Measurement Error in a Temperature Probe

Consider the temperature probe sketched in Fig. 1 (from Ref. 1). The probe has an overall length of L = 20 cm and diameter D = 1.25 cm. It is inserted through a duct wall such that a portion of its length, referred to as the immersion length, L_i, is in contact with the water stream whose temperature, $T_{\infty i}$, is to be determined. The convection coefficients over the immersion length and the region outside the wall are $h_i = 1100\ W/m^2$-C and $h_o = 10\ W/m^2$-C, respectively. The probe is an aluminum alloy that has a thermal conductivity of 177 W/m-C and it is in poor thermal contact with the duct wall. The probe is approximately a solid circular rod -- which, as part of this application, can be thought of as a pin-shaped fin.

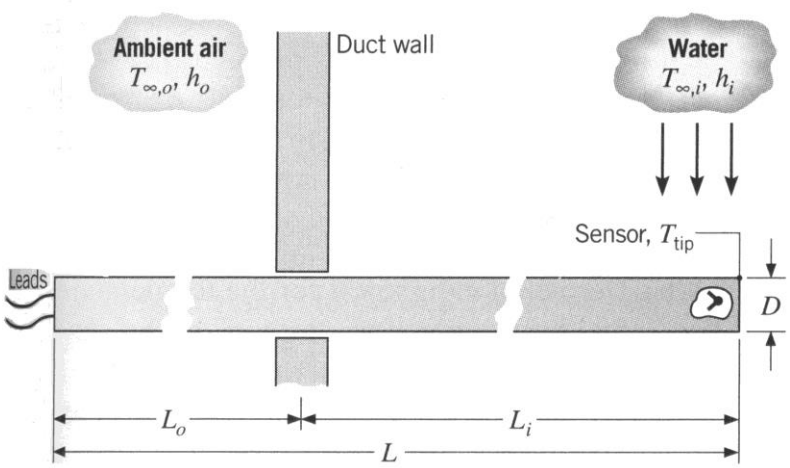

Fig. 1 Sketch of a pin-shaped fin as a temperature probe.

We can develop an expression for evaluating the measurement error, $\Delta T_{err} = T_{tip} - T_{\infty i}$, which is the difference between the tip temperature, T_{tip}, and the water temperature, $T_{\infty i}$, as follows:

First, let's define a coordinate system with the origin at the duct wall and treat the probe as two pin fins extending inward and outward from the duct. Note here that both fins have the same base temperature, T_b, and that positive x is to the right for the inside fin and that positive x is to the left for the outside fin. Now, from our study of heat transfer from extended surfaces (see **rect1d_fin_1.pdf**), we already know how to evaluate the temperature profile and overall heat transfer from a fin with a constant cross-sectional area, a fixed base temperature, and a convection environment at the tip.

Writing the appropriate expressions for the two fins inside and outside the duct wall gives:

Inside Duct:

$$\frac{T_i(x) - T_{\infty i}}{T_b - T_{\infty i}} = \left. \frac{\cosh m(L-x) + \frac{h}{mk}\sinh m(L-x)}{\cosh mL + \frac{h}{mk}\sinh mL} \right|_{inside} \qquad (1)$$

$$q_{fi} = M \left. \frac{\sinh mL + \frac{h}{mk}\cosh mL}{\cosh mL + \frac{h}{mk}\sinh mL} \right|_{inside} \tag{2}$$

with $m_i = \left. \sqrt{\frac{hP}{kA_c}} \right|_{inside}$ and $M_i = (T_b - T_{\infty i})\left. \sqrt{kA_c hP} \right|_{inside}$

Outside Duct:

$$q_{fo} = M \left. \frac{\sinh mL + \frac{h}{mk}\cosh mL}{\cosh mL + \frac{h}{mk}\sinh mL} \right|_{outside} \tag{3}$$

with $m_o = \left. \sqrt{\frac{hP}{kA_c}} \right|_{outside}$ and $M_o = (T_b - T_{\infty o})\left. \sqrt{kA_c hP} \right|_{outside}$

Now, clearly, the heat transfer from the two fins at x = 0 must be identical, or

$$q_{fo} = -q_{fi} \tag{4}$$

where the negative sign is needed since the heat flow in the formal development in **rect1d_fin_1.pdf** was assumed to go from the base of the fin towards the tip and, for the two fins of interest here, these coordinate systems are in opposite directions.

