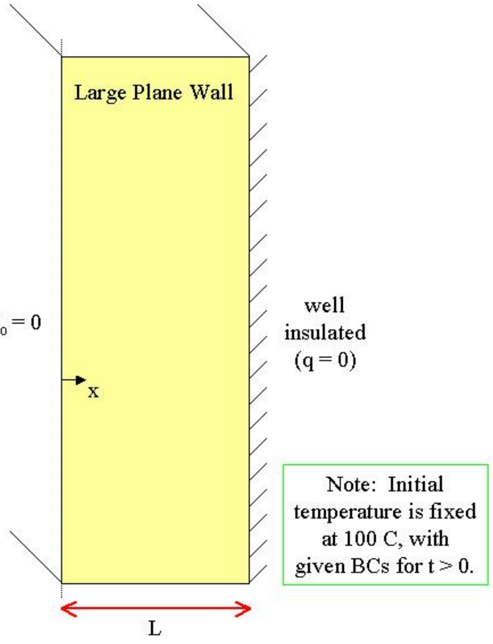
Evaluating and Plotting Space-Time Temperature Distributions

We have already seen several examples of heat transfer applications. As another example, consider the case where a large plane wall is initially at a constant temperature of 100 C. Then, at t = 0, the left side of the wall at x = 0 is exposed to an ice bath and the right side at x = L is perfectly insulated. The ice bath forces the left wall temperature to be essentially at 0 C. A rough sketch illustrating this situation is shown at the right.


Large Plane Wall
$T_0 = 0$
x
well insulated ($q = 0$)
Note: Initial temperature is fixed at 100 C, with given BCs for $t > 0$.
L

This problem can be formulated mathematically as

$$\frac{\partial}{\partial t}T(x,t) = \alpha\frac{\partial^2}{\partial x^2}T(x,t) \qquad (1)$$

with

IC: $T(x,0) = T_i = 100$ C (constant initial temperature)

BC#1: $T(0,t) = 0$ C (left surface at 0 C)

BC#2: $\left.\frac{\partial}{\partial x}T(x,t)\right|_{x=L} = 0$ (no heat transfer at right surface)

where α is the thermal diffusivity and L is the thickness of the wall. For this application, we will assume that the planar object of interest is a large rectangular copper plate with a thickness of L = 10 cm, where the thermal diffusivity of copper is $\alpha = 1.15$ cm^2/s (note that we will use centimeters as the base unit for distance in this problem -- thus, the value of α is given in consistent units). The goal here is to evaluate and plot the space-time temperature distribution in the copper plate.

A classical analytical technique for solving problems of this type is the Separation of Variables (SOV) method (useful for linear homogeneous PDEs with homogeneous BCs). The details of the SOV method are beyond the scope of this course [see an Advanced Applied Mathematics text, for example, if you are interested], but the result for this problem can be written succinctly as follows:

$$T(x,t) = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x)\, e^{-\alpha\lambda_n^2 t} \qquad (2)$$

where

$$\lambda_n = \frac{(2n-1)\pi}{2L} \quad \text{and} \quad a_n = \frac{4T_i}{(2n-1)\pi} \qquad (3)$$

Thus, we see that, with the solution given as an infinite series, our real task involves evaluating and plotting the infinite series given by eqns. (2) and (3).

Using an algorithm similar to that developed in the last example (without the recurrence treatment to get term n+1 written in terms of term n -- see pg. 2 of **infinite_series.pdf**), eqns. (2) and (3) were evaluated and the T(x,t) distribution was plotted in various ways within Matlab program **planewall_1.m**. A listing of **planewall_1.m** is given in Table 1. The results of the program are summarized in Figs. 1 and 2, where both 2-D and 3-D surface/contour plots were used to display the space-time temperature profile in a number of ways. The 2-D plot in Fig. 1 gives a quantitative perspective, with several T(x) profiles at different time points, and the two 3-D views in Fig. 2 give a good qualitative overview of the full space-time temperature distribution in the copper plate.

