Differential Equations (92.236)
Applications: First Order Systems

Introduction

A series of examples that include some model development from basic principles and at least one specific example that uses the resultant differential equations has been developed to give you more practice and experience with using your knowledge of mathematics and differential equations in practical applications. Here we focus on physical systems that involve first order ODEs. These examples use a mix of analytical and numerical techniques and some include simulation and plotting of the solutions within Matlab. These examples simply provide a larger inventory of worked problems than available in your text and they also illustrate the wide range of applications for the methods discussed in Chapters 1 and 2 of Edwards and Penny.

The currently available application areas and specific examples involving first order ODEs are listed below:

Newton’s Second Law
   Bombs Away
Torricelli’s Law
   Design of a Water Clock
Newton’s Law of Cooling
   When did he die?
Radioactive Decay
   Old Moon Rocks
Mixture Problems
   Two Salty Tanks
2-D Velocity Profile
   River Crossing
Population Models
   A Fish Story
Newton’s Second Law

Model Development

If we define $\vec{p} = m\vec{v}$ as the momentum of an object with mass $m$ and velocity vector $\vec{v}$, then Newton's Second Law of motion says that the rate of change of momentum is equal to the sum of all the external forces acting on the object. In equation form, this can be written as

$$\frac{d}{dt} \vec{p} = \frac{d}{dt} (m\vec{v}) = \vec{F} = \sum_i \vec{F}_i \tag{1}$$

This representation is a little more complicated than the usual statement of Newton's Second Law, which is often written as

$$F = ma \tag{2}$$

where the unit of force is the Newton (N) [with $N = \text{kg-m/s}^2$].

To systematically reduce eqn. (1) to eqn. (2) we make a series of assumptions and definitions, as follows:

1. Assume 1-D motion -- this removes the vector notation, giving

$$\frac{d}{dt} (mv) = F$$

2. Assume that the object has constant mass (i.e. $\frac{dm}{dt} = 0$), giving

$$\frac{d}{dt} (mv) = m\frac{dv}{dt} + \frac{dm}{dt} v = m\frac{dv}{dt}$$

3. Now define velocity as the rate of change of position and acceleration as the rate of change of velocity, or

$$v = \frac{dx}{dt} \quad \text{and} \quad a = \frac{dv}{dt}$$

Now with these definitions, eqn. (1) becomes eqn. (2).

Some Examples

There is a rich variety of examples and realistic physical situations that use either eqn. (1) or eqn. (2) as the basis for the mathematical modeling and simulation of the given system -- a falling parachutist, a low altitude rocket, a boat’s movement in the water, a car moving down the road, and so on. Resistance to motion (i.e. friction) is usually a complicating feature, and various levels of approximation are often necessary to make a particular problem tractable. For example, some common approximations for air resistance upon a falling parachutist are:
**No Air Resistance**

The only external force is the gravity force, $F_g = -mg$. Therefore Newton’s Law gives

$$ma = m \frac{dv}{dt} = -mg$$

Therefore,

$$\frac{dv}{dt} = -g$$

and, from the definition of velocity as the rate of change of position, we have

$$\frac{dy}{dt} = v$$

These are separable equations in this simple situation, and are easily solvable using standard methods.

**Air Resistance Proportional to Speed**

In this case the force balance gives

$$F = F_g + F_a$$

and, for $F_a = -kv$, we have

$$m \frac{dv}{dt} = -mg - kv$$

Therefore, the mathematical model for this situation is

$$\frac{dv}{dt} + \frac{k}{m} v = -g$$

which is now a first order linear ODE (which is still easily solvable). Note that the negative sign in the air resistance term implies that the friction force is in a direction opposite to the direction of motion.

