Ave Velocity: $\boxed{V_{ave} = \frac{1}{A}\int v\,dA}$

Laminar Flow in Pipe (1-D)

$$V = u(r) = u_0\left[1-\left(\frac{r}{R}\right)^2\right]$$

$$A = \pi R^2 \quad \text{and} \quad dA = 2\pi r\,dr$$

$$\therefore V_{ave} = \frac{2u_0}{R^2}\int_0^R \left(1-\frac{r^2}{R^2}\right) r\,dr$$

$$= \frac{2u_0}{R^2}\int_0^R \left(r - \frac{1}{R^2}r^3\right)dr$$

$$= \frac{2u_0}{R^2}\left[\frac{r^2}{2} - \frac{1}{R^2}\frac{r^4}{4}\right]_0^R = \frac{2u_0}{R^2}\underbrace{\left[\frac{R^2}{2} - \frac{R^2}{4}\right]}_{\frac{R^2}{4}}$$

$$\boxed{V_{ave} = \frac{u_0}{2}}$$ ans

for laminar flow in circular pipe

Turbulent Flow in Pipe (1-D)

$$V = u(r) = u_0\left(1-\frac{r}{R}\right)^m \quad \text{with } m \simeq \frac{1}{7}$$

$$\therefore V_{ave} = \frac{2u_0}{R^2}\int_0^R \left(1-\frac{r}{R}\right)^m r\,dr$$

Now, to perform this integral, we can take advantage of the integral from the Standard Integral tables

$$\int x(ax+b)^n dx = \frac{1}{a^2(n+2)}(ax+b)^{n+2} - \frac{b}{a^2(n+1)}(ax+b)^{n+1} \qquad n \neq -1,-2$$

let focus on $I = \int_0^R \left(1-\frac{r}{R}\right)^m r\,dr$

$$I = \int_0^R \left(\frac{R-r}{R}\right)^m r\,dr$$

or $$\boxed{I = \frac{1}{R^m}\int_0^R (R-r)^m\, r\,dr}$$

now this is in the form needed for the above integral expression for our case $a = -1 \quad b = R$ and $n = m$

$$\therefore \quad I = \frac{1}{R^m}\int_0^R (R-r)^m r\,dr = \left[\frac{1}{(-1)^2(m+2)}(R-r)^{m+2} - \frac{R}{(-1)^2(m+1)}(R-r)^{m+1}\right]_0^R \frac{1}{R^m}$$

$$= \frac{1}{R^m}\left[(0-0) - \left(\frac{R^{m+2}}{m+2} - \frac{R\,R^{m+1}}{m+1}\right)\right]$$

$$= \frac{1}{R^m}\left[\frac{1}{m+1} - \frac{1}{m+2}\right]R^{m+2}$$

$$= R^2\left[\frac{m+2-m-1}{(m+1)(m+2)}\right] = \boxed{\frac{R^2}{(m+1)(m+2)}}$$

