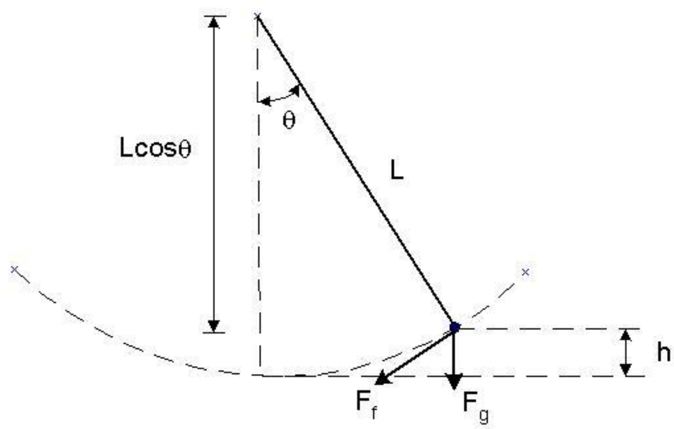
Pendulum Dynamics

Consider a simple pendulum with a massless arm of length L and a point mass, m, at the end of the arm. Assuming that the friction in the system is proportional to the negative of the tangential velocity, Newton's second law gives:



$$ma_r = \sum_i F_{r_i} \quad \text{radial direction} \qquad (1)$$

$$ma_t = \sum_i F_{t_i} \quad \text{tangential direction} \qquad (2)$$

For a pendulum of fixed length, the radial acceleration is zero; thus there is no motion in the radial direction -- since the sum of the forces in the radial direction always balance (i.e. the tension in the pendulum arm must balance the radially outward centrifugal force and gravity force components). Thus, of interest here is the tangential force balance given in eqn. (2).

Focusing on the tangential direction, we see that the left hand side (LHS) of eqn. (2) can be written as

$$ma_t = m\frac{dv_t}{dt} = mL\frac{d\omega}{dt} = mL\frac{d^2\theta}{dt^2} = mL\theta'' \qquad (3)$$

where v_t = tangential velocity = $L\omega$

ω = angular velocity = $d\theta/dt$

θ = angular position

On the RHS we have the friction force and a force due to gravity. Since the friction force is proportional to the tangential velocity, we have

$$F_{f_t} = -cv_t = -cL\frac{d\theta}{dt} = -cL\theta' \qquad (4)$$

where c is the drag or friction coefficient (proportionality constant). For the gravity force, we need the component in the direction tangent to the curve traced by the point mass at the point (L, θ), or

$$F_{g_t} = -mg\sin\theta \qquad (5)$$

Therefore, the equation of motion for a simple pendulum with a fixed arm of length L and point mass, m, is given by

$$mL\theta'' + cL\theta' + mg\sin\theta = 0 \qquad (6)$$

This is a 2nd order nonlinear ODE. With given values for the constants and with a set of specified initial conditions, $\theta(0)$ and $\theta'(0)$, this nonlinear equation can be solved numerically using a standard ODE solver (such as ***ode45*** in Matlab).

If, however, the pendulum motion only produces relatively small angles then, for $\theta \approx 0$, $\sin\theta \approx \theta$, and we have a linearized version of the equation of motion for the pendulum,

$$mL\theta'' + cL\theta' + mg\theta = 0 \tag{7}$$

This is now a linear constant coefficient system that is relatively easy to solve analytically.

To demonstrate, let's assume the following numerical values for the linearized model:

$m = 1$ kg, $c = 2$ kg/s, $L = 1$ m, and $g = 10$ m/s^2

along with initial conditions (ICs)

$\theta(0) = \pi/6$ radians and $\theta'(0) = 0$ rad/s

which says that the pendulum is initially at rest at an angle of 30 degrees. With these values, the initial value problem (IVP) becomes

$$\theta'' + 2\theta' + 10\theta = 0 \quad \text{with} \quad \theta(0) = \pi/6 \quad \text{and} \quad \theta'(0) = 0 \tag{8}$$

The unique solution to this IVP is given as follows:

1. Assume a solution of the form $\theta(t) = e^{rt}$ to develop the characteristic equation and its roots,

$$r^2 + 2r + 10 = 0$$

and

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 4(10)}}{2} = -1 \pm 3j$$

2. The general solution is then given by

$$\theta(t) = e^{-t}\left(c_1 \cos 3t + c_2 \sin 3t\right) \tag{9}$$

and

$$\theta'(t) = e^{-t}\left(-3c_1 \sin 3t + 3c_2 \cos 3t\right) - e^{-t}\left(c_1 \cos 3t + c_2 \sin 3t\right)$$

or

$$\theta'(t) = e^{-t}\left[\left(3c_2 - c_1\right)\cos 3t - \left(c_2 + 3c_1\right)\sin 3t\right] \tag{10}$$

3. Now applying the ICs gives the unique solution, as follows:

IC#1: $\theta(0) = \frac{\pi}{6} = (1)\left(c_1 + 0\right)$ or $c_1 = \frac{\pi}{6}$

IC#2: $\theta'(0) = 0 = (1)\left[\left(3c_2 - c_1\right) - 0\right]$ or $c_2 = \frac{1}{3}c_1 = \frac{1}{3}\left(\frac{\pi}{6}\right)$

which gives the angular position versus time as

$$\theta(t) = \frac{\pi}{6} e^{-t} \left(\cos 3t + \frac{1}{3} \sin 3t \right) \tag{11}$$

and the angular velocity as

$$\omega(t) = \theta'(t) = -\left(\frac{10}{3} \right) \frac{\pi}{6} e^{-t} \sin 3t = -\frac{5}{9} \pi e^{-t} \sin 3t \tag{12}$$

These functions, $\theta(t)$ and $\omega(t) = \theta'(t)$, give the desired angular position and angular velocity versus time. These represent the state of the system at any time.

