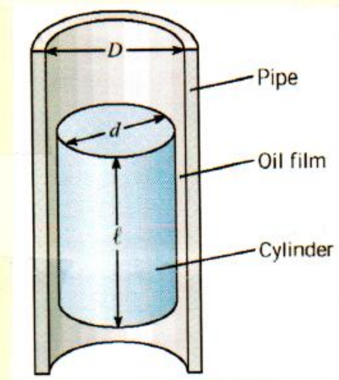
(2.39) A solid circular cylinder of diameter d and length ℓ slides inside a vertical smooth pipe that has an inside diameter D as shown in the sketch. The small space between the cylinder and the pipe is lubricated with an oil film that has a viscosity μ. Assume that the cylinder has a weight W and is concentric with the pipe as it falls. Also assume that the velocity profile within the small gap is linear.



ⓐ Under the above conditions, derive a formula for the steady state velocity of the cylinder in the vertical pipe.

→ force balance on cylinder

$$m\frac{dv}{dt} = F_{fric} - W$$

$F_{fric} = \tau A$ ← surface area (shear stress; sign is opposite to direction of motion)

$W = mg$ ⇐ given as W

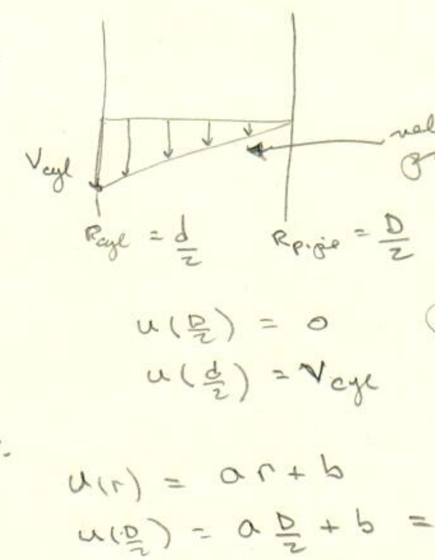
→ in steady state the viscous friction term just balances the cylinder's weight (F_{fric} is in +y direction)

↑ F_{fric} ↓ W

$$\therefore \quad \tau A = W$$

but $\tau = -\mu \left.\frac{du}{dr}\right|_{r=\frac{d}{2}}$ → r

$$\tau = -\mu\left(-\frac{2V_{cyl}}{D-d}\right)$$



$$A = \pi d \ell$$

$u(\frac{D}{2}) = 0$

$u(\frac{d}{2}) = V_{cyl}$ (no slip conditions)

$$\therefore \quad \mu\left(-\frac{2V_{cyl}}{D-d}\right)(\pi d \ell) = W$$

$$\boxed{V_{cyl} = -\frac{W(D-d)}{2\mu(\pi d \ell)}}$$

∴

$u(r) = ar + b$

$u(\frac{D}{2}) = a\frac{D}{2} + b = 0$

$\therefore b = -\frac{aD}{2}$

$u(\frac{d}{2}) = a\frac{d}{2} + b = V_{cyl}$

$a\left(\frac{d}{2} - \frac{D}{2}\right) = V_{cyl}$

or $a = -\frac{2V_{cyl}}{D-d}$

$$u(r) = -\frac{2V_{cyl}}{D-d}\left(r - \frac{D}{2}\right)$$

note: from simple geometry for a linear profile

$$\frac{du}{dr} = \frac{\Delta u}{\Delta r} = \frac{-V_{cyl}}{\frac{D-d}{2}} = -\frac{2V_{cyl}}{D-d}$$

$$= \frac{0 - V_{cyl}}{\frac{D}{2} - \frac{d}{2}}$$

b. Find the Terminal velocity of the cylinder using the following data:

$d = 10\text{ cm}$ $D = 10.05\text{ cm}$ $\ell = 20\text{ cm}$

$W = 20\text{ N}$ oil ⇒ SAE 20W at 10°C with $\mu = 0.3\ N\text{-}s/m^2$ ← give as part of problem

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from Fig A.2 in Crowe $\mu_{\text{SAE 20W at 10°C}} \simeq 0.3\ N\text{-}s/m^2$

$$\therefore\ V_{cyl} = \frac{-W(D-d)}{2\mu\pi d\ell}$$

$$= \frac{-20\,N(.05\,cm)}{2\left(0.3\,\frac{N\text{-}s}{m^2}\right)(\pi)(0.1\,m)(20\,cm)}$$

$$= -\frac{20(0.05)}{2(0.3)\pi(0.1)(20)}\ m/s$$

$$= 0.265\ m/s \Rightarrow \boxed{0.27\ \frac{m}{s}}$$