Now putting this back into the expression for V_{ave}, we have

$$V_{ave} = \frac{2u_0}{R^2}\left[\frac{R^2}{(m+1)(m+2)}\right]$$

$$\boxed{V_{ave} = \frac{2u_0}{(m+1)(m+2)}}$$ ans

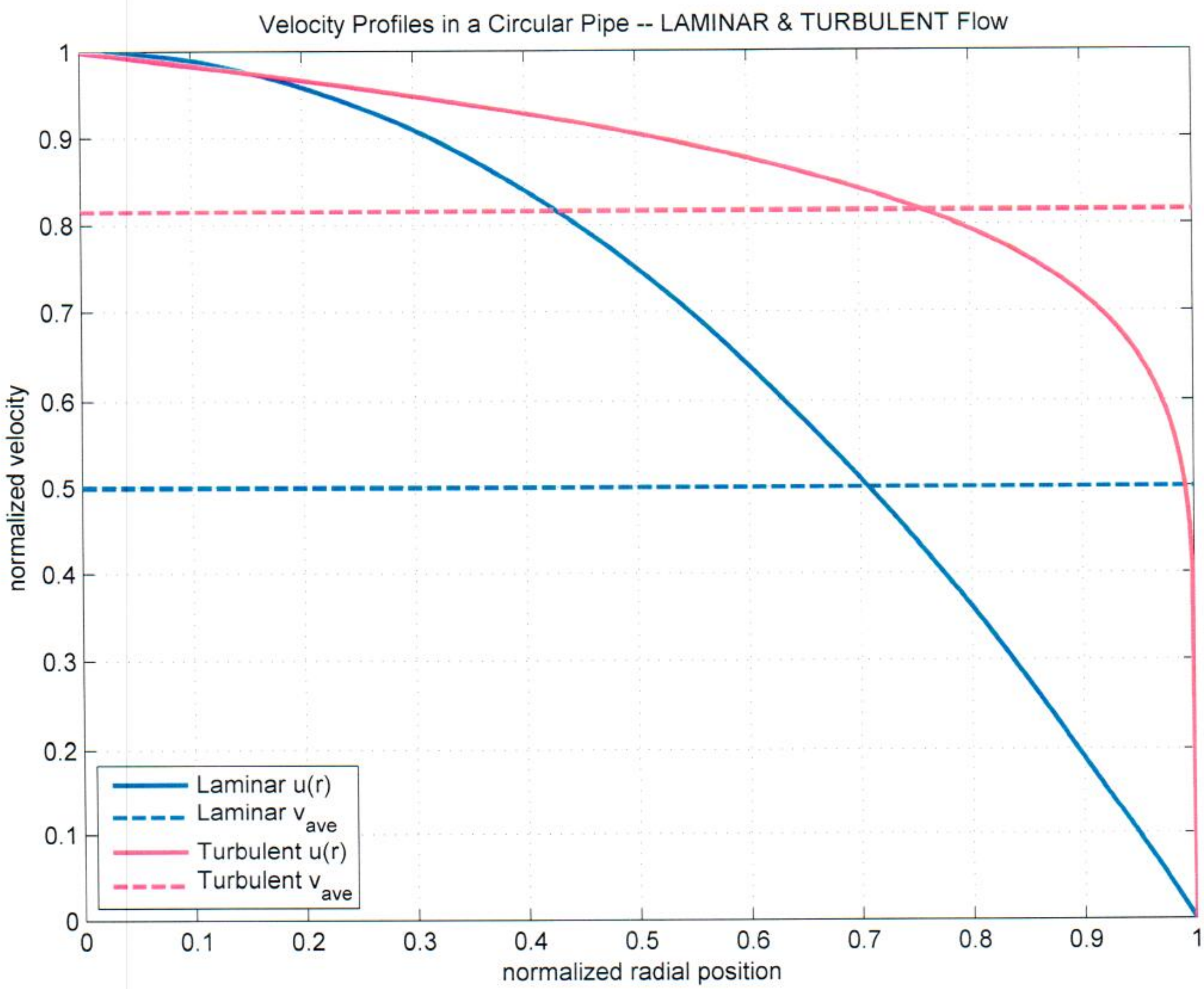
for turbulent flow in a circular pipe

Now for the case where $m = \frac{1}{7}$, we have

$$V_{ave} = \frac{2u_0}{\left(1+\frac{1}{7}\right)\left(2+\frac{1}{7}\right)} = \frac{2u_0}{\left(\frac{8}{7}\right)\left(\frac{15}{7}\right)} = 2u_0\left(\frac{49}{120}\right)$$

$$\boxed{V_{ave} = \frac{49\,u_0}{60}} \qquad \boxed{V_{ave} \approx 0.817\,u_0}$$

ans



Velocity Profiles in a Circular Pipe -- LAMINAR & TURBULENT Flow
normalized velocity
normalized radial position
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Laminar u(r)
Laminar v_ave
Turbulent u(r)
Turbulent v_ave

```
%
%   PIPE_AVEVEL.M  Plot velocity profile in pipe geometry
%
%   This file simply plots the axial velocity profile in a circular pipe.
%   Both laminar and turbulent flow conditions are represented.
%
%   File prepared by J. R. White, UMass-Lowell (last update: April 2016)
%

      clear all;   close all;   nfig = 0;
%
%   set some parameters (normalized profiles: Uo = 1  and  z = r/R from 0 to 1)
      Uo = 1;    z = linspace(0,1,501);
%
%   let's do LAMINAR FLOW first
      vl = Uo*(1 - z.*z);    vlave = Uo/2;
%
%   now lets do the TURBULENT FLOW case
      mt = 1/7;     vt = Uo*(1 - z).^mt;     vtave = 2*Uo/((mt+1)*(mt+2));
%
%   now plot the results
      nfig = nfig+1;    figure(nfig)
      plot(z,vl,'b-',[0 1],[vlave vlave],'b--',z,vt,'r-',[0 1],[vtave vtave],'r--', ...
           'LineWidth',2),grid
      title('Velocity Profiles in a Circular Pipe -- LAMINAR & TURBULENT Flow')
      ylabel('normalized velocity'),xlabel('normalized radial position')
      legend('Laminar u(r)','Laminar v_{ave}','Turbulent u(r)','Turbulent v_{ave}', ...
             'Location','SouthWest')

%   end of file
```

Kinetic Energy Correction Factor $\boxed{\alpha = \frac{1}{V_{ave}^3 A} \int_A V^3 \, dA}$