Upon substitution into eqn. (4), we have

$$M_o \frac{\sinh m_o L_o + \frac{h_o}{m_o k}\cosh m_o L_o}{\cosh m_o L_o + \frac{h_o}{m_o k}\sinh m_o L_o} = -M_i \frac{\sinh m_i L_i + \frac{h_i}{m_i k}\cosh m_i L_i}{\cosh m_i L_i + \frac{h_i}{m_i k}\sinh m_i L_i}$$

or, defining the constants C_o and C_i by comparison to the above equation, we have

$$M_o C_o = -M_i C_i \tag{5}$$

Now, expanding eqn. (5) gives

$$(T_b - T_{\infty o})\sqrt{kA_c h_o P}\; C_o = -(T_b - T_{\infty i})\sqrt{kA_c h_i P}\; C_i$$

and solving this expression for the fin's base temperature, T_b, gives

$$T_b \sqrt{kA_c h_o P}\; C_o + T_b \sqrt{kA_c h_i P}\; C_i = T_{\infty o}\sqrt{kA_c h_o P}\; C_o + T_{\infty i}\sqrt{kA_c h_i P}\; C_i$$

or $$T_b = \frac{T_{\infty o}\sqrt{h_o}\; C_o + T_{\infty i}\sqrt{h_i}\; C_i}{\sqrt{h_o}\; C_o + \sqrt{h_i}\; C_i} \tag{6}$$

Now, evaluating eqn. (1) for the fin temperature distribution on the inside at $x = L_i$ gives

$$\Delta T_{err} = T_{tip} - T_{\infty i} = (T_b - T_{\infty i}) \frac{1}{\cosh m_i L_i + \frac{h_i}{m_i k} \sinh m_i L_i} \tag{7}$$

which is the desired expression that represents an estimate of the measurement error. Since, in a temperature measurement system, we would like the tip temperature to be exactly at the fluid temperature, $T_{\infty i}$, to be measured, the difference between the inside tip temperature and the fluid temperature is indeed the measurement error.

Now, the goal here is to study the sensitivity of the temperature error, ΔT_{err}, relative to several parameters, as follows:

immersion length:	$0.1 \le L_i/L \le 0.9$	
thermal conductivity of rod:	reference case --	k = 177 W/m-C (Al alloy)
	low k case --	k = 20 W/m-C (stainless steel)
velocity of water:	reference case --	h_i = 1100 W/m^2-C
	low flow --	h_i = 550 W/m^2-C

To facilitate the analysis process, we will evaluate ΔT_{err} vs. L_i/L within a function file. Within the function subprogram, given the seven design/operational parameters: $T_{\infty o}$, h_o, $T_{\infty i}$, h_i, L, D, and k, we simply evaluate the above equations, as follows:

1. Calculate the fin cross sectional area, A_c, and perimeter, P.
2. Compute m_o and m_i.
3. Set the range for the immersion length, $0.1L \le L_i \le 0.9L$, and compute $L_o = L - L_i$ and L_i/L.
4. Evaluate the equation constants, C_o and C_i, which are defined via eqn. (5), or

$$C_o = \frac{\sinh m_o L_o + \frac{h_o}{m_o k} \cosh m_o L_o}{\cosh m_o L_o + \frac{h_o}{m_o k} \sinh m_o L_o} \quad \text{and} \quad C_i = \frac{\sinh m_i L_i + \frac{h_i}{m_i k} \cosh m_i L_i}{\cosh m_i L_i + \frac{h_i}{m_i k} \sinh m_i L_i}$$

5. Compute the base temperature, T_b, using eqn. (6).
6. Compute ΔT_{err} from eqn. (7).
7. Pass back ΔT_{err} and L_i/L to the main program for further plotting and analysis.