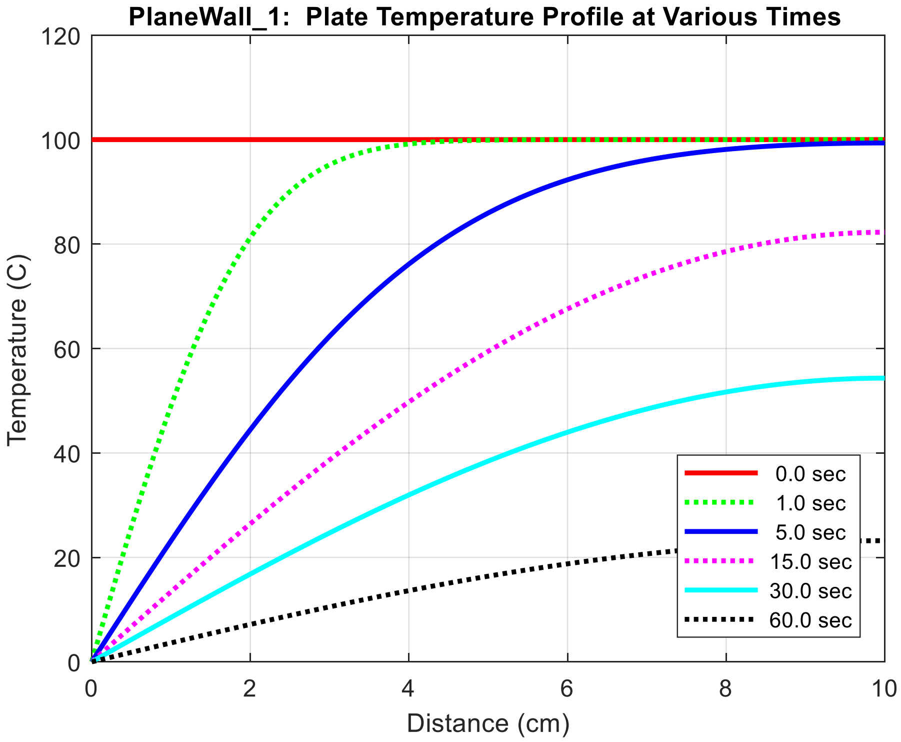


Fig. 1 Temperature profiles at various times in copper plate (2-D plot).

With an ice bath on the left surface and no heat transfer on the right side, we know that the all the initial excess energy in the copper plate must leave through the left face, which will give a positive temperature gradient, with the highest temperature (and zero slope) on the right side of the copper plate. With increasing time, the right-side temperature will decrease, eventually coming to the same temperature as the left side -- that is, the whole plate will approach a temperature of zero to be in equilibrium with the ice bath temperature. This general behavior is exactly that observed in Figs. 1 and 2, and we see that it takes a little over two minutes (120 sec) to essentially reach equilibrium.

