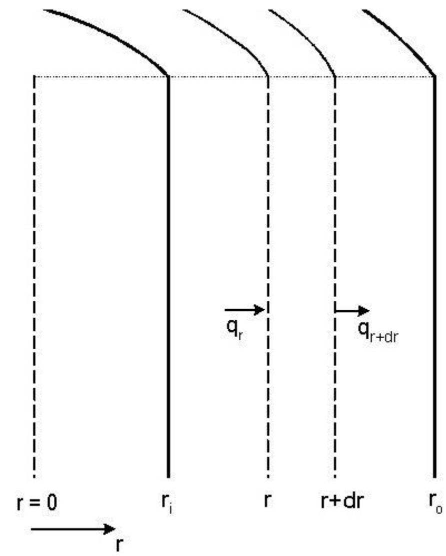
Pipe Insulation Considerations



Consider heat transfer in the radial direction of a long annular region -- such as the walls of a thick pipe or the insulation surrounding a circular conduit.

If we focus on steady-state operation and applications with no internal heating, a simple energy balance on the control volume shown in the sketch can be written as

$$\frac{dE_{CV}}{dt} = \dot{E}_{in} - \dot{E}_{out} + \dot{Q}_{gen}$$

but, since the rate of change is zero for steady state and there is no internal energy generation, this simplifies to

$$\dot{E}_{in} - \dot{E}_{out} = 0$$

or, using the notation from the sketch, we have

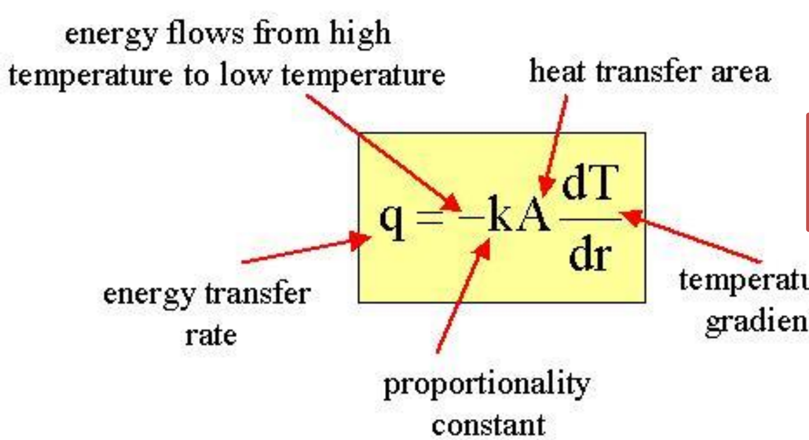
$$q_r - q_{r+dr} = 0 \qquad (1)$$

Now, in a solid, energy transfer is via conduction heat transfer, which is given by Fourier's Law. As illustrated below, Fourier's law says that the energy transfer rate is proportional to the heat transfer area and the negative gradient of the temperature profile. Mathematically, this can be written as

$$q = -kA\frac{dT}{dr} \qquad (2)$$

The proportionality constant, k, is a material property known as the thermal conductivity. By focusing on units in eqn. (2), we can determine the units of k, as follows:

$$W = \frac{J}{s} = (\text{units of } k)(m^2)\left(\frac{{}^oC}{m}\right) \qquad \text{or} \qquad (\text{units of } k) = \frac{W}{m - {}^oC}$$



Fourier's Law for 1-D radial heat flow in a solid

Now, putting Fourier's law into the steady-state energy balance in eqn. (2) gives

$$-\left(kA\frac{dT}{dr}\right)\bigg|_r + \left(kA\frac{dT}{dr}\right)\bigg|_{r+dr} = 0$$

Rearranging, dividing both sides by dr, and then taking the limit as dr $\rightarrow$ 0, gives

$$\lim_{dr\to 0}\frac{\left(kA\frac{dT}{dr}\right)\Big|_{r+dr} - \left(kA\frac{dT}{dr}\right)\Big|_r}{dr} = 0$$

But the LHS is simply the definition of the continuous derivative of kAdT/dr, or

$$\frac{d}{dr}\left(kA\frac{dT}{dr}\right) = 0 \qquad (3)$$

This equation is appropriate for 1-D radial steady-state heat conduction with no heat source.