To complete the desired sensitivity study, we can simply call this function three times as follows:

Reference Data:	$T_{\infty o}$ = 20 C	h_o = 10 W/m^2-C	
	$T_{\infty i}$ = 80 C	h_i = 1100 W/m^2-C	
	L = 0.20 m	D = 0.0125 m	k = 177 W/m-C

Case 1: use reference data

Case 2: low k case -- use reference data but with k = 20 W/m-C

Case 3: low flow case -- use reference data but with h_i = 550 W/m^2-C

The main program can then plot ΔT_{err} vs. L_i/L for all three cases on the same plot for easy visualization of the analysis results.

The solution algorithm described above was implemented into Matlab programs **pin_fin_1.m** and **pin_fin_1a.m**, where the first file is the main program and the **pin_fin_1a.m** file is the function subprogram, respectively. These two files are listed in Tables 1 and 2. Since the key aspect of this problem as a Matlab demo is the use of function files, you should make a special note of the syntax for the function file and the way the function is called from the main program (three times in this example). Note also that all communication to and from the function file is via the input and output argument lists (no ***global*** parameters are defined here). Recall that one of the key differences of a function file and a script file is that all the variables within the function file are local, and they are not available within the Matlab workspace. Thus, upon completion of the function file, **pin_fin_1a.m**, only the two output arguments, **LiL** and **DTerr**, are available for use in the main program. Notice also how the use of the function file simplifies the readability and overall structure of the program -- where the parameter definition, computation, and program output sections of the main program are clearly defined, with all the detailed computations being done in the function file. The function file is called three times from the main program, with a single change in the input argument list each time. In this way, we can repeat a whole series of computations with minimal programming effort.

Table 1 Listing of the main program pin_fin_1.m.

```
%
%   PIN_FIN_1.M   Function Evaluation, Plotting, and Use of Function Files
%                     Heat Transfer Analysis of a Temperature Probe
%
%   This example evaluates and plots the measurement error associated with a
%   pin-shaped temperature probe that is inserted through a wall into a fluid.
%   Given a set of design parameters (Tinfo, ho, Tinfi, hi, k, etc.), a function
%   subprogram, PIN_FIN_1A.M, is called to evaluate DTerr = Ttip - Tinfi as a
%   function of the fraction of the probe that is immersed into the fluid whose
%   temperature is to be measured.
%
%   This main routine simply calls PIN_FIN_1A a number of times to perform the
%   desired sensitivity analysis (vary hi and k).  It also plots the final
%   results of the analysis for easy visualization.
%
%   This example emphasizes function evaluation, plotting, and the use of
%   user-defined function files in Matlab.
%
%   The basic idea for this problem came from Prob. 3.110 on pg. 173 in the text
%   "Fundamentals of Heat and Mass Transfer" 5th Ed by Incropera and Dewitt
%   (2002, John Wiley & Sons).
%
%   File prepared by J. R. White, UMass-Lowell (last update: Sept. 2017)
%

     clear all,  close all,  nfig = 0;
%
%   identify basic problem data
     Tinfo = 20;                        % outside temperature (C)
     ho = 10;                           % outside heat transfer coeff (W/m^2-C)
     Tinfi = 80;                        % inside temperature (C)
     hi = [1100 550];                   % outside heat transfer coeffs (W/m^2-C)
     k = [177 20];                      % probe's thermal conductivity (W/m-C)
     L = 0.2;                           % overall probe length (m)
     D = 0.0125;                        % probe diameter (m)
%
```

```
%   compute DTerr vs Li/L for reference case
      [LiL1,DTerr1] = pin_fin_1a(Tinfo,ho,Tinfi,hi(1),k(1),L,D);
%
%   compute DTerr vs Li/L for low thermal conductivity case
      [LiL2,DTerr2] = pin_fin_1a(Tinfo,ho,Tinfi,hi(1),k(2),L,D);
%
%   compute DTerr vs Li/L for low heat transfer coefficent case
      [LiL3,DTerr3] = pin_fin_1a(Tinfo,ho,Tinfi,hi(2),k(1),L,D);
%
%   plot DTerr vs Li/L for all three cases
      nfig = nfig+1;   figure(nfig)
      plot(LiL1,DTerr1,'r-',LiL2,DTerr2,'g--',LiL3,DTerr3,'b-.','LineWidth',2),grid
      title(['PIN\_FIN\_1:  \DeltaT_{err} vs L_i/L for Several Cases (L = ', ...
            num2str(L*100),' cm)'])
      xlabel('Immersion Length Ratio (L_i/L)')
      ylabel('\DeltaT_{err}  (^oC)')
      legend('Ref. Case','Low k Case','Low h_i Case','Location','SouthEast')
%
%  end of problem
```