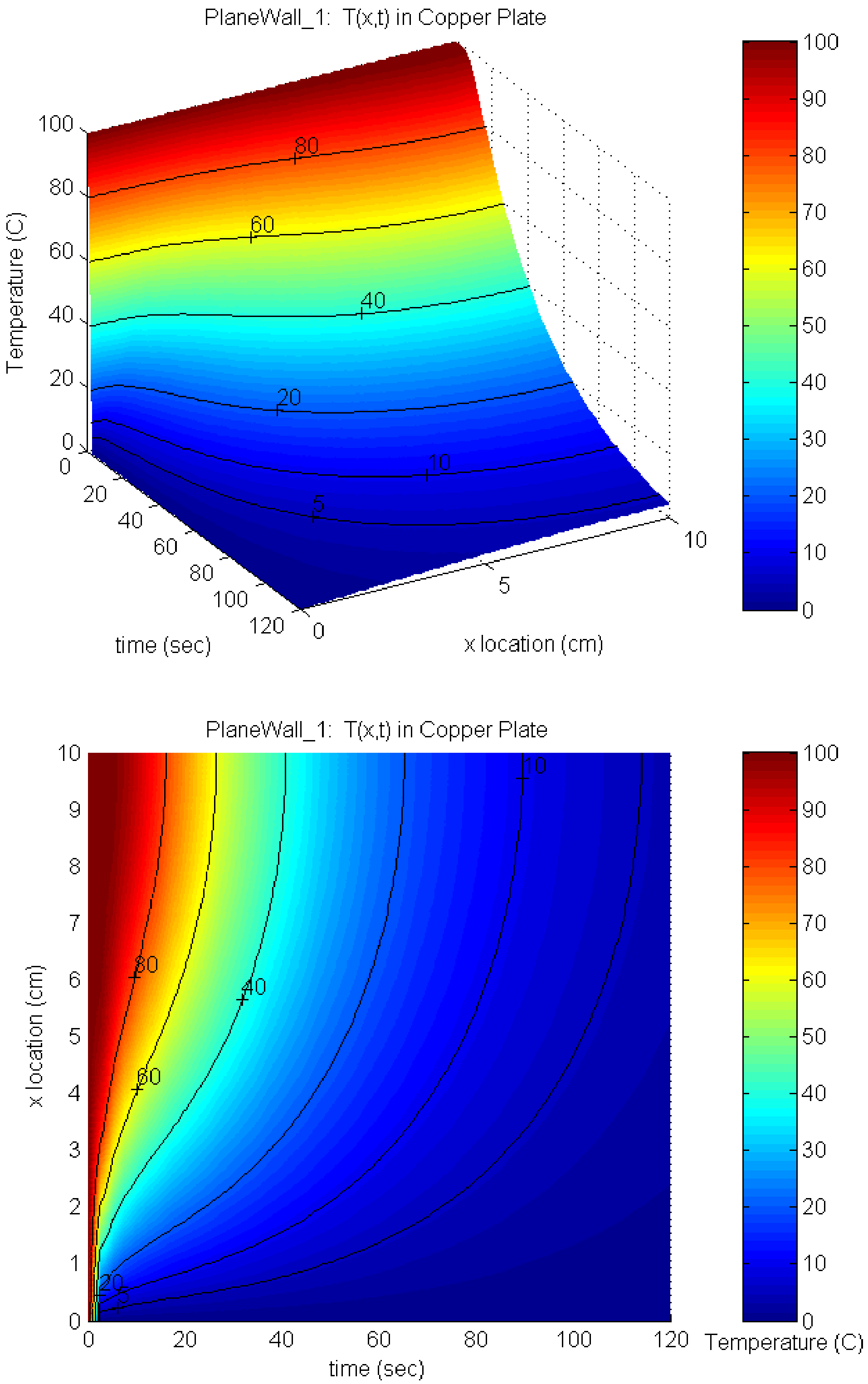


Fig. 2 Space-time temperature distribution in copper plate (various 3-D views).

