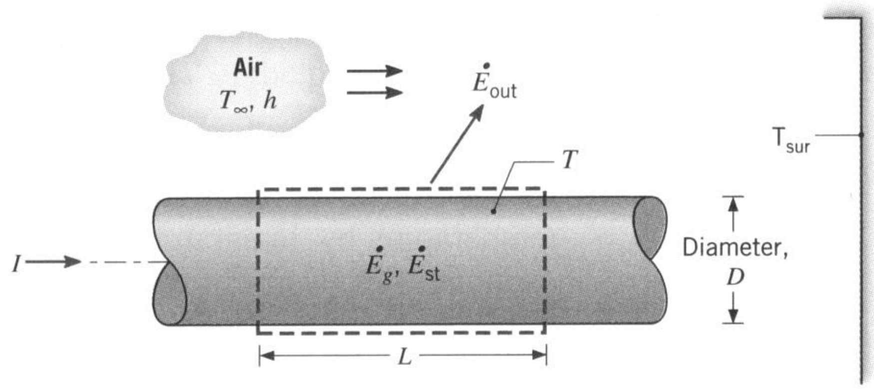
Energy Balance on a Conducting Rod

Consider a long conducting rod of diameter D and electrical resistance per unit length, R_L. The rod is initially in thermal equilibrium with the ambient air and its surroundings. This situation changes, however, when an electrical current, I, is passed through the rod.



Considering the above diagram, we can perform an energy balance on the indicated control volume (CV) as described in the following discussion. For a long small-diameter rod, we can assume that the temperature throughout the rod is nearly uniform, with the temperature at any time, t, denoted as T(t). Thus, assuming a uniform temperature at time t, an energy balance gives

$$\begin{bmatrix} \text{rate of energy} \\ \text{storage in CV} \end{bmatrix} = \begin{bmatrix} \text{energy generation rate} \\ \text{within CV by ohmic heating} \end{bmatrix} - \begin{bmatrix} \text{net energy flow rate} \\ \text{out of CV by} \\ \text{convection and radiation} \end{bmatrix}$$

or, in symbols,

$$\dot{E}_{st} = \dot{E}_g - \dot{E}_{out} \tag{1}$$

The energy within the control volume (CV) at any time can be written as

$$E_{CV} = \rho V u = \text{(mass/vol)(vol)(internal energy/mass) = energy in CV} \tag{2}$$

and, by definition of the internal energy, we have

$$du = c\,dT \tag{3}$$

where c is the specific heat, which quantifies the amount of increase in internal energy per unit mass that is observed for a one degree increase in temperature (with units of J/kg-C). Thus, assuming constant properties, we can write the rate of energy storage in the CV as

$$\dot{E}_{st} = \frac{d}{dt} E_{CV} = \rho c V \frac{dT}{dt} \tag{4}$$

For the energy generation term, we know from basic electricity that ohmic heating is given by

$$\dot{E}_g = P = I^2 R_L L \tag{5}$$

where, in this case, R_L is the resistance per unit length and the units in eqn. (5) are watts = $(\text{amps})^2(\text{ohms})$.

Now, assuming both convective and radiative losses to the surroundings, the energy loss rate can be written as

$$\dot{E}_{out} = hA\left(T - T_\infty\right) + \varepsilon\sigma A(T^4 - T_{sur}^4) \qquad (6)$$

where A is the heat transfer surface area, T_∞ is the bulk fluid temperature, and T_{sur} represents the temperature of the surroundings for radiation heat transfer (note that absolute temperatures need to be used for radiation heat transfer problems). The remaining variables in the radiative term include the material's emissivity, ε, and the Stefan-Boltzmann constant, $\sigma = 5.67\text{x}10^{-8}$ W/m^2-K^4.

Finally, putting all these terms into the balance equation gives

$$\rho c V \frac{dT}{dt} = I^2 R_L L - hA\left(T - T_\infty\right) - \varepsilon\sigma A(T^4 - T_{sur}^4) \qquad \text{with} \quad T(0) = T_o \qquad (7)$$

where, for this problem, $A = \pi DL$ and $V = \pi D^2 L/4$.

Thus, given the constant material properties (ρ, c, ε, and R_L), the geometry (D), and the environmental conditions (h, T_∞, and T_{sur}), we should be able to determine T(t) given the current, I, in the system. This requires solving the IVP defined in eqn. (7) subject to the initial condition, $T(0) = T_o$. Note that, if the initial current is zero, then the initial steady state temperature, T_o, is given by the solution to the nonlinear equation

$$hA\left(T_o - T_\infty\right) + \varepsilon\sigma A(T_o^4 - T_{sur}^4) = 0 \qquad (8)$$

If T_∞ and T_{sur} are identical, then $T_o = T_\infty = T_{sur}$. Otherwise, the above nonlinear equation will need to be solved for T_o.