Table 2 Listing of the function subprogram pin_fin_1a.m.

```
%
%   PIN_FIN_1A.M     Function for Computing DTerr vs Li/L
%                        (called from PIN_FIN_1.M)
%
%   Given a set of design parameters (Tinfo, ho, Tinfi, hi, k, etc.), this routine
%   computes the measurement error versus immersion length of a pin-shaped probe
%   that is inserted through a wall into a fluid.  The purpose of the probe is to
%   measure the bulk fluid temperature.  However, because of the various heat
%   transfer mechanisms involved, the probe's tip temperature may be slightly
%   different from the free stream fluid temperature.  This routine estimates the
%   measurement error for a given set of design parameters.
%
%   Input Variables:
%      Tinfo, ho -- outside temperature and heat transfer coeff
%      Tinfi, hi -- inside temperature and heat transfer coeff
%      k         -- probe material thermal conductivity
%      L, D      -- probe's total length and diameter
%
%   Output Variables:
%      LiL       -- values of Li/L
%      DTerr     -- difference between the inside tip temperature and the fluid
%                   temperature for each value of LiL
%
%   File prepared by J. R. White, UMass-Lowell (last update: Sept. 2017)
%

      function [LiL,DTerr] = pin_fin_1a(Tinfo,ho,Tinfi,hi,k,L,D)
      Ac = pi*D^2/4;   P = pi*D;
      mo = sqrt(ho*P/(k*Ac));   mi = sqrt(hi*P/(k*Ac));
      Li = 0.1*L:0.025*L:0.9*L;
      Lo = L-Li;   LiL = Li/L;
      co1 = sinh(mo*Lo) + (ho/(mo*k))*cosh(mo*Lo);
      co2 = cosh(mo*Lo) + (ho/(mo*k))*sinh(mo*Lo);
      co = co1./co2;
      ci1 = sinh(mi*Li) + (hi/(mi*k))*cosh(mi*Li);
      ci2 = cosh(mi*Li) + (hi/(mi*k))*sinh(mi*Li);
      ci = ci1./ci2;
      Tb1 = Tinfo*sqrt(ho)*co + Tinfi*sqrt(hi)*ci;
      Tb2 = sqrt(ho)*co + sqrt(hi)*ci;
      Tb = Tb1./Tb2;
      DTerr = (Tb - Tinfi)./ci2;
%
%   end of function
```

The results from the **pin_fin_1** program are given in Fig. 2. The three curves on this plot correspond to the three cases described above. Focusing on the reference case first, we see that the measurement error decreases to essentially zero as the immersion length increases. This is consistent with our understanding of heat transfer in a fin arrangement since, as the length of the inside fin increases, we would expect the fin's tip temperature to approach the temperature of the environment. Thus, in this case, fin efficiency is not the issue, and a larger immersion length is desirable (within the geometry constraints of the system under measurement).