Table 1 Listing of the planewall_1.m program.

```
%
%   PLANEWALL_1.M     Heat Transfer 1-D Planar Wall (Copper Plate)
%
%   The program implements the analytical solution using the Separation of Variables
%   (SOV) method to a relatively simple heat transfer problem.  The result of the
%   SOV method (see notes) is an infinite series expansion, and the goal here is
%   to evaluate and plot the T(x,t) distribution given by the infinite series.
%
%   The physical problem models 1-D space-time heat transfer in a large planar wall.
%   The wall is initially at a constant temperature of 100 C.  At time t = 0, the
%   left face is exposed to an ice bath (0 C) and the right surface is perfectly
%   insulated (no heat transfer or T' = 0).  The object of the simulation is to
%   compute and visualize the space-time temperature profile within the wall (in
%   this case, a copper plate 10 cm thick).
%
%   This represents another example of evaluating infinite series...
%
%   File prepared by J. R. White, UMass-Lowell (last update: Oct. 2017)
%


      clear all,   close all,   nfig = 0;


%
%   problem data
      alf = 1.15;             % cm^2/s thermal diffusivity for copper
      L = 10;                 % cm     thickness of plate
      Ti = 100;               % C      initial temp throughout plate
      Nx = 101;               % number of x values
      x = linspace(0,L,Nx)';  % vector of points to evaluate function
      nmax = 50;              % max number of nonzero terms in expansion
      tol = 0.001;            % tolerance to stop series evaluation
%
%   calc ln = (2n-1)*pi/2L  and   an = 4*Ti/(2n-1)*pi)  for n = 1,2,3,...
      ln = zeros(1,nmax);   an =  zeros(1,nmax);
      d = pi/(2*L);    c = 4*Ti/pi;
      for n = 1:nmax
        ln(n) = (2*n-1)*d;    an(n) = c/(2*n-1);
      end
%
%   choose type of plot to be produced
      ic = menu('Choose type of plot', ...
                'Show T(x) for several different times (2-D plot)  ', ...
                'Show T(x,t) in a variety of surface plots (3-D plots) ');
%
%   now evaluate series expansion for several different times
      if ic == 1
        tt = [0 1 5 15 30 60];        % use this for simple 2-D plot
      else
        tt = linspace(0,120,49);      % use this for 3-D plots (more time points)
      end
      Nt = length(tt);   T = zeros(Nx,Nt);
%
%   since the initial temperature profile is known, let's just set it here
      T(:,1) = Ti*ones(size(x));
%
%   now loop over remaining times
      for j = 2:Nt
        t = tt(j);   cc = -alf*t;   mrerr = 1.0;   n = 0;   Tx = zeros(size(x));
        while mrerr > tol   &&   n < nmax
          n = n+1;  Txn = an(n)*exp(cc*ln(n)*ln(n))*sin(ln(n)*x);
          Tx = Tx + Txn;
          i = find(Tx);  rerr = Txn(i)./Tx(i);   mrerr = max(abs(rerr));
        end
        T(:,j) = Tx;
        disp([' Needed ',num2str(n),' terms for convergence at t = ',num2str(t),' s'])
      end
```

```
%
%   *****  Create 2-D plot  *****
      if ic == 1
%
%   set color and marker code for creating 2-D plots
      Ncm = 6;  scm = ['r-';'g:';'b-';'m:';'c-';'k:'];
      h = zeros(Nt,1);  st = zeros(Nt,9);
%
%   plot curves of T(x,t) for various times
      nfig = nfig+1;  figure(nfig)
      for j = 1:Nt
        h(j) = plot(x,T(:,j),scm(j,:),'LineWidth',2); hold on
        st(j,:) = sprintf('%5.1f sec',tt(j));
      end
      title('PlaneWall\_1:  Plate Temperature Profile at Various Times')
      grid,xlabel('Distance (cm)'),ylabel('Temperature (C)')
      legend(h,char(st),'Location','SouthEast')
%
      end
%
%   *****  Create various 3-D plots  *****
      if ic == 2
%
      nfig = nfig+1;  figure(nfig);  colormap(jet)

      surf(tt,x,T), shading interp; colorbar, view(60,30), hold on
      [cc,hh] = contour3(tt,x,T, [80 60 40 20 10 5]);
      clabel(cc), set(hh,'EdgeColor','k')
      axis('tight')
      title('PlaneWall\_1:  T(x,t) in Copper Plate')
      ylabel('x location (cm)'),   xlabel('time (sec)')
      zlabel('Temperature (C)'),   hold off
%
      nfig = nfig+1;  figure(nfig);  colormap(jet)
      surf(tt,x,T), shading interp; colorbar, view(2), hold on
      [cc,hh] = contour3(tt,x,T, [80 60 40 20 10 5]);
      clabel(cc), set(hh,'EdgeColor','k')
      axis('tight')
      title('PlaneWall\_1:  T(x,t) in Copper Plate')
      ylabel('x location (cm)'),   xlabel('time (sec)')
      zlabel('Temperature (C)'),  gtext('Temperature (C)'),  hold off
%
      end
%
%   end of example
```

The program implementation for this problem was quite straightforward. The first part of the program sets the base variables and calculates λ_n and a_n, as given in eqn. (3), for 50 terms. The user then selects either the 2-D or 3-D plotting option, where this decision simply identifies how many time points will be used in the series evaluation (with more points needed for the 3-D plots for good quality plots). The infinite series is then evaluated for all values of x for each value of time, where a check on the utility of continuing with additional terms (up to 50 terms maximum) is made -- with the relative tolerance set at 0.001 for this application.

Once the computation of the series expansion for the temperature distribution is complete, the 2-D array of temperatures, $\mathbf{T} \rightarrow T(x,t)$, can be plotted, as desired. Here, we implemented the capability for a simple 2-D plot, where T(x) is plotted at several different times, and the option to produce two different qualitative 3-D views of the full space-time distribution. Other options are available, of course, but the specific representations chosen here give a pretty good overview of the complete space-time temperature distribution in the copper plate.

Well, this completes another example of evaluating infinite series -- where for this application, we were interested in visualizing the space-time temperature distribution in a planar wall configuration. A very similar approach can be used for other applications involving series expansions. These types of problems occur quite frequently in many useful engineering applications -- especially those that can be solved analytically using the Separation of Variables method.

In closing, I should comment that I find it quite difficult to visualize the physical behavior of a system that is described by mathematical equations like those given in eqn. (2) and (3). Thus, the ability to evaluate and plot functions written as an infinite series can be very useful in lots of applications. Hopefully, you will also find this visualization technique to be useful in your analysis of other systems of interest in your work!!!