_∞ = 30 C h = 500 W/m^2-C

and, as part of our analysis, we will look at three different values of thermal conductivity, k, to represent the use of three different fin materials, where

$$k_1 = 5\frac{W}{m-C}, \qquad k_2 = 50\frac{W}{m-C}, \qquad k_3 = 200\frac{W}{m-C}$$

Using these values, our goal is to compute and plot the temperature profiles for the three different fin materials. In addition, we will create a short table of results that includes θ_L, q_f, ε_f, and η_f for the different fin materials, as well as a short table of temperatures that show T(x) for several discrete x_i values. Specifically, the three profiles will be on a single well-labeled plot and the numerical data will be formatted into a table that includes sufficient data so that one can easily draw appropriate conclusions.

The goals identified in the previous paragraph have been realized in Matlab file **rect1d_fin_1.m**. A listing of this program is given in Table 1. The program is quite straightforward, with the three standard sections that include data specification, computation, and presentation of results. The computational section includes appropriate array arithmetic and a loop that treats the three different fin materials. The plot section is also standard, with nothing much different from our previous examples (note how the labels for use in the ***legend*** command are generated in the computational section and stored in a ***cell*** array).

Table 1 Listing of the rect1d_fin_1.m program.

```
%
%   RECT1D_FIN_1.M   Function Evaluation, Plotting, and Table Formation
%                 Heat Transfer Analysis of a Rectangular Fin Arrangement
%
%   This file does some computational analysis for a rectangular fin with a fixed
%   base or wall temperature and a convective environment at the tip.  The
%   equations programmed for the temperature profiles assume that the energy flow
%   in the fin is essentially one-dimensional along the length of the fin.
%   Several performance parameters are also tabulated for the fins, including
%   the thermal conductivity, the base and tip temperatures, the total heat
%   transferred from the fin, the fin effectiveness, and the overall fin efficiency.
%
%   The goal of this file is to illustrate several of Matlab's basic capabilities,
%   including function evaluation and plotting and the use of the fprintf command
%   to create a summary table of results in an easy-to-read format.  It also shows
%   how to use a for...end loop to perform a parametric study involving a single
%   parameter (in this case we look at three different values of the fin's
%   thermal conductivity).  Also, of course, you should gain a little further
%   understanding of some simple heat transfer principles (conduction and
%   convection heat transfer).
%
%   The basic idea for this problem came from the text "Fundamentals of Heat
%   and Mass Transfer" 5th Ed by Incropera and Dewitt (2002, John Wiley & Sons).
%
%   File prepared by J. R. White, UMass-Lowell  (last update: Sept. 2017)
%

     clear all,  close all,  nfig = 0;
%
%   identify basic problem data
     w = 1;                           % unit width of fin (m)
     thk = 0.01;                      % fin thickness (m)
     L = 0.05;                        % fin length  (m)
     Tb = 200;                        % fin base temperature (C)
     Tinf = 30;                       % environment temperature (C)
     h = 500;                         % heat transfer coeff  (W/m^2-C)
     k = [5 50 200];                  % fin thermal conductivities (W/m-C)
%
%   compute some derived parameters (note the use of 'dot arithmetic' where needed)
     P = 2*w + 2*thk;                 % perimeter
     Ac = w*thk;                      % cross section area (conduction area)
     Af = P*L + Ac;                   % total fin surface area (for convection)
     qmax = h*Af*(Tb - Tinf);         % max feat transfer in fin
     qmin = h*Ac*(Tb - Tinf);         % heat transfer if no fin is present
