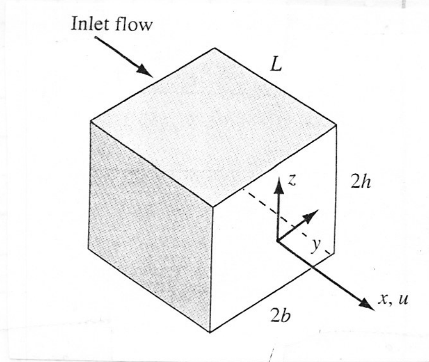
3.18) pg 186

An incompressible fluid flows steadily through the rectangular duct shown in the diagram. The exit velocity profile is given approximately by

$$u(y,z) = U_{max}\left(1 - \frac{y^2}{b^2}\right)\left(1 - \frac{z^2}{h^2}\right)$$



(a) Does this profile satisfy the correct boundary conditions for viscous fluid flow? Explain.

(b) Find an analytical expression for the volume flow Q at the exit.

(c) If the inlet flow rate is 300 ft³/min, estimate U_{ave} and U_{max} at the exit in m/s for b = h = 10 cm.

(a) The proper B.C. at all surfaces is V = 0 (the "no-slip" condition)

at $y = \pm b$ $u = 0$ and at $z = \pm h$ $u = 0$

$\therefore$ The B.C. are satisfied

(b) For incompressible flow

1-D flows | in this case, $\vec{V}$ is only in the x-direction

$$Q = \int_A \vec{V}\cdot\hat{n}\,dA = \int u\,dA$$

$$= \int_{-h}^{h}\int_{-b}^{b} U_{max}\left(1 - \frac{y^2}{b^2}\right)\left(1 - \frac{z^2}{h^2}\right)dy\,dz$$

$$= U_{max}\left[\int_{-h}^{h}\left(1 - \frac{z^2}{h^2}\right)dz\right]\left[\int_{-b}^{b}\left(1 - \frac{y^2}{b^2}\right)dy\right]$$

$$z - \frac{z^3}{3h^2}\Big|_{-h}^{h} = \left(h - \frac{h}{3}\right) - \left(-h + \frac{h}{3}\right) = 2h - \frac{2h}{3} = \frac{4}{3}h$$

$$\therefore Q = U_{max}\left(\frac{4}{3}h\right)\left(\frac{4}{3}b\right)$$

$$\boxed{Q = \frac{16}{9}bh\,U_{max}}$$

(c) From Part b, we immediately see that

$$u_{ave} = \frac{Q}{A} = \frac{\frac{16}{9} bh\, U_{max}}{(2h)(2b)}$$

$$\boxed{u_{ave} = \frac{4}{9} u_{max}}$$

and for $Q = 300\ ft^3/min \times \left(\frac{1m}{3.281 ft}\right)^3 \times \frac{1 min}{60 sec}$

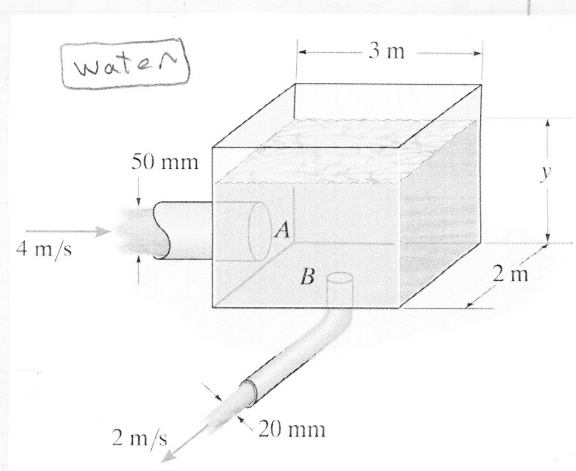
$$= 0.142\ m^3/s = Q_{in} = Q_{out}$$

and $b = h = 10 cm = 0.1 m$

we have $u_{ave} = \frac{0.142 m^3/s}{(0.2m)(0.2m)} = \boxed{3.54\ m/s}$ ans

and $u_{max} = \frac{9}{4} u_{ave} = \boxed{7.96\ m/s}$ ans

The Reynolds Transport Theorem (RTT) was developed in class with the following result:



$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{cv} b\rho\, dr^3 + \int_{cs} b\rho \vec{v}\cdot\hat{n}\, dA$$

Within the context of the continuity eqn applied to the flow situation in the diagram, answer the following questions.