Laminar Flow in Pipe (1-D)

$$V = u(r) = u_o \left[1 - \left(\frac{r}{R}\right)^2 \right]$$

where $\boxed{V_{ave} = \frac{u_o}{2}}$

$$A = \pi R^2 \quad \text{and} \quad dA = 2\pi r \, dr$$

$$\therefore \quad \alpha = \frac{2\pi u_o^3}{u_o^3/8 \; \pi R^2} \int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right)^3 r \, dr$$

$$\alpha = \frac{16}{R^2} \int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right]^3 r \, dr$$

note $r = 0 \quad x = 1$
$r = R \quad x = 0$

To evaluate this, let $x = 1 - \left(\frac{r}{R}\right)^2$

Then $dx = -2\left(\frac{r}{R}\right)\frac{dr}{R} = -\frac{2r\,dr}{R^2}$

or $r\,dr = -\frac{R^2 dx}{2}$

$$\therefore \alpha = \frac{16}{R^2}\left(-\frac{R^2}{2}\right)\int_1^0 x^3 \, dx$$

$$\alpha = -8\int_1^0 x^3 \, dx = -\frac{8}{4} x^4 \Big|_1^0 = -2(0-1) = 2$$

$\therefore \boxed{\alpha = 2.0}$ ans (for laminar flow in circular pipe)

Turbulent Flow in Pipe (1-D)

$$V = u(r) = u_0 \left(1 - \frac{r}{R}\right)^m \quad \text{with } m = \frac{1}{7}$$

$$v_{ave} = \frac{2u_0}{(m+1)(m+2)}$$

$$\alpha = \frac{2\pi u_0^3}{\frac{8u_0^3 \quad \pi R^2}{(m+1)^3(m+2)^3}} \int_0^R \left(1 - \frac{r}{R}\right)^{3m} r\,dr$$

$$= \frac{(m+1)^3(m+2)^3}{4R^2} \int_0^R \left(1 - \frac{r}{R}\right)^{3m} r\,dr$$

To evaluate this, let's make the following substitutions

$$n = 3m \quad \text{and} \quad \left(1 - \frac{r}{R}\right)^n = \left(\frac{R-r}{R}\right)^n = \frac{1}{R^{3m}}(R-r)^n$$

or $$\alpha = \frac{(m+1)^3(m+2)^3}{4R^2 R^{3m}} \int_0^R (R-r)^n r\,dr \quad \text{with } n = 3m$$

Now, using the Integral Tables

$$\int x(ax+b)^n dx = \frac{1}{a^2(n+2)}(ax+b)^{n+2} - \frac{b}{a^2(n+1)}(ax+b)^{n+1} \quad n \neq -1, -2$$

for our case $a = -1 \quad b = R \quad x = r \quad dx = dr$

$$\therefore I = \int_0^R (R-r)^n r\,dr$$

$$= \left[\frac{1}{(-1)^2(n+2)}(R-r)^{n+2} - \frac{R}{(-1)^2(n+1)}(R-r)^{n+1}\right]\Bigg|_0^R$$

$$= (0 - 0) - \left(\frac{R^{n+2}}{n+2} - \frac{R(R)^{n+1}}{n+1}\right)$$

$$= R^{n+2}\left(-\frac{1}{n+2} + \frac{1}{n+1}\right) = R^{n+2}\left[\frac{-n-1+n+2}{(n+1)(n+2)}\right]$$

$$I = R^{n+2}\left[\frac{1}{(n+1)(n+2)}\right] = \boxed{\frac{R^{3m+2}}{(3m+1)(3m+2)}}$$

Now, putting this into our expression for α, we have

$$\alpha = \frac{(m+1)^3(m+2)^3}{4R^{3m+2}} \times \frac{R^{3m+2}}{(3m+1)(3m+2)}$$

$$\boxed{\alpha = \frac{(m+1)^3(m+2)^3}{4(3m+1)(3m+2)}}$$

Now for the case where $m = \frac{1}{7}$, we have

$$\alpha = \frac{\left(\frac{8}{7}\right)^3\left(\frac{15}{7}\right)^3}{4\left(\frac{10}{7}\right)\left(\frac{17}{7}\right)} = \frac{8^3\, 15^3\, 7^2}{4(10)(17)(7)^6} \quad \underline{7^4}$$

$$= \frac{(512)(3375)}{(680)(2401)}$$

$$= 1.0584 \qquad \boxed{\therefore\ \alpha \simeq 1.06}$$ for turbulent flow in circular pipe

Summary

laminar flow $\alpha = 2.0$ Turbulent flow $\alpha = 1.06$

also note this assumes that fully developed flow — which is often not the case

This we cannot ignore — leads to a factor of two error

This is often ignored. The uncertainty in our calculations are often 10–20%. thus a 6% correction factor is often in the noise region