Note also that, if we are interested in the new steady state temperature after current I is applied, the dT/dt term in eqn. (7) is simply set to zero, giving

$$I^2 R_L L - hA\left(T_{ss} - T_\infty\right) - \varepsilon\sigma A(T_{ss}^4 - T_{sur}^4) = 0 \qquad (9)$$

Solution of this nonlinear problem using standard root-finding software will give the desired steady state temperature, T_{ss}, after the initial transient has stabilized.

As a specific example, consider the following material and geometry parameters and the stated environmental conditions:

bare copper wire: D = 1 mm, ε = 0.8, R_L = 0.4 ohm/m

environmental conditions: h = 100 W/m^2-K, $T_\infty = T_{sur}$ = 25 C

with the goal of computing and plotting the steady state temperature as a function of the applied current over the range 0 < I < 10 A. In addition, we would also like to tabulate the various components of the energy balance equation for several values of I in the indicated range.

Well, as we have seen in the Lesson 5 Lecture Notes, this is a standard root finding problem that is easily solved using Matlab's built-in ***fzero*** routine. A main program, **conducting_rod_1.m**, and function file, **conducting_rod_1a.m** (for use by ***fzero***), were written to solve this problem, with a complete listing of both files given in Table 1. The key parts to the program are easily identifiable, with the initialization of the problem parameters, the single ***for … end*** loop containing the call to ***fzero*** for solving for T_{ss} for each value of current, a post processing section that computes the components of the energy balance equation once the correct temperatures are available and, finally, the output section that does the desired plotting and balance table generation.