1. What is b?

the continuity eqn represents the conservation of mass.

$$\therefore B_{sys} = \underset{\uparrow \text{mass}}{m} \quad \text{and} \quad \boxed{b = \frac{dB}{dm} = 1}$$

ans

2. Determine $\frac{dB_{sys}}{dt}$

the basic defn of a "system" is that it has fixed mass

$$\boxed{\therefore \frac{dB_{sys}}{dt} = 0}$$

3. Determine the value of $\int_{cs} b\rho\vec{v}\cdot\hat{n}\, dA$

the term $\int_{cs} b\rho\vec{v}\cdot\hat{n}\, dA$ = net mass flowrate out of cv (across the CS) ← control surface

for uniform, 1-D flows, this reduces to

$$\int_{cs} b\rho\vec{v}\cdot\hat{n}\, dA = \sum_{i \text{ outlets}} (\rho V A)_i - \sum_{i \text{ inlets}} (\rho V A)_i$$

$$= \dot{m}_{out} - \dot{m}_{in}$$

here $\dot{m}_{in} = \rho A V|_{in} = \left(1000 \frac{kg}{m^3}\right)\left(\frac{\pi}{4}\right)(0.05 m)^2 (4 m/s)$

$$= \boxed{7.854 \text{ kg/s}}$$

$$\dot{m}_{out} = \rho A V|_{out} = \left(1000\,\frac{kg}{m^3}\right)\left(\frac{\pi}{4}\right)(0.02\,m)^2\left(2\,\frac{m}{s}\right)$$

$$= \boxed{0.628\ kg/s}$$

$$\therefore \int_{cs} b\rho\vec{V}\cdot\hat{n}\,dA = \underset{net}{\dot{m}_{out}} = 0.628 - 7.854$$

$$= \boxed{-7.226\ kg/s} \quad \text{ans}$$

a negative here implies a net inflow instead of outflow

4. Determine $\frac{d}{dt}\int_{cv} b\rho\, dr^3$

here $\frac{d}{dt}\left(\int_{cv} b\rho\, dr^3\right) = \frac{d}{dt} m_{cv}$ mass inside CV

with $dB_{sys}/dt = 0$, The original eqn for RTT goes

$$\frac{d}{dt}\int_{cv} b\rho\, dr^3 = -\int_{cs} b\rho\vec{V}\cdot\hat{n}\,dA$$

$$\frac{dm_{cv}}{dt} = -\ \text{net mass flow rate out of CV}$$

$$= -(-7.226\ kg/s)$$

$$= \boxed{+7.226\ kg/s} \quad \text{ans}$$

the mass within the CV is increasing at a rate of 7.226 kg/s

This should give a better perspective of the meaning of the RTT as applied to a mass balance problem

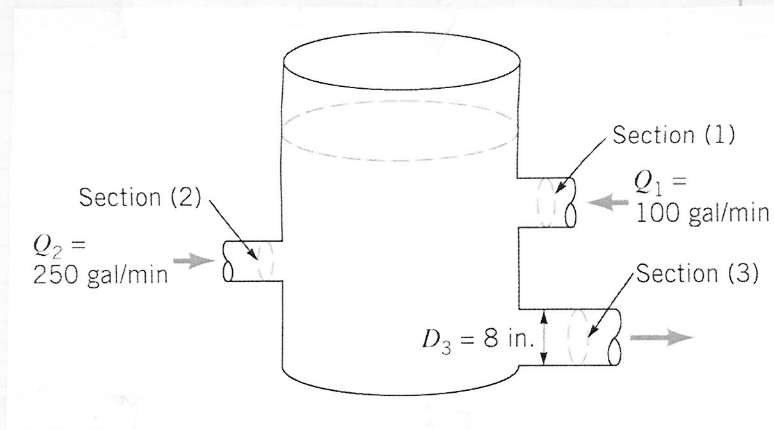
→ from a single CV perspective

$$\frac{dm_{cv}}{dt} = -(\dot{m}_{out} - \dot{m}_{in}) = \dot{m}_{in} - \dot{m}_{out}$$

