Differential Equations (92.236)

Homework Assignment #4   Spring 2007

Linear First-Order Equations

Problem 1

Solve the following linear ODEs:

a. $xy' + 5y = 7x^2$ with $y(2) = 5$

b. $ty' + 2y = t^2$

c. $2y' + t(2y - 1) = 0$

d. $xy' = 2y + x^3 \cos x$

e. $(x^2 + 4)y' + 3xy = x$ with $y(0) = 1$

f. $y' = 2xy + 3x^2 e^{x^2}$ with $y(0) = 5$

Problem 2

A 100 gallon tank is initially full of water. Pure water flows into the tank at a rate of 1 gallon per minute (1 gpm). At the same time, another input stream containing brine with 0.25 lbm of salt per gallon flows into the tank. Stream #2 also has a flow rate of 1 gpm.

If the tank is well-mixed and the outlet flow rate of the mixture is 2 gpm, find an expression for the amount of salt in the tank versus time.

Problem 3

a. Consider a two-tank system. Tank 1 initially contains 100 gallon of pure ethyl alcohol and Tank 2 contains 100 gallons of pure water. Pure water flows into Tank 1 at q = 10 gal/min, the well-mixed solution in Tank 1 flows into Tank 2 at the same rate, and the well-mixed solution in Tank 2 flows out of Tank 2 and is discarded. The flow rate is constant throughout the system. Determine the alcohol content versus time in the two tanks. What is the maximum amount of alcohol in Tank 2 and at what time does this condition occur?

   Hint: Carry along the variable q in the analysis for Part a so that q appears directly in the final equations for $V_1(t)$ and $V_2(t)$, where these represent the alcohol volumes versus time for the two tanks, respectively. Then, to answer the questions in Part a, you can simply substitute the indicated numerical value for q. Doing this will save time on Part b, since you can simply substitute the three desired flow rates given into the general equations derived in Part a (see below).

b. Also make a plot of the alcohol volume in both tanks versus time using Matlab and address how the V(t) curves behave as a function of the flow rate in the system. In particular, you should let q = 10, 30, and 50 gal/min and generate a curve for each case, all plotted on the same set of axes. Thus, you should have two plots, one for $V_1(t)$ and one for $V_2(t)$, and each plot should have three properly-labeled curves that correspond to the three different flow flows indicated above. Do the curves make sense? Explain.

As documentation, turn in your derivations and numerical results for Part a and copies of the two plots, a listing of the Matlab program that generated your results, and a brief description that addresses the suitability of your results for Part b of this problem!

Problem 4 (Optional Extra Credit – up to 15 points)

Technetium 99m is one of the most important radioisotopes used for medical diagnostics. Tc99m decays via isomeric transition to the ground state of Tc99 with a half-life of about 6 hours and emits a 0.14 MeV photon that can be easily detected by medical imaging equipment. It is formed from the β^- decay of Molybdenum 99 which has a half-life of 66 hours. The production/decay scheme is as follows:

$$\text{Mo99} \xrightarrow{\beta^-} \begin{cases} \text{Tc99m} & (0.875) \\ \quad\downarrow \text{IT} & \\ \text{Tc99} & (0.125) \end{cases}$$

Note that the isotope used for medical diagnostics is Tc99m and it is formed with a yield of 0.875 (i.e. about 12.5 % of the Mo99 disintegrations go directly to Tc99 at the ground state).

The 6 hr half-life of Tc99m, while beneficial in reducing the patient's exposure to radiation, is disadvantageous in terms of maintaining adequate supplies. Therefore, Tc99m generators, which periodically extract Tc from Mo, are maintained and centrally located for ease of distribution. In your work, assume that the extraction process is 80% efficient (that is, 80% of the available Tc99m is removed from the generator at each extraction step).

Our goal is to study the kinetics of the Tc99m generator and to determine the number of stages and optimal times to perform the extraction process. To develop the defining equations, let isotope 1 denote Mo99 and isotope 2 refer to the Tc99m. Also, for convenience, we will write the balance equations with units of activity, where $A(t) = \lambda N(t)$ is the activity of a radioactive isotope with decay constant λ and N(t) atoms present at time t. The curie (Ci) is the most common unit used to represent activity (1 Ci = $3.7\text{x}10^{10}$ decays/sec).

The balance equation for Mo99 only includes a loss term from radioactive decay and it can be written as

$$\frac{d}{dt}N_1(t) = -\lambda_1 N_1(t) = -A_1(t)$$

or multiplying by λ_1 gives a balance equation for the activity of Mo99,

$$\frac{d}{dt}A_1(t) = -\lambda_1 A_1(t) \qquad (1)$$

Similarly, focus on the Tc99m population shows a production path from the decay of Mo99 and a loss rate associated with its decay to the ground state of Tc99. Putting these production and loss rates within a balance equation for Tc99m gives

$$\frac{d}{dt}N_2(t) = \gamma A_1(t) - \lambda_2 N_2(t) = \gamma A_1(t) - A_2(t)$$

where γ is the yield of Tc99m from the decay of Mo99. As before, multiplying by λ_2 gives

$$\frac{d}{dt}A_2(t) = \lambda_2 \gamma A_1(t) - \lambda_2 A_2(t) \tag{2}$$

which represents the desired balance equation for the activity of Tc99m versus time.

With the above background information, consider the following problems/analyses:

a. Solve eqns. (1) and (2) for the activity of Mo99 and Tc99m versus time. If the initial conditions for the two activities are A_{1o} = 100 Ci and A_{2o} = 5 Ci, what is the best time to extract the Tc99m from the Tc99m generator? Explain! (**Hint:** It is best to answer this question by simply observing a plot of the respective activities versus time. Also, an estimate, rounded to the nearest hour, is all that is needed here).

b. At the time determined from Part a, assume that 80% of the Tc99m available is extracted from the generator for use in a local hospital (assume that this can be performed very quickly). However, the Mo99 in the generator still continues to decay to Tc99m, allowing subsequent extractions at later times. Under this scenario, when would it be best to perform the second extraction? What is the activity of this batch of Tc99m? Explain the logic used here and show your supporting analysis (again, preferably in graphical form).

c. Simulate the above optimal process several times, each time removing 80% of the Tc99m present until the minimum usable Tc99m activity level of 20 Ci is reached. How many production-extraction cycles are possible with the original 100 Ci Mo99 source? Create a single plot of the activities of Mo99 and Tc99m versus time to illustrate the complete production-extraction process for a single Mo99 batch. Does this plot of the activities within the generator make sense? Can you visualize the actual processes that affect these quantities? Explain.

Documentation for this extra credit problem should include a description of the overall solution strategy, the details of the actual solution technique, and the key results of your analyses including the requested plots and comparisons. Also include any program listings (or tabular data if using a spreadsheet) used in generating the summary plots. This problem requires a fair amount of thought and analysis as well as the ability to solve some simple first order ODEs. Thus, it represents a good self evaluation of your personal progress thus far in this course. It should be a challenging and rewarding project. Good luck and have fun…