Equation (3) is a 2nd order ODE that can be solved quite easily. Multiplying by dr and integrating gives

$$d\left(kA\frac{dT}{dr}\right) = 0\,dr \qquad \text{or} \qquad kA\frac{dT}{dr} = c_1 \qquad (4)$$

Now, before continuing, we note that in cylindrical geometry, the heat transfer area, A, is a function of location r, or

$$A(r) = 2\pi rL \qquad (5)$$

where L represents the length of the pipe (this is often treated as unity so that the analysis is "per unit length of pipe").

It is also important to note that, in general, the thermal conductivity, k, is a function of temperature; that is, k $\rightarrow$ k(T). However, in many applications, especially when the temperature variation is small, it is appropriate to simplify the analysis and assume constant material properties. Here we will make this assumption -- k(T) = k = constant.

With these relationships, eqn. (4) becomes

$$\frac{dT}{dr} = \frac{c_1}{2\pi kL}\frac{1}{r} \qquad (6)$$

This 1st order ODE is separable and integration gives

$$T(r) = \frac{c_1}{2\pi kL}\ln r + c_2 \qquad (7)$$

where, as expected, there are two arbitrary coefficients in the general solution to the original 2nd order ODE.

Now, to obtain a unique solution, we need to impose two conditions specific to the problem of interest. As indicated above, we will focus our attention in this problem on the insulation surrounding a circular conduit. In particular, let's consider an annular region of insulation over the domain $r_i \le r \le r_o$, with the following boundary conditions (BCs):

$$T(r_i) = T_i \qquad \text{and} \qquad -kA\left.\frac{dT}{dr}\right|_{r_o} = hA\left(T - T_\infty\right)\Big|_{r_o} \tag{8}$$

where the first condition specifies a fixed inside temperature and the second statement says that the heat transfer by conduction at the outer surface must equal the heat transfer by convection, as described by Newton's Law of Cooling. Here h is the heat transfer coefficient and T_∞ is the bulk fluid temperature. The heat transfer coefficient is generally a function of the geometry and the fluid environment. As before, we can determine the units of h by forcing units consistency in the equation for Newton's Law of Cooling, or

$$W = (\text{units of } h)\left(m^2\right)\left({}^{o}C\right) \qquad \text{or} \qquad (\text{units of } h) = \frac{W}{m^2 - {}^{o}C}$$

Now applying the problem-specific BCs in eqn. (8) to the general solution in eqn. (7) gives

at r_i
$$T_i = \frac{c_1}{2\pi kL}\ln r_i + c_2 \tag{9a}$$

at r_o
$$-k\frac{c_1}{2\pi kL}\frac{1}{r_o} = h\left(\frac{c_1}{2\pi kL}\ln r_o + c_2 - T_\infty\right) \tag{9b}$$

and rearranging these gives a system of two equations for two unknowns,

$$\frac{\ln r_i}{2\pi kL}c_1 + c_2 = T_i$$

$$\frac{\ln r_o + k/hr_o}{2\pi kL}c_1 + c_2 = T_\infty$$

We can solve for c_1 by subtracting these two equations to give

$$\left[\frac{\ln r_i}{2\pi kL} - \frac{\ln r_o + k/hr_o}{2\pi kL}\right]c_1 = T_i - T_\infty$$

or

$$\frac{1}{2\pi kL}\left[\ln\frac{r_i}{r_o} - \frac{k}{hr_o}\right]c_1 = T_i - T_\infty$$

which gives

$$c_1 = \frac{\left(T_i - T_\infty\right)\left(2\pi kL\right)}{\ln\frac{r_i}{r_o} - \frac{k}{hr_o}} \tag{10}$$

Now, putting this into eqn. (9a) gives

$$c_2 = T_i - \frac{\ln r_i}{\ln\frac{r_i}{r_o} - \frac{k}{hr_o}}\left(T_i - T_\infty\right) \tag{11}$$

Putting these expressions for c_1 and c_2 into the general solution in eqn. (7) gives the unique solution for this particular set of BCs, or

$$T(r) = T_i + \frac{T_i - T_\infty}{\ln\frac{r_i}{r_o} - \frac{k}{hr_o}}\ln\frac{r}{r_i} \tag{12}$$

We can evaluate and plot this profile to try to understand the heat transfer process for this situation. Before doing this, however, let's use normalized variables to reduce the number of parameters in the system under study.

In particular, let's define a normalized temperature profile, u(r), as

$$u(r) = \frac{T(r) - T_\infty}{T_i - T_\infty} \tag{13}$$

where at $r = r_i$, $\quad u(r_i) = 1$

at $r = r_o$, $\quad u(r_o) = \frac{T(r_o) - T_\infty}{T_i - T_\infty} \le 1$

This implies that u(r) decreases from unity at the inside of the annular region to some value that is usually less than unity at the outer surface.