Table 1 Listing of conducting_rod_1.m and conducting_rod _1a.m.

```
%
%   CONDUCTING_ROD_1.M      A Root Finding Example
%      Steady State Temperature vs Current in an Bare Conducting Wire
%
%   This file solves for the steady state temperature in a circular conducting rod
%   that is heated internally via ohmic heating and that loses energy by both
%   convective and radiation heat transfer.  The analysis is done for a unit length
%   of rod or wire.  A range of currents is used so that a plot of Tss vs I can be made.
%
%   The goal of this problem is to show how to solve nonlinear equations using the
%   built-in fzero routine in Matlab.  It also demonstrates how, with a simple for loop,
%   to do a parametric study for a single variable -- in this case, we use a range of
%   values of the current, I, to show how the steady state temperature varies with I.
%   For each value of I, a nonlinear energy balance equation must be solved (here we use
%   fzero to do this) to determine the correct steady state temperature.
%
%   A table of results is also produced that shows the various components of the energy
%   balance eqn for each value of I.
%
%   Note: When including radiation heat transfer, the computations need to be done
%   using absolute temperatures.  Here, however, the final plot and table display
%   temperature in C  --  the units conversion is done in the plot and fprintf
%   commands.
%
%   The basic idea for this problem comes from Example 1.3 on pgs. 17-19
%   in the text, Fundamentals of Heat and Mass Transfer, by Incropera and DeWitt
%   5th Ed (2002, John Wiley & Sons).
%
%   File prepared by J. R. White, UMass-Lowell (last update:  Nov. 2017)
%

      clear all;   close all;   nfig = 0;
      global D emiss R_L sig h Tinf Tsur Isq
%
%   set parameters for the problem
      D = 0.001;            % wire diameter (m)
      emiss = 0.8;          % emissivity (dimensionless)
      R_L = 0.4;            % resistance per unit length (ohm/m)
      sig = 5.67e-8;        % Stefan-Boltzmann constant (W/m^2-K^4)
      h = 100;              % heat transfer coeff (W/m^2-K)
      Tinf = 25+273.15;     % bulk fluid temperature (K)
      Tsur = Tinf;          % temperature of surroundings for radiation HT (K)
%
      NI = 21;  I = linspace(0,10,NI); % set range for wire current (amps)
      Tss = zeros(size(I));            % allocate space for storage of SS temperatures
      Isq = I(1)*I(1);                 % I^2 for first value of current
      Tg = Tinf;                       % initial guess of Tss for first current
      Tss(1) = fzero(@conducting_rod_1a,Tg);  % find actual Tss for first value of I
%   Note: The above step was not really necessary for this case, since for I = 0
%         and Tsur = Tinf, Tss = Tinf (so we could simply set Tss(1) = Tinf.  However,
%         for the general case, this is not true...
%
%   find SS temperature for each current (implicit)
      for n = 2:NI
        Isq = I(n)*I(n);
        Tss(n) = fzero(@conducting_rod_1a,Tss(n-1));
```

```
      end
%
%   find SS temperature for each current assuming no radiation heat transfer (explicit)
       Tssa = Tinf + I.*I*R_L/(h*pi*D);  % this is often a poor approx (not so here)
%
%   plot Tss vs I
      nfig = nfig+1;   figure(nfig)
      plot(I,Tss-273.15,'r-',I,Tssa-273.15,'b--','LineWidth',2), grid on
      rr = axis;   rr(3) = 0;   axis(rr);
      title('Steady State Wire Temperature vs Current')
      xlabel('Current (amps)'),ylabel('Steady State Temperature (^oC)')
      legend('T_{ss} with radiation','T_{ss} with no radiation', ...
             'Location','NorthWest')
%
%   evaluate components of balance equation (note 'dot arithmetic' where appropriate)
      A = pi*D;                              % surface area per unit length
      gen = I.*I*R_L;                        % energy generation rate
      conv = h*A*(Tss - Tinf);               % loss rate by convection
      rad = emiss*sig*A*(Tss.^4 - Tsur^4);   % loss rate via radiation
      bal =  (gen - conv - rad);             % SS energy balance eqn
%
%   write summary table of results
      fprintf(1,'\n\n');
      fprintf(1,'           Summary Results from the CONDUCTING_ROD_1.M Program \n');
      fprintf(1,'\n');
      fprintf(1,'    Basic Design Parameters      \n');
      fprintf(1,'      Wire Diameter (m):                     %8.3f    \n',D);
      fprintf(1,'      Emissivity (dimensionless):            %8.2f    \n',emiss);
      fprintf(1,'      Resistance per Unit Length (ohm/m):    %8.2f    \n',R_L);
      fprintf(1,'      Stefan-Boltzmann Constant (W/m^2-K^4): %9.2e    \n',sig);
      fprintf(1,'      Heat Transfer Coeff (W/m^2-K):         %8.2f    \n',h);
      fprintf(1,'      Bulk Fluid Temperature (C):            %8.2f    \n',Tinf-273.15);
      fprintf(1,'      Temperature of Surroundings (C):       %8.2f    \n',Tsur-273.15);
      fprintf(1,'\n');
      fprintf(1,'    Components of Energy Balance vs. Current  \n');
      fprintf(1,'        I    Generation    Convection   Radiation     Balance      SS Temp \n');
      fprintf(1,'       (A)      (W/m)         (W/m)        (W/m)        (W/m)          (C)   \n');
      for n = 1:NI
        fprintf(1,'     %5.1f  %8.2f      %8.2f     %8.2f     %8.3f     %8.2f    \n', ...
                  I(n),gen(n),conv(n),rad(n),bal(n),Tss(n)-273.15);
      end
%
%   end of problem
```

```
%
%   CONDUCTING_ROD_1A.M   fzero function file for use with CONDUCTING_ROD_1.M
%
%   This file simply evaluates a function of the form F(T).  It is used by
%   the fzero routine to find the value T such that F(T) = 0.  The function
%   actually represents the steady state energy balance for a circular conducting
%   rod that is heated internally via ohmic heating and that loses energy by both
%   convective and radiation heat transfer.  The analysis is done for a unit length
%   of rod or wire.
%
      function F = conducting_rod_1a(T)
      global D emiss R_L sig h Tinf Tsur Isq
      A = pi*D;                              % surface area per unit length
      gen = Isq*R_L;                         % energy generation rate
      conv = h*A*(T - Tinf);                 % loss rate by convection
      rad = emiss*sig*A*(T^4 - Tsur^4);      % loss rate via radiation
      F =  (gen - conv - rad);               % SS energy balance eqn
%
%   end of function
```

As expected, as the current increases, the steady state temperature also increases, and this is clearly shown in Fig. 1. Also, as we might expect, as the temperature increases, the contribution due to radiation heat transfer also increases as seen in the tabulated output from the **conducting_rod_1.m** program. However, since the convective heat transfer coefficient is fairly large (i.e., h = 100 W/m^2-K), we see from Table 2 that the radiation heat transfer term never plays a dominant part in the overall energy balance (always less than 10% of total). Thus, for this problem, most of the energy generated by ohmic heating within the rod is removed by convection to the environment. This suggests that we could get a rough estimate of T_{ss} by simply ignoring the radiation term, and explicitly evaluating the simplified eqn. (9) for T_{ss} vs. I. This approximate result is shown in Fig. 1 along with the more accurate T_{ss} vs. I curve and, indeed, for this problem, the no radiative heat transfer approximation gives a reasonable result. However, since ***fzero*** is so easy to use, there is really no need to make this kind of simplifying assumption for this problem.