$$\frac{dm_{cv}}{dt} = \dot{m}_{in} - \dot{m}_{out}$$

$$= 7.854 - 0.628 = 7.226\ kg/s$$

Water enters a cylindrical tank through two pipes at a rate of 250 gpm and 100 gpm. The level of the tank remains constant.



Determine the average velocity of the flow through the 8″ inside diameter exit pipe.

Since the water level is constant, then $\frac{dM_{cv}}{dt} = 0$

$\therefore \boxed{\dot{m}_{out} = \dot{m}_{in}}$ steady flow

Also, since water is incompressible, $\rho = \text{const}$, and

$$Q_{out} = Q_{in}$$

$$\therefore Q_3 = Q_1 + Q_2 = 100 + 250 = \boxed{350 \text{ gpm}}$$

Exit area $A_3 = \frac{\pi D^2}{4} = \frac{\pi}{4}\left(\frac{8}{12}\right)^2 = 0.3491 \text{ ft}^2$

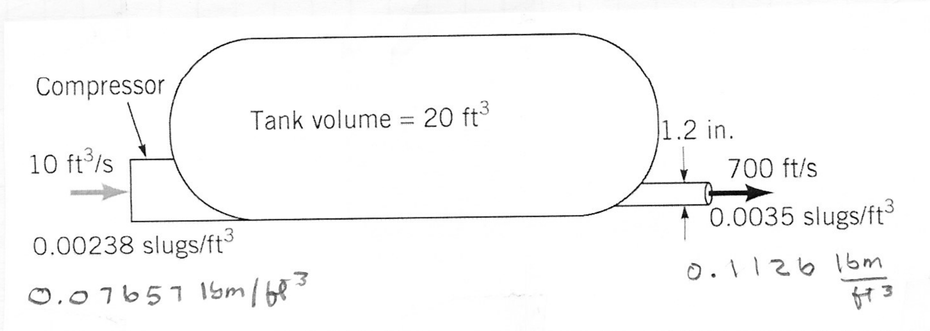
and $Q_3 = V_3 A_3$

$$\text{or } V_3 = \frac{Q_3}{A_3} = \frac{350 \text{ gal/min} * \frac{2.228 \times 10^{-3} \text{ ft}^3/\text{s}}{\text{gpm}}}{0.3491 \text{ ft}^2}$$

$$\frac{1 \text{ ft}^3}{7.48 \text{ gal}} * \frac{1 \text{ min}}{60 \text{ s}}$$

or $V_3 = 2.234 \text{ ft/s}$

$= \boxed{2.23 \text{ ft/s}}$ ans



Air at standard conditions enters the compressor shown at a rate of 10 ft^3/s. It leaves the tank through a 1.2" diameter pipe with a uniform speed of 700 ft/s. The inlet and outlet densities and the tank volume are given (as shown)

Determine the rate of change of mass of air within the tank in lbm/s.