Putting eqn. (12) into eqn. (13) gives

$$u(r) = 1 + \frac{\ln r/r_i}{\ln\frac{r_i}{r_o} - \frac{k}{hr_o}} \qquad \text{or} \qquad u(r) = 1 - \frac{\ln r/r_i}{\ln\frac{r_o}{r_i} + \frac{k}{hr_o}} \tag{14}$$

which clearly shows the trend described above.

From this expression we can easily see that the three key variables are:

k = thermal conductivity of material

- rock wool insulation has $k \approx 0.04$ W/m-C
- window glass has $k \approx 0.80$ W/m-C
- aluminum metal has $k \approx 200$ W/m-C

h = heat transfer coefficient (varies considerably)

- natural convection: $h_{air} \approx 5 - 20$ W/m^2-C and $h_{water} \approx 50 - 500$ W/m^2-C
- forced convection: $h_{air} \approx 5 - 500$ W/m^2-C and $h_{water} \approx 200 - 10000$ W/m^2-C
- boiling water: $h \approx 2000 - 50000$ W/m^2-C

r_o/r_i = geometry of system (design dependent)

To see the effects of these variables on the normalized temperature profile, let's plot u(r) from eqn. (14) for the following cases, where we only change one parameter at a time to more easily isolate its effect on the system response:

Case 0 -- Reference Case

$k = 0.1$ W/m-C (reasonably good insulation material)

$h = 5$ W/m^2-C (natural convection in air)

$r_i = 1$ cm and $r_o = 1.635$ cm (only about ¼ inch of insulation)

Case 1 -- Increased Thermal Conductivity

$k_1 = 1.0$ W/m-C (still an insulator but not nearly as good as Ref. Case)

Case 2 -- Increased Heat Transfer Coefficient

$h_2 = 50$ W/m^2-C (forced convection in air)

Case 3 -- More Insulation (increased r_o/r_i)

$r_{o3} = 6.08$ cm (nearly 2 inches of insulation)

Plotting u(r) for these cases should allow us to get a good qualitative understanding of how the changes implied here affect the heat transfer process. In addition, we can also quantify any differences that may be observed by comparing the heat transfer, q, for the three different perturbation cases relative to the reference case.

We have two different expressions for q -- one based on conduction heat transfer and one that is related to the convection heat transfer process:

$$q_{cond} = -kA(r)\frac{dT}{dr} \qquad \text{(Fourier's Law)} \qquad (15)$$

$$q_{conv} = hA(r_o)\left(T(r_o) - T_\infty\right) \qquad \text{(Newton's Law of Cooling)} \qquad (16)$$

where $T(r_o)$ and $A(r_o)$ are the outer surface temperature and outer surface area, respectively. Note that the expression for q_{cond} is valid at any r in the domain of interest.

Before actually evaluating these expressions, let's write them in terms of the normalized temperature, u(r). From eqn. (13), we have

$$T(r) = T_\infty + \left(T_i - T_\infty\right)u(r) \qquad (17)$$

and

$$\frac{dT}{dr} = \left(T_i - T_\infty\right)\frac{du}{dr} \qquad (18)$$

Putting these into eqns. (15) and (16) gives

$$q_{cond} = -k2\pi rL\left(T_i - T_\infty\right)\frac{du}{dr} \qquad (19)$$

$$q_{conv} = h2\pi r_o L\left(T_i - T_\infty\right)u(r_o) \qquad (20)$$

where du/dr is given explicitly as

$$\frac{du}{dr} = -\frac{1}{\ln\frac{r_o}{r_i} + \frac{k}{hr_o}}\frac{1}{r} \tag{21}$$

Now let's compute q_{cond} and q_{conv} for the reference case and the three perturbation cases, and look at the relative values of q_n/q_{ref} -- which represents q for case n divided by the heat transfer for the reference case. For example, looking explicitly at the convection heat transfer for case n, we have

$$\left.\frac{q_n}{q_{ref}}\right|_{conv} = \frac{h_n 2\pi r_{o_n} L\left(T_i - T_\infty\right) u_n(r_{o_n})}{h_{ref} 2\pi r_{o_{ref}} L\left(T_i - T_\infty\right) u_{ref}(r_{o_{ref}})}$$

or

$$\left.\frac{q_n}{q_{ref}}\right|_{conv} = \frac{h_n r_{o_n} u_n(r_{o_n})}{h_{ref} r_{o_{ref}} u_{ref}(r_{o_{ref}})} \tag{22}$$

where we see that the length, L, of the annular section and the temperature difference, $T_i - T_\infty$, conveniently cancel from the final expression -- which allows us to concentrate on the real design variables of interest (k, h, and r_o).