     Bi = h*thk./k;                   % Biot number
     m = sqrt(h*P./(k*Ac));           % constant in derived equations (see notes)
     M = sqrt(k*Ac*h*P)*(Tb - Tinf);  % constant in derived equations (see notes)
     hmk = h./(m.*k);                 % constant in derived equations (see notes)
     bot = cosh(m*L) + hmk.*sinh(m*L); % constant in derived equations (see notes)
     qf = M.*(sinh(m*L) + hmk.*cosh(m*L))./bot;   % total fin heat transfer
%
%   compute temp profile for different materials (i.e. different thermal conductivities)
     Nk = length(k);                  % number of different materials
     Nx = 51;  x = linspace(0,L,Nx)';  % independent spatial variable
     T = zeros(Nx,Nk);                % allocate space for temperature profiles
     lstr = cell(1,Nk);               % initialize space for legend labels
     for n = 1:Nk
       T(:,n) = Tinf + (Tb - Tinf)*(cosh(m(n)*(L-x)) + hmk(n)*sinh(m(n)*(L-x)))/bot(n);
       lstr(n) = {['k = ',num2str(k(n),'%3i'),' W/m-C']};
     end
%
%   plot temp profiles
     nfig = nfig+1;   figure(nfig)
     plot(x*100,T(:,1),'r-.',x*100,T(:,2),'b--',x*100,T(:,3),'g-','LineWidth',2),grid
     title('T(x) Profiles in Fin for Different Values of k ')
     xlabel('Spatial Position (cm)')
     ylabel('Temperature (^oC)')
     legend(lstr)
%
```

```
%   write summary table of results (note that this assumes Nk = 3)
      fprintf(1,'\n\n');
      fprintf(1,'          Summary Results from the RECT1D_FIN_1.M Program \n');
      fprintf(1,'\n');
      fprintf(1,'    Case-Independent Parameters      \n');
      fprintf(1,'  Base Temperature (C):               %8.2f    \n',Tb);
      fprintf(1,'  Environment Temperature (C):        %8.2f    \n',Tinf);
      fprintf(1,'  Heat Transfer Coeff (W/m^2-C):      %8.2f    \n',h);
      fprintf(1,'  Fin Length (cm):                    %8.2f    \n',L*100);
      fprintf(1,'  Fin Thickness (cm):                 %8.2f    \n',thk*100);
      fprintf(1,'  Min Heat Transfer possible (W):     %8.2f    \n',qmin);
      fprintf(1,'  Max Heat Transfer possible (W):     %8.2f    \n',qmax);
      fprintf(1,'\n');
      fprintf(1,'    Case-Dependent Parameters          Case #1    Case #2   Case #3 \n');
      fprintf(1,'  Thermal Conductivity (W/m-C):       %8.2f   %8.2f  %8.2f  \n',k);
      fprintf(1,'  Tip Temperature (C):                %8.2f   %8.2f  %8.2f  \n',T(Nx,:));
      fprintf(1,'  Heat Transfer per unit width (W):   %8.2f   %8.2f  %8.2f  \n',qf);
      fprintf(1,'  Fin Effectiveness (dimensionless):  %8.2f   %8.2f  %8.2f  \n',qf/qmin);
      fprintf(1,'  Fin Efficiency (dimensionless):     %8.2f   %8.2f  %8.2f  \n',qf/qmax);
      fprintf(1,'  Biot Number (dimensionless):        %8.2f   %8.2f  %8.2f  \n',Bi);
      fprintf(1,'\n');
      fprintf(1,'    Actual Temperature Profiles, T(x) in C, for the Three Cases \n');
      fprintf(1,'\n');
      fprintf(1,'        Position      Case #1   Case #2   Case #3 \n');
      fprintf(1,'          (cm)          T(x)      T(x)      T(x)  \n');
      for i = 1:2:Nx
        fprintf(1,'         %5.1f      %8.1f  %8.1f  %8.1f  \n',x(i)*100,T(i,:));
      end
%
%  end of problem
```

The only unique feature, and the real point of this Matlab example, is the use of the ***fprintf*** command to create summary tabular results for this problem. We have used this command before, but only to print out a few key results. Here, our goal was to create a full page of results that fully documents this particular analysis. Your texts for this course give a good overview of the ***fprintf*** command syntax and, of course, the Matlab ***help*** facility gives very detailed information. You certainly need to review these information sources! However, I feel that the best way to learn how to effectively use the ***fprintf*** function is to see its use in a real example. The last portion of **rect1d_fin_1.m** provides this demonstration, where the output is sent to the screen (fid = 1 in all the ***fprintf*** statements, where unit 1 is the display screen).