Once the state is known, we can compute other quantities for the system as desired. For example, the total energy in this mechanical system is given by the sum of the potential and kinetic energies, or

$$E_{tot} = E_p + E_k = mgh + \frac{1}{2} mv^2 \tag{13}$$

or

$$E_{tot} = mgL\left(1 - \cos\theta\right) + \frac{1}{2} mL^2 \left(\theta'\right)^2 \tag{14}$$

Thus, with $\theta(t)$ and $\theta'(t)$ known, we can easily find $E_p(t)$, $E_k(t)$, and $E_{tot}(t)$. This essentially completes our analytical development, and we are now ready to evaluate and plot some of these expressions to help us visualize the overall behavior of this particular system.

Extra! Extra!

Although it is a little more tedious, it is also usually much more insightful if the above analytical manipulations can be done with symbolic variables instead of numerical values. To give us full flexibility with this problem, we repeat the above development for the more general case, as follows:

We start by restating the IVP as

$$\theta'' + \frac{c}{m}\theta' + \frac{g}{L}\theta = 0 \qquad \text{with} \quad \theta(0) = \theta_o \quad \text{and} \quad \omega(0) = \theta'(0) = \omega_o \tag{15}$$

The characteristic equation is

$$r^2 + \frac{c}{m} r + \frac{g}{L} = 0 \tag{16}$$

with the roots given by

$$r_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m} \right)^2 - \frac{g}{L}} \tag{17}$$

Now, one assumption that we will make about the pendulum is that it will oscillate about its equilibrium point ($\theta_{eq} = 0$ and $\omega_{eq} = 0$) after some initial perturbation at time zero. This means that the friction, although nonzero, is not so great as to cause over-damping. In mathematical terms, this means that we assume that

$$\frac{g}{L} > \left(\frac{c}{2m}\right)^2$$

For this case, we will have complex conjugate roots,

$$r_{1,2} = \alpha \pm \beta \tag{18}$$

where

$$\alpha = -\frac{c}{2m} \qquad \text{(real part)} \tag{19}$$

$$\beta = \sqrt{\frac{g}{L} - \left(\frac{c}{2m}\right)^2} \qquad \text{(imaginary part)} \tag{20}$$

with the corresponding general solution for the pendulum position given as

$$\theta(t) = e^{\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right) \tag{21}$$

and the pendulum angular velocity given by

$$\omega(t) = \theta'(t) = e^{\alpha t}\left(-c_1\beta \sin\beta t + c_2\beta\cos\beta t\right) + \alpha e^{\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right)$$

or $$\omega(t) = e^{\alpha t}\left[\left(\beta c_2 + \alpha c_1\right)\cos 3t + \left(\alpha c_2 - \beta c_1\right)\sin\beta t\right] \tag{22}$$

Now, with the formal expressions for θ(t) and ω(t) in eqns. (21) and (22), we can apply the initial conditions, as follows:

IC#1: $$\theta(0) = \theta_o = c_1 \qquad \text{or} \qquad c_1 = \theta_o \tag{23}$$

IC#2: $$\omega(0) = \omega_o = \beta c_2 + \alpha c_1 \qquad \text{or} \qquad c_2 = \frac{\omega_o - \alpha c_1}{\beta} \tag{24}$$

Finally, putting these into the general solution gives the unique solution for the linearized pendulum model.

As a summary, we simply rewrite the above equations as part of an algorithm for implementing these equations into Matlab (or any other computational tool). The solution algorithm requires the following steps:

1. Define problem parameters: m, c, g, L, θ_o, and ω_o.
2. Compute real and complex parts of the roots of the characteristic equation:

$$\alpha = -\frac{c}{2m} \qquad \text{and} \qquad \beta = \sqrt{\frac{g}{L} - \left(\frac{c}{2m}\right)^2}$$

3. Compute the equation coefficients:

$$c_1 = \theta_o \qquad \text{and} \qquad c_2 = \frac{\omega_o - \alpha c_1}{\beta}$$

$$d_1 = \beta c_2 + \alpha c_1 \qquad \text{and} \qquad d_2 = \alpha c_2 - \beta c_1$$

4. Discretize the time domain variable over an appropriate domain [t = linspace(0,5,101), for example, will define a discrete time vector with 101 points evenly spaced with 0 and 5 seconds as the end points].
5. Evaluate the angular position and velocity at each time point (use vector arithmetic):

$$\theta(t) = e^{\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right)$$

$$\omega(t) = e^{\alpha t}\left(d_1 \cos 3t + d_2 \sin\beta t\right)$$

6. Evaluate the individual components of the total energy in the system [as implied in eqns. (13) and (14)].
7. Plot and interpret the solutions as needed.

This algorithm is really quite straightforward to implement in Matlab. Note that there is no reason to substitute long algebraic expressions into every equation. Just as we do in mathematics, we define intermediate variables as needed to simplify the math. In programming, we simply do the same thing when actually coding the equations. Of course, we must be sure that all the intermediate terms are computed before they are used in subsequent evaluations. Following the above algorithm, step by step, should guarantee success in our goal for evaluating and visualizing the dynamics of this simple linearized pendulum model.

Please see **pendulum_1.m** for the actual implementation of the above algorithm…