Similarly, for conduction heat transfer, the ratio of the heat transfer for case n to the reference case becomes

$$\left.\frac{q_n}{q_{ref}}\right|_{cond} = \frac{k_n \Big/ \left(\ln\frac{r_{o_n}}{r_i} + \frac{k_n}{h_n r_{o_n}}\right)}{k_{ref} \Big/ \left(\ln\frac{r_{o_{ref}}}{r_i} + \frac{k_{ref}}{h_{ref} r_{o_{ref}}}\right)} \tag{23}$$

where we note that this latter expression is independent of the radial position, r.

Okay, at this point we have all the development needed to do the desired comparisons. As a way to summarize the key elements of the required evaluations, let's outline a brief algorithm for implementing the above analysis into Matlab, as follows:

1. Identify the basic problem parameters for the reference case.
2. Discretize the spatial domain and evaluate the normalized temperature profile given in eqn. (14) for the reference case.
3. Set the perturbed variable for each case and repeat Step 2 for the three perturbation cases.
4. Plot the results from Steps 2 and 3. Note that Case 3 has a different spatial domain than the other cases. Thus, for ease of comparison, we will plot Cases 0 – 2 in one subplot and Cases 0 and 3 in another (see below).
5. Evaluate the ratios for convection and conduction heat transfer defined in eqns. (22) and (23) and tabulate the results in a short summary table.

6. Interpret the results…

The above algorithm was implemented into a Matlab program called **pipe_insulation_1.m**. The key results of the analysis are summarized in Fig. 1 and Table 1 (this is a direct listing of the code output). A complete listing of the file that produced these results is given in Table 2. As expected, the normalized temperature profiles decrease from unity at the inside radius of the insulation to some lower value at the outside of the insulation where, for the reference case, the temperature drop is almost 30% of the actual maximum temperature difference, $\Delta T = T_i - T_\infty$. For the thin insulation thickness, the profiles are nearly linear with nearly constant slopes; but the slopes for Cases 0 – 2 are quite different. And, for Case 3 with its nearly 2″ of insulation, the temperature gradient continually decreases as the normalized temperature approaches its minimum value at the outer surface of the pipe.

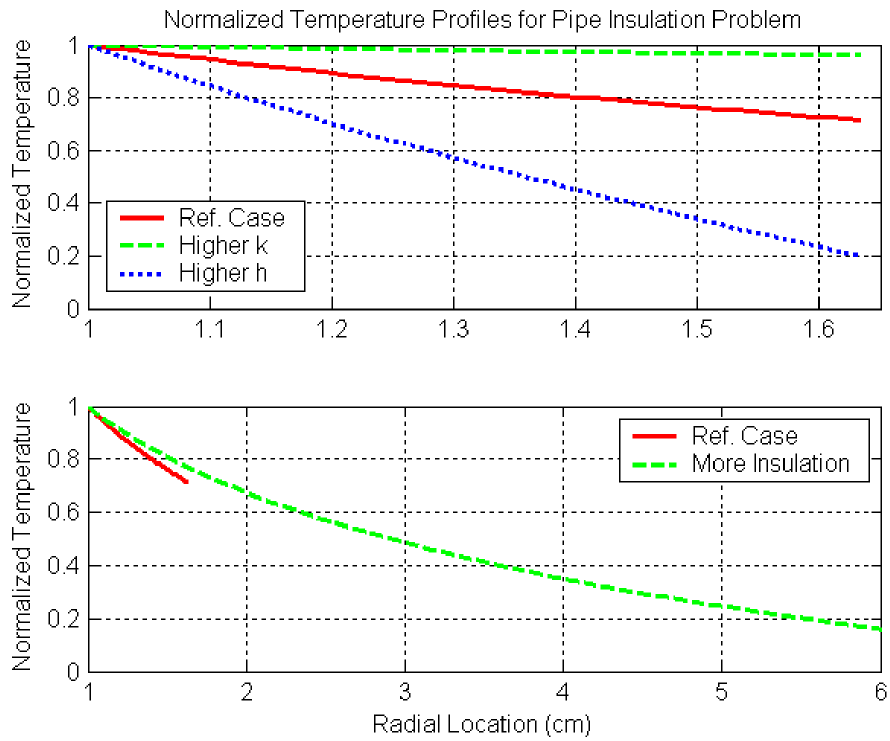


Fig. 1 Normalized temperature profiles in the pipe insulation for several cases.