The results from **rect1d_fin_1.m** are summarized in Table 2 and in Fig. 1. The figure is a copy of the single plot produced in the program and Table 2 is an image of the tabular output produced with the use of ***fprintf*** in the last section of **rect1d_fin_1.m**. Since a high thermal conductivity implies that energy can be conducted more easily, the relative results from both Table 2 and Fig. 1 are as expected. Clearly, the T(x) profiles indicate that there is less of a temperature variation in the fin with increasing k. In addition, the total heat transfer, the fin effectiveness, and the fin efficiency are all enhanced when using a fin material with a higher thermal conductivity. Thus, you can correctly conclude that real fins are made of materials with high k -- the higher the better!

However, even with a high k, the fin analyzed here is not very efficient, with the best case having an efficiency less than 70%. This is because the temperature drop from the base to the tip is too high -- since the area near the tip with a temperature of 120 C loses much less energy per unit length than a similar area near the fin's base (which is near 200 C). Thus, for good efficiency (i.e. effective use of the fin material), we desire to minimize the temperature decrease along the fin's length, where the best possible scenario is when $T(x) \approx T_b$ everywhere. In this case, one easy way to increase the efficiency would be to decrease the length of the fin. This would, of course, decrease the total heat transfer area and the total energy transfer of a single fin, but multiple fins could be used to counter this effect, and keep the efficiency high.

Table 2 Tabular results from rect1d_fin_1.m.

```
          Summary Results from the RECT1D_FIN_1.M Program

   Case-Independent Parameters
 Base Temperature (C):                 200.00
 Environment Temperature (C):           30.00
 Heat Transfer Coeff (W/m^2-C):        500.00
 Fin Length (cm):                        5.00
 Fin Thickness (cm):                     1.00
 Min Heat Transfer possible (W):       850.00
 Max Heat Transfer possible (W):      9435.00

   Case-Dependent Parameters          Case #1    Case #2   Case #3
 Thermal Conductivity (W/m-C):           5.00      50.00    200.00
 Tip Temperature (C):                   30.16      59.19    121.72
 Heat Transfer per unit width (W):    1208.08    3766.37   6449.51
 Fin Effectiveness (dimensionless):      1.42       4.43      7.59
 Fin Efficiency (dimensionless):         0.13       0.40      0.68
 Biot Number (dimensionless):            1.00       0.10      0.03

   Actual Temperature Profiles, T(x) in C, for the Three Cases

       Position      Case #1   Case #2   Case #3
         (cm)         T(x)      T(x)      T(x)
          0.0        200.0     200.0     200.0
          0.2        157.9     185.6     193.7
          0.4        126.3     172.5     187.8
          0.6        102.5     160.5     182.1
          0.8         84.5     149.5     176.8
          1.0         71.0     139.6     171.8
          1.2         60.9     130.5     167.1
          1.4         53.2     122.2     162.6
          1.6         47.5     114.7     158.4
          1.8         43.2     107.9     154.5
          2.0         39.9     101.7     150.8
          2.2         37.5      96.0     147.3
          2.4         35.6      90.9     144.1
          2.6         34.2      86.3     141.2
          2.8         33.2      82.2     138.4
          3.0         32.4      78.5     135.9
          3.2         31.8      75.1     133.6
          3.4         31.4      72.1     131.5
          3.6         31.0      69.5     129.6
          3.8         30.8      67.2     127.9
          4.0         30.6      65.2     126.4
          4.2         30.4      63.5     125.0
          4.4         30.3      62.0     123.9
          4.6         30.3      60.8     123.0
          4.8         30.2      59.9     122.3
          5.0         30.2      59.2     121.7
```

