Differential Equations (92.236)

Homework Assignment #9   Spring 2007

Mathematical Models and Autonomous Systems

Problem 1

Any population contained in a finite living area with finite resources has a maximum size that it can sustain -- call this P_{max}. In such a system, a relatively simple but plausible statement of population growth is that the **absolute growth rate** (i.e. absolute birth rate - absolute death rate) is proportional to the difference between the limiting population and the population size at time t.

a. Under the above scenario, **formally derive/explain** an appropriate population balance equation for this system. Also solve the resultant first-order equation and show that its solution can be written as

$$\frac{P(t) - P_{max}}{P_o - P_{max}} = e^{-kt}$$

where k is the proportionality constant discussed above and P_o is the initial population.

b. Assume that the finite living space is Planet Earth and that the particular species of interest is the human race. Current belief is that P_{max} is about 20 billion people (i.e. this is an estimate of the maximum population that the earth can sustain). The world population was about 4.5 billion people in 1980 and about 6.0 billion people in 1999. With these data points and the simple model from Part a., estimate the number of people on Planet Earth in the year 2025.

Problem 2

a. Consider the following population model for a particular species of fish in a small pond:

$$\frac{dP}{dt} = 2P - \frac{P^2}{50}$$

where P(t) is the number of fish in the pond at time t.

For this system, if the initial fish population is 20 fish, what is the value of P(t) as t becomes large? What if the initial population is $P_o = 200$ fish? Explain your results and show how you arrived at these conclusions.

b. After a long time, the fish/pond system in Part a is perturbed by allowing limited fishing (i.e. harvesting). If the number of fish removed from the pond per year is 42, the mathematical model now becomes

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 42$$

For this new system, what is the fish population as t becomes large? Again, explain the logic here!!!

Problem 3

A rabbit farmer raises rabbits for most of the local pet stores in the area. If we let t denote time in years, then P(t) represents the number of rabbits on the farm at any time t.

Under current breeding conditions at the farm, the birth rate per unit population is given by

$$\beta = 2.22 - 0.001P$$

The first term in this expression represents the nominal normalized birth rate (i.e. 2.22 rabbits per year per rabbit) and the second term accounts for a controlled decrease in normalized birth rate when the population gets large.

Assume that the farm is well managed so that the death rate is relatively low at only 0.02 deaths per year per rabbit (i.e. 2 deaths occur per year for every 100 rabbits). Also assume that the rabbit farm supplies about 210 rabbits per year (on the average) to the local marketplace.

a. With the above description, formally **develop** and **explain** a mathematical model that describes the rabbit population versus time. Show that this model can be written, after some manipulation, as

$$\frac{dP}{dt} = -0.001(P-100)(P-2100)$$

b. Draw the phase line for the ODE given in Part a and address the stability of the critical points for this autonomous system. Also carefully sketch a series of solution curves for various values of $P(0) = P_o$. In particular, if $P_o = 600$ rabbits, what is the value of P(t) as t becomes large?

Problem 4

Consider the nonlinear autonomous ODE:

$$\frac{dx}{dt} = 4x - x^3 \qquad \text{with} \quad x(0) = x_o$$

We are interested in both qualitative and quantitative solutions to this system.

a. As part of the qualitative analysis, draw the phase line for this autonomous system and carefully sketch the solution curves for the following set of initial conditions:

$x_o = -2$ $\quad x_o = -1$ $\quad x_o = 0$ $\quad x_o = 2$ $\quad x_o = 3$

b. For the quantitative component, find the unique solution for the case where $x_o = 3$.

c. Consider your solutions to Parts a and b. For the case where $x_o = 3$, what is x(t) as t becomes large? Do the qualitative and quantitative solutions agree? Explain.

Problem 5

a. A motorboat starts from rest. The motor provides a constant acceleration of 4 ft/s^2, but the water causes a deceleration of $v^2/400$ ft/s^2. With this information, determine an analytical expression for v(t).

b. What is the value of the boat's speed at 10 seconds? Also determine the limiting speed of the boat for the conditions stated in Part a.

Problem 6 (Note: This problem is optional and is worth up to 10 extra points)

Consider an object with initial velocity v_o that encounters a medium whose resistance is proportional to the object's velocity to the 3/2 power. The mathematical model for the object is then given by

$$\frac{dv}{dt} = -kv^{3/2} \qquad \text{with} \quad v(0) = v_o$$

a. For this system, show that the distance traveled within the medium is given by (with $x_o = 0$),

$$x(t) = \frac{2\sqrt{v_o}}{k}\left(1 - \frac{2}{kt\sqrt{v_o} + 2}\right)$$

where k is the resistance coefficient (i.e. the proportionality constant implied in the above problem description).

b. Also, for this system, find an expression for the total distance traveled within the medium. Explain your logic!!!