Table 1 Listing of summary printed output from pipe_insulation_1.m

```
>> pipe_insulation_1

       Summary Results for the Pipe Insulation Problem

  Case #1 (higher k)       : qcond1/qcondref =  1.348  and  qconv1/qconvref =  1.348
  Case #2 (higher h)       : qcond2/qcondref =  2.793  and  qconv2/qconvref =  2.793
  Case #3 (more insulation): qcond3/qcondref =  0.804  and  qconv3/qconvref =  0.804
```

Table 2 Listing of the pipe_insulation_1.m program.

```
%
%   PIPE_INSULATION_1.M    Function evaluation and plotting in Matlab
%             A Simple Heat Transfer Analysis: Circular Pipe with Insulation
%
%   This file evaluates and plots the temperature profiles in a simple annular
%   region in the range ri < r < ro, where ri and ro are the inside and outside
%   radii of a solid annulus.  The inside boundary has a fixed temperature, Ti.
%   The outside surface is exposed to a convective environment, with heat
%   transfer coefficient h and bulk fluid temperature Tinf.
%
%   A normalized temperature is defined (see notes) so that the resulting
%   expression highlights only three design parameters  --  k, h, and ro  --  and
%   a series of cases is analyzed to determine the effect of each of these
%   parameters on the temperature profile and on the overall heat transfer from
%   this particular system (relative to the reference case).
%
%   The goals of this exercise are:
%     1. to become comfortable with function evaluation and plotting in Matlab,
%     2. to evaluate and edit some important quantitative characteristics of
%        a particular system for comparison purposes (ie. sensitivity studies), and
%     3. to hopefully learn a little bit about conduction and convection heat
%        transfer in a simple 1-D system.
%
%   The basic idea for this problem was obtained from the discussion in Chapter 2
%   of the Heat Transfer text by J. P. Holman (5th Ed.)  (1981, McGraw-Hill).
%
%   File prepared by J. R. White, UMass-Lowell  (last update: August 2017)
%

      clear all,  close all,  nfig = 0;
%%
%   identify basic problem data (reference data)
      kref = 0.1;                % thermal conductivity of insulation (W/m-C)
      href = 5;                  % heat transfer coeff for reference cooling (W/m^2-C)
      ri = 0.01;                 % inside radius of insulation (m)
      roref = 0.01635;           % outside radius of insulation (m) (about 1/4 inch)
%%
%   compute reference normalized temperature profile (use vector arithmetic)
      Nr = 50;  r = linspace(ri,roref,Nr);
      uref = 1 - log(r/ri)./(log(roref/ri) + kref/(href*roref));
%
%   now compute the normalized profile for three separate perturbation cases
%   case 1: high thermal conductivity
      k1 = 1.0;     % thermal conductivity of (relatively poor) insulation (W/m-C)
      u1 = 1 - log(r/ri)./(log(roref/ri) + k1/(href*roref));
%   case 2: higher heat transfer coeff
      h2 = 50;      % heat transfer coeff for high cooling rate (W/m^2-C)
      u2 = 1 - log(r/ri)./(log(roref/ri) + kref/(h2*roref));
%   case 3: more insulation
      ro3 = 0.0608; % outside radius of insulation (m) (about 2 inches)
      r3 = linspace(ri,ro3,Nr);
      u3 = 1 - log(r3/ri)./(log(ro3/ri) + kref/(href*ro3));
%%
%   now let's plot these comparisons
      nfig = nfig+1;   figure(nfig)
      subplot(2,1,1),plot(r*100,uref,'r-',r*100,u1,'g--',r*100,u2,'b:','LineWidth',2),grid
      title('Normalized Temperature Profiles for Pipe Insulation Problem')
      ylabel('Normalized Temperature')
      v1 = axis;  v1(2) = 1.65;  axis(v1)
      legend('Ref. Case','Higher k','Higher h','Location','SouthWest')
%
      subplot(2,1,2),plot(r*100,uref,'r-',r3*100,u3,'g--','LineWidth',2),grid
      xlabel('Radial Location (cm)'),ylabel('Normalized Temperature')
      legend('Ref. Case','More Insulation')
      v2 = axis;  v2(2) = 6;  axis(v2)
%%
%   just for fun, let's compute the heat transfer for each of these cases
```

```
%   Note: we will compute q two different ways for each case just to show
%         that they are equivalent...
%               qcond = conduction    and   qconv = convection
%   Since there is no internal heat source qcond is not a function of r (see notes).
%   Also, the right boundary condition forces qcond = qconv.
%   Let's see if this all works as expected...
%
%   ref case
      qcondref = kref/(log(roref/ri) + kref/(href*roref));
      qconvref = href*roref*uref(Nr);
%   case 1
      qcond1 = k1/(log(roref/ri) + k1/(href*roref));
      qconv1 = href*roref*u1(Nr);
%   case 2
      qcond2 = kref/(log(roref/ri) + kref/(h2*roref));
      qconv2 = h2*roref*u2(Nr);
%   case 3
      qcond3 = kref/(log(ro3/ri) + kref/(href*ro3));
      qconv3 = href*ro3*u3(Nr);
%   compute and edit desired ratios
      ratio1 = [qcond1/qcondref qconv1/qconvref];
      ratio2 = [qcond2/qcondref qconv2/qconvref];
      ratio3 = [qcond3/qcondref qconv3/qconvref];
      fprintf(1,'\n       Summary Results for the Pipe Insulation Problem \n\n');
      fprintf(1,'  Case #1 (higher k)       : qcond1/qcondref = %6.3f  and  qconv1/qconvref = 
%6.3f \n',ratio1);
      fprintf(1,'  Case #2 (higher h)       : qcond2/qcondref = %6.3f  and  qconv2/qconvref = 
%6.3f \n',ratio2);
      fprintf(1,'  Case #3 (more insulation): qcond3/qcondref = %6.3f  and  qconv3/qconvref = 
%6.3f \n',ratio3);
%
%  end of problem
```