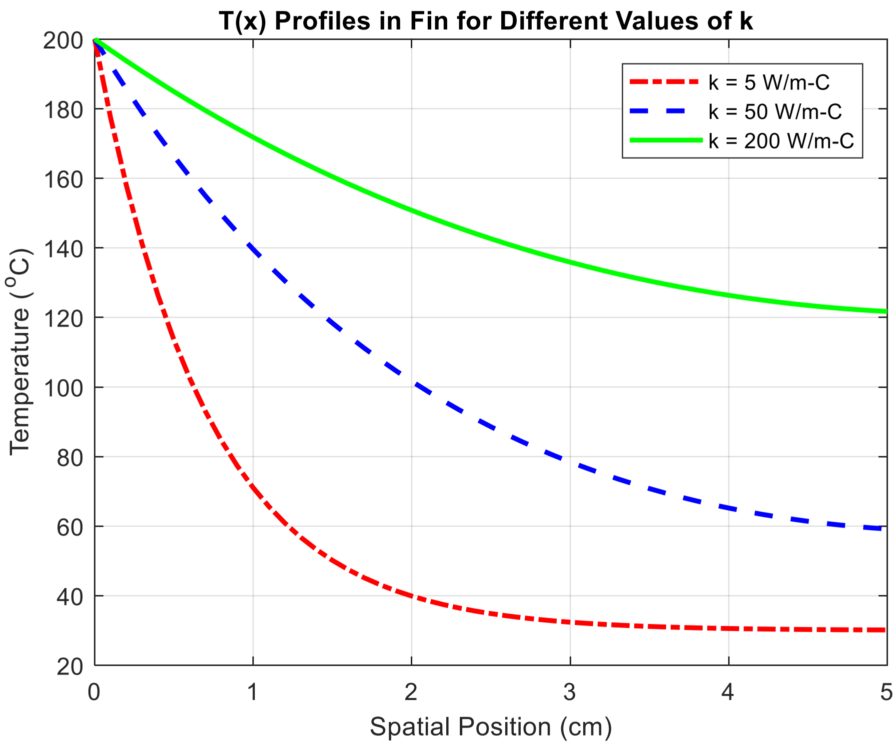


Fig. 1 Temperature profiles in the rectangular fin for different fin materials.

One final point that should be mentioned here is the appropriateness of the 1-D approximation that was made in the development of the equations used here. The 1-D approximation essentially assumes that the temperature variation through the thickness of the fin (the z-direction in our original sketch for a general extended surface) is negligible -- that is $T(x,z) \approx T(x)$. The goodness of the approximation is characterized by the Biot number, which is a ratio of the conduction resistance to the convection resistance in the direction of interest (the transverse direction in this case), or

$$Bi = \frac{R_{cond}}{R_{conv}} = \frac{t/kA_s}{1/hA_s} = \frac{ht}{k} \qquad (16)$$

Typically, if Bi << 1, then the resistance to conduction heat transfer is small compared to the resistance to convection heat transfer. When this happens, the temperature distribution through the thickness of the fin is nearly flat and the 1-D approximation is very good. If the Biot number is close to or greater than unity, then you should seriously question the results of a 1-D analysis - since a high Biot number implies that a 2-D analysis is probably needed. As a test of this, the Biot number was computed in **rect1d_fin_1.m** and printed along with the other numerical results. As apparent in Table 2, the Biot number for Case 1 with k = 5 W/m-C is exactly unity. Thus, the results given here for Case 1 are probably not very reliable. The other two cases, with Bi = 0.10 and 0.03, should be quite accurate relative to a more detailed 2-D analysis.

Well, we have come to the end of another example. This treatment of a 1-D rectangular fin represents a standard approach to problem solving, where we have done the formal model development from base principles, solved the resultant BVP using analytical techniques, and then analyzed the results of a parametric study that addressed how the thermal conductivity affects the heat transfer process. Matlab was used in the analysis portion of our study to evaluate the resultant equations, plot the temperature profiles, and tabulate summary results from the parametric study -- where we highlighted the use of the ***fprintf*** command in Matlab to prepare the formatted tabular data. Later in the semester, we will bypass the analytical solution step and show you how to use Matlab to directly solve the BVP numerically. Thus, we will revisit this example when we get to the subject of Numerical Solution of ODEs. For now, you should leave this application with a better understanding of some heat transfer fundamentals, a good example of the usefulness of the ***fprintf*** command, and another illustration of a solid approach to solving problems in engineering design and analysis…

Reference: The basic idea for this problem came from the text "Fundamentals of Heat and Mass Transfer" 5th Ed. by Incropera and Dewitt, John Wiley & Sons, 2002. The two sketches used at the beginning of this example also came from this reference (pgs. 129 and 130).