$\therefore$ we must use the term

$\frac{\alpha V^2}{2g}$ instead of $\frac{V^2}{2g}$ for laminar flow problems in the energy eqn (does not hurt to include for both)

Momentum Flux Correction Factor $\boxed{\beta = \frac{1}{V_{ave}^2 A} \int V^2 dA}$

Laminar Flow in Pipe (1-D)

$$V = u(r) = u_0 \left[1 - \left(\frac{r}{R}\right)^2 \right] \qquad \boxed{V_{ave} = \frac{u_0}{2}}$$

$$A = \pi R^2 \quad \text{and} \quad dA = 2\pi r\, dr$$

$$\therefore \quad \beta = \frac{2\pi u_0^2}{\frac{u_0^2}{4} \pi R^2} \int_0^R \left[1 - \left(\frac{r}{R}\right)^2 \right]^2 r\, dr$$

$$= \frac{8}{R^2} \int_0^R \left[1 - \left(\frac{r}{R}\right)^2 \right]^2 r\, dr$$

as before, let $x = 1 - \left(\frac{r}{R}\right)^2$ where $r = 0 \quad x = 1$; $r = R \quad x = 0$

Then $dx = -2\left(\frac{r}{R}\right)\frac{dr}{R} = -\frac{2r\,dr}{R}$

or $r\,dr = -\frac{R^2 dx}{2}$

$$\therefore \quad \beta = \frac{8}{R^2}\left(-\frac{R^2}{2}\right) \int_1^0 x^2\, dx$$

$$= -4 \int_1^0 x^2\, dx = -\frac{4}{3} x^3 \Big|_1^0$$

$$= -\frac{4}{3}(0 - 1) = \frac{4}{3}$$

$$\therefore \quad \boxed{\beta = \frac{4}{3}} \text{ ans}$$

for laminar flow in circular pipe

Turbulent Flow in Pipe (1-D)

$$v = u(r) = u_0\left(1 - \frac{r}{R}\right)^m \qquad \text{with } m \simeq \frac{1}{7}$$

$$v_{ave} = \frac{2u_0}{(m+1)(m+2)}$$

$$\beta = \frac{2\pi u_0^2}{\frac{4u_0^2 \pi R^2}{(m+1)^2(m+2)^2}} \int_0^R \left(1-\frac{r}{R}\right)^{2m} r\,dr$$

$$= \frac{(m+1)^2(m+2)^2}{2R^2} \int_0^R \left(1-\frac{r}{R}\right)^{2m} r\,dr$$

Following the same procedure as before, we have

$$n = 2m \quad \text{and} \quad \left(1-\frac{r}{R}\right)^n = \left(\frac{R-r}{R}\right)^n = \frac{1}{R^{2m}}(R-r)^n$$

or

$$\beta = \frac{(m+1)^2(m+2)^2}{2R^2 R^{2m}} \int_0^R (R-r)^n r\,dr \qquad \text{with } n = 2m$$

Now using The Integral Tables

$$\int x(ax+b)^n dx = \frac{1}{a^2(n+2)}(ax+b)^{n+2} - \frac{b}{a^2(n+1)}(ax+b)^{n+1} \quad n \neq -1, -2$$

for the current case $a = -1$ $b = R$ $x = r$ and $dx = dr$

$$\therefore \quad I = \int_0^R (R-r)^n r\,dr = R^{n+2}\left[\frac{1}{(n+1)(n+2)}\right]$$

see development for calc of α

$$\boxed{I = \frac{R^{2m+2}}{(2m+1)(2m+2)}}$$

$$\therefore \quad \beta = \frac{(m+1)^2(m+2)^2}{2R^{2m+2}} \frac{R^{2m+2}}{(2m+1)(2m+2)}$$

$$\boxed{\beta = \frac{(m+1)^2(m+2)^2}{2(2m+1)(2m+2)}}$$

Now for the case when $m = \frac{1}{7}$, we have

$$\beta = \frac{\left(\frac{8}{7}\right)^2\left(\frac{15}{7}\right)^2}{2\left(\frac{9}{7}\right)\left(\frac{16}{7}\right)} = \frac{8^2\, 15^2\, 7^2}{2(9)(16)\,7^4}$$

$$= \frac{(64)(225)}{(18)(16)(49)} = \frac{2(225)}{9(49)} = \frac{450}{441} = 1.020$$

$$\boxed{\therefore\ \beta = 1.02}$$ for turbulent flow in a circular pipe

Summary

laminar flow $\beta = \frac{4}{3}$

This may be important. If neglected, it leads to about a 30% error, which is quite large

turbulent flow $\beta = 1.02$

This small correction factor is often neglected

$\therefore$ we should use the term

$\beta \dot{m} V$ instead of $\dot{m} V$ for laminar flow problem

in the momentum eqn (again it does not hurt to include in both)