From Table 1, we see that increasing the thermal conductivity by a factor of 10 (Case 1 vs. the Reference Case) only increased the total heat transfer by about 35%. This can be explained by the fact that the higher k leads to a lower slope. Since the heat transfer is proportional to the product of k and dT/dr, we see that most of the increased k is offset by the decreased gradient, with the product only increasing by about 35%.

A similar trend is also observed for Case 2 relative to the Reference Case. Here, the heat transfer coefficient is increased by a factor of 10, yet the total heat transfer is only increased by a factor of 2.79. Because forced convection enhances heat transfer, the temperature gradient in the insulation is increased and the outer surface temperature is decreased relative to the reference case. At the surface, it is the product of h and $(T(r_o) - T_\infty)$ that gives the heat flux and again, we see that an increase in one term is somewhat offset by a decrease in the other -- although an increase in heat transfer by a factor of 2.79 is a significant amount.

Finally, as expected, we see that an increase in the insulation thickness leads to a decrease in the heat transfer through the insulation, with a reduction of about 20% relative to the reference case. Figure 1 shows that the temperature gradient at r_i is reduced, thus leading to an overall decrease in the conduction heat transfer entering the region. An alternate view is that the increased insulation leads to a lower outer surface temperature, which results in less convection heat transfer at the outer surface. Of course, these two views are equivalent (since there is no energy production in the insulation, all the energy leaving by convection at the outer surface must enter at the inner surface by conduction).

The Matlab coding for this heat transfer analysis is quite straightforward where, once again, we are focusing on simple function evaluation and plotting within the Matlab environment. The only new feature that was not addressed in any real detail in the previous examples was the use of the ***fprintf*** command to print the heat transfer ratios to the screen (see the last few lines of code in Table 2). This command is used to send formatted print to the screen or to an external file. It is discussed in some detail in later chapters of your texts, as well as in the Matlab ***help*** facility (please see these references if more detail is desired at this time). For now, we will simply demonstrate its use via example, and we will discuss it and use it to a greater extent in future examples and assignments.

This completes our first example involving some simple heat transfer processes -- in this case, 1-D radial conduction in a long annular configuration with convection at the system boundary. Matlab was used to help visualize and quantify the equations that were developed, and to hopefully give some insight into one important application area -- heat transfer in an insulated pipe configuration. We will also come back to the general subject of heat transfer several times throughout the semester, since it offers a rich variety of examples that illustrate many of the techniques to be discussed over the course of the semester.