The continuity eqn gives

$$\frac{d m_{cv}}{dt} = \dot{m}_{in} - \dot{m}_{out}$$

here

$$\dot{m}_{in} = \rho_{in} Q_{in} = \left(0.07657 \frac{lbm}{ft^3}\right)\left(10 \frac{ft^3}{s}\right)$$

or $\dot{m}_{in} = 0.7657 \frac{lbm}{s}$

$$\dot{m}_{out} = \rho_{out} Q_{out} = \rho_{out} A_{out} V_{out}$$

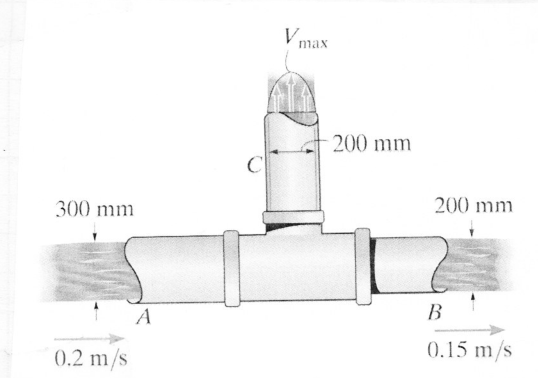
$$= \left(0.1126 \frac{lbm}{ft^3}\right)\left(\frac{\pi}{4}\right)\left(\frac{1.2}{12} ft\right)^2 (700 \, ft/s)$$

$$= 0.6191 \, lbm/s$$

$$\therefore \frac{d m_{cv}}{dt} = 0.7657 - 0.6191 = 0.1466 \frac{lbm}{s}$$

$$\boxed{\frac{d m_{air}}{dt} = 0.147 \frac{lbm}{s}}$$ ans

Oil flows in the pipe segment as shown in the sketch.



Determine the max velocity, V_{max}, of the oil as it emerges at point C if the velocity distribution in pipe C is given by

$$V_c = V_{max}(1 - 100r^2)$$

where r is in meters measured from the pipe centerline.

This is a steady flow problem and oil is incompressible

$\therefore \quad Q_{out} = Q_{in}$

or $\quad Q_C + Q_B = Q_A$

$\quad Q_C = Q_A - Q_B$

$$\boxed{Q_C = 9.428 \times 10^{-3} \ \frac{m^3}{s}}$$

$Q_A = V_A A_A = (0.2 \tfrac{m}{s})(\tfrac{\pi}{4})(0.3)^2 \ m^2$

$= 1.414 \times 10^{-2} \ m^3/s$

$Q_B = V_B A_B = (0.15)(\tfrac{\pi}{4})(0.2)^2$

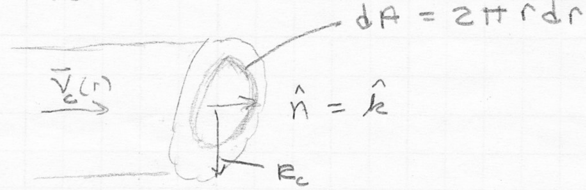
$= 4.712 \times 10^{-3} \ m^3/s$

However we can also write Q_C as

$Q_C = \int \vec{V} \cdot \hat{n} \, dA$

$= \int_0^{R_c} V_{max}(1 - 100r^2) \, 2\pi r \, dr$

$= 2\pi V_{max} \underbrace{\int_0^{R_c} (1 - 100r^2) \, r \, dr}$



1-D flow

$\vec{V} \cdot \hat{n} = V_c(r)$

$$\text{Int} = \frac{r^2}{2} - 100\frac{r^4}{4} \Big|_0^{R_c}$$

$= R_c^2 \left(\frac{1}{2} - 25 R_c^2\right)$

$= (0.1)^2 \left[\frac{1}{2} - 25(0.1)^2\right]$

$= 2.5 \times 10^{-3} \ m^2$

$= 2\pi V_{max} (2.5 \times 10^{-3})$

$= 0.01571 \, V_{max}$

or finally

$$V_{max} = \frac{Q_C}{0.01571} = \frac{9.428 \times 10^{-3} \ m^3/s}{0.01571 \ m^2}$$

$$\boxed{V_{max} = 0.600 \ \frac{m}{s}}$$

ans

$$V_{ave} = \frac{Q_C}{A} = \frac{9.428 \times 10^{-3} \ m^3/s}{\frac{\pi}{4}(0.2 \ m)^2}$$

$\boxed{V_{ave} = 0.300 m}$