Reference: The basic idea for this problem came from Example 1.3 on pgs. 17-19 in the text "Fundamentals of Heat and Mass Transfer" 5th Ed. by Incropera and Dewitt, John Wiley & Sons, 2002. The sketch used at the beginning of this example also came from this reference (pg. 17).

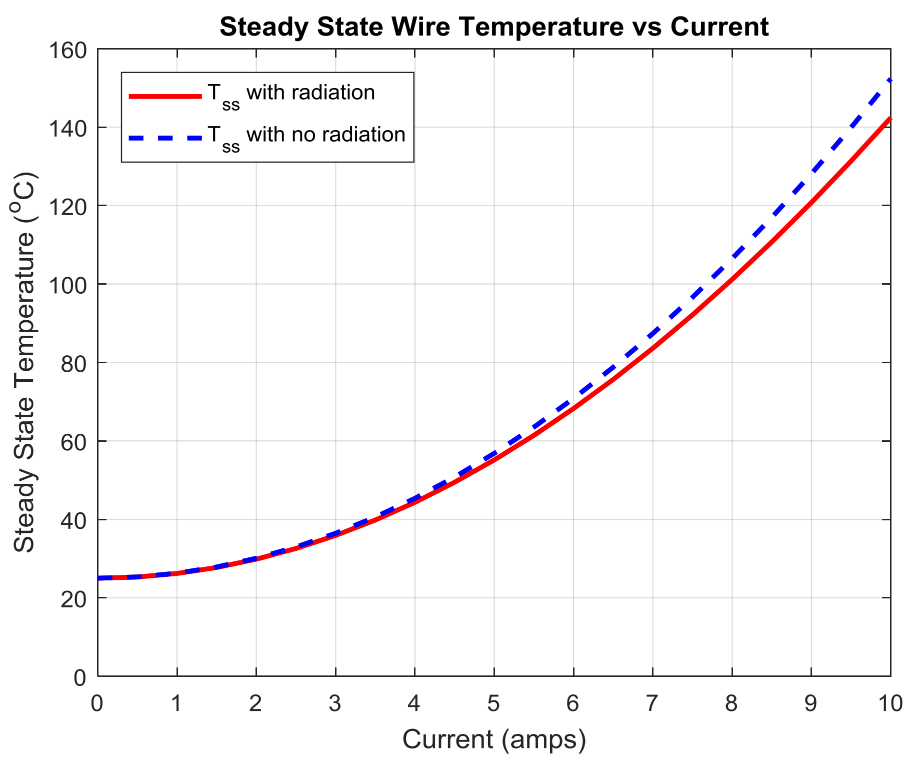


Fig. 1 Steady-state rod temperature versus applied current.

Table 2 Energy balance components from the conducting_rod_1.m program.

```
       Summary Results from the CONDUCTING_ROD_1.M Program

Basic Design Parameters
  Wire Diameter (m):                           0.001
  Emissivity (dimensionless):                   0.80
  Resistance per Unit Length (ohm/m):           0.40
  Stefan-Boltzmann Constant (W/m^2-K^4): 5.67e-008
  Heat Transfer Coeff (W/m^2-K):              100.00
  Bulk Fluid Temperature (C):                  25.00
  Temperature of Surroundings (C):             25.00

Components of Energy Balance vs. Current
    I    Generation    Convection   Radiation     Balance      SS Temp
   (A)     (W/m)         (W/m)        (W/m)        (W/m)         (C)
   0.0      0.00          0.00         0.00        0.000        25.00
   0.5      0.10          0.10         0.00        0.000        25.30
   1.0      0.40          0.38         0.02        0.000        26.21
   1.5      0.90          0.86         0.04       -0.000        27.73
   2.0      1.60          1.52         0.08        0.000        29.85
   2.5      2.50          2.38         0.12        0.000        32.58
   3.0      3.60          3.43         0.17        0.000        35.91
   3.5      4.90          4.66         0.24        0.000        39.83
   4.0      6.40          6.08         0.32       -0.000        44.35
   4.5      8.10          7.68         0.42       -0.000        49.45
   5.0     10.00          9.47         0.53       -0.000        55.15
   5.5     12.10         11.44         0.66       -0.000        61.42
   6.0     14.40         13.59         0.81       -0.000        68.26
   6.5     16.90         15.92         0.98        0.000        75.66
   7.0     19.60         18.42         1.18        0.000        83.62
   7.5     22.50         21.09         1.41        0.000        92.13
   8.0     25.60         23.93         1.67        0.000       101.17
   8.5     28.90         26.93         1.97       -0.000       110.73
   9.0     32.40         30.09         2.31       -0.000       120.79
   9.5     36.10         33.41         2.69       -0.000       131.35
  10.0     40.00         36.88         3.12        0.000       142.38
```