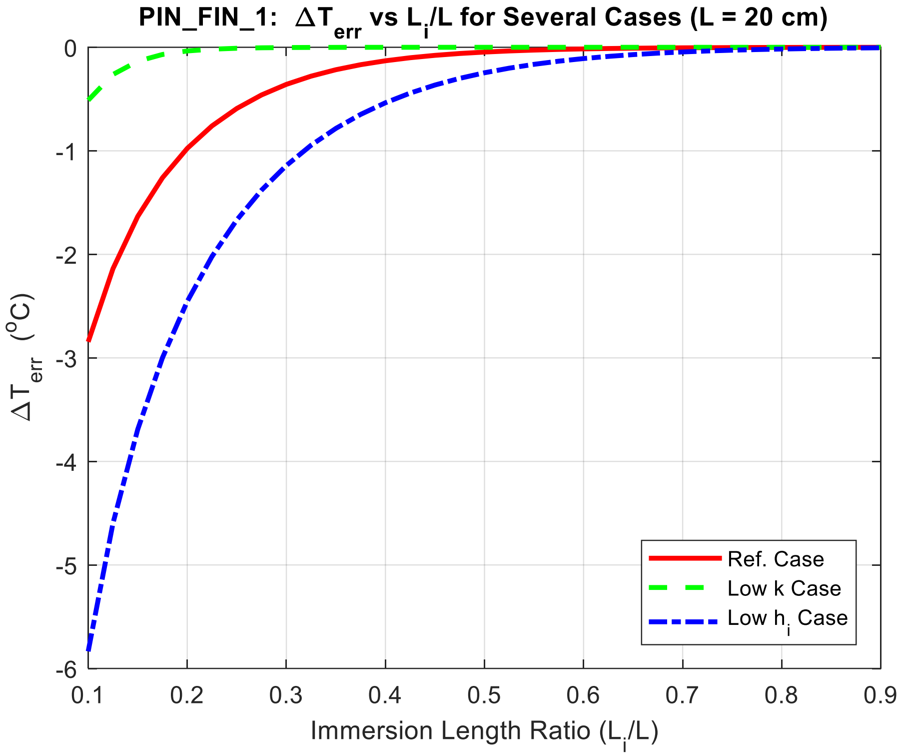

Fig. 2 Summary results from the temperature probe sensitivity study.

Fig. 2 also shows that a decrease in the fin's thermal conductivity leads to a lower overall measurement error. Since, in this case, we do not want to enhance conduction along the length of the fin, a lower k improves the accuracy of the temperature measurement. Also, as expected, decreasing the fluid velocity decreases the heat transfer coefficient, which leads to a greater temperature difference between the fluid and the fin -- giving a larger measurement error. In summary, Fig. 2 shows that a lower k and a larger L_i and h should lead to better performance for the pin-shaped temperature probe. These considerations could assist in the design of a simple temperature probe and/or in the correction of actual data measured from the device.

This concludes our example for using a pin-shaped fin as a simple temperature probe. There may be better ways to measure the temperature of a moving fluid, but this application meshes nicely with our previous heat transfer example (see **rect1d_fin_1.pdf**) and it re-enforces some of the fundamental heat transfer concepts introduced earlier. In addition, it is a problem where the use of a function file in Matlab is ideal for simplifying the overall analysis. Function programs are, in fact, an essential component of every programming language, and we will see many occasions where they are needed for the design of a well-structured program. We will also use function files quite extensively in the second part of this course, when focusing on using numerical methods as part of our problem-solving toolbox. In fact, much of Matlab's built-in computational capability requires user-defined functions to identify the function to integrate, the ODE to solve, the function to minimize, etc. etc.. Thus, we will routinely use this aspect of programming within Matlab -- so it is advisable that you become comfortable as soon as possible with the use of this particular programming device…

Reference: The idea for this problem came from Problem 3.111 on pg. 173 of the text "Fundamentals of Heat and Mass Transfer" 5th Ed. by Incropera and Dewitt, John Wiley & Sons, 2002. The sketch of the basic concept for the temperature probe shown in Fig. 1 also came from this reference.