**Air Resistance Proportional to the Square of the Speed**

In this case the friction force can be written as $F_a = \pm kv^2 = -kv |v|$ ($+kv^2$ for downward motion and $-kv^2$ for upward motion). Thus, the full mathematical model can be written as

$$m \frac{dv}{dt} = -mg \pm kv^2 = -mg - kv |v|$$

or

$$\frac{dv}{dt} = -g - \frac{k}{m} |v|$$

In this latter expression, the magnitude of velocity, $|v|$, is always positive.
Analytical solutions for problems using this air friction model are somewhat more complicated. For example, consider the *downward motion case*,

\[ \frac{dv}{dt} = -g + \frac{k}{m} v^2 \]

This can be manipulated algebraically as follows:

\[ \frac{dv}{dt} = \frac{k}{m} \left( v^2 - \frac{gm}{k} \right) = a\left( v^2 - b^2 \right) \]

where \( a \) and \( b \) are simply constants defined by

\[ a = \frac{k}{m} \quad \text{and} \quad b^2 = \frac{gm}{k} \]

The above 1st order ODE is separable and can be written as

\[ \frac{dv}{b^2 - v^2} = -adt \]

Integrating both sides (using standard integral tables for the LHS) gives

\[ \frac{1}{2b} \ln \left| \frac{v + b}{v - b} \right| = -at + c'' \]

and a little algebra gives the final general solution

\[ \ln \left| \frac{v + b}{v - b} \right| = -2abt + c' \]

or

\[ \frac{v + b}{v - b} = ce^{-2abt} \]

Applying the initial condition, at \( t = 0 \), \( v(0) = v_0 \), gives \( c = \frac{v_0 + b}{v_0 - b} \) and the unique solution becomes

\[ \frac{v + b}{v - b} = \left( \frac{v_0 + b}{v_0 - b} \right) e^{-2abt} \]

To write this as an explicit solution of time [i.e. \( v = f(t) \)], we need to do a little more algebra. If we define a new variable \( z(t) \), where

\[ z(t) = \left( \frac{v_0 + b}{v_0 - b} \right) e^{-2abt} \]

then the solution can be written as
\[ \frac{v + b}{v - b} = z \]

or

\[ v + b = zv - zb \]
\[ v(1-z) = -b(1+z) \]
\[ v = -b \left( \frac{1+z}{1-z} \right) \]

and the final form for the velocity versus time becomes

\[ v(t) = -b \left[ \frac{1+z(t)}{1-z(t)} \right] \]

The point here is that as the math model complexity increases (in this case, the \( v^2 \) term makes this system nonlinear), so does the solution scheme (usually). In many cases, no analytical solution is possible and we must resort to numerical techniques for the system simulations.

**Variable Gravity Case**

In the above cases the gravity force was assumed to be constant since, for motion near the earth’s surface, \( g \) does not change significantly. However if the distances traveled are large, then a variable gravity force must be considered. *Newton’s Law of Gravitation* gives the gravity force as

\[ F_g = \frac{GMm}{r^2} \]

where \( G = 6.67260 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg} \), \( M \) and \( m \) are point masses (or homogeneous spheres), and \( r \) is the distance between the point masses (or the centers of the homogeneous spheres). Note that at the Earth’s surface, \( y = r = R \) is the radius of the Earth, and the gravitational acceleration is

\[ g = \frac{GM}{R^2} \approx 9.8 \text{ m/s}^2 \]

As a specific example, we can ignore air friction for simplicity, and Newton’s 2\(^{nd}\) Law, \( F = F_g \), gives the following mathematical model:

\[ m \frac{dv}{dt} = -m \left( \frac{GM}{y^2} \right) \]

But, from the definition of velocity as the rate of change of position, we have

\[ \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v \frac{dv}{dy} \]

Therefore, the above model can be rewritten with the explicit dependence on time removed (i.e. \( v \) is now only an explicit function of \( y \)), giving
\[
mv \frac{dv}{dy} = -m \left( \frac{GM}{y^2} \right)
\]

This ODE is separable,

\[
v dv = -GM y^{-2} dy
\]

and both sides can be integrated to gives the general solution,

\[
\frac{v^2}{2} = \frac{Gm}{y} + c
\]

With a specific set of initial conditions, one can easily evaluate this expression for the constant \(c\) and determine a unique relationship for \(v(y)\).
Bombs Away

Problem Description:
A bomb is dropped from a balloon hovering at an altitude of 800 ft. A gun emplacement is located on the ground directly below the balloon. The gun fires a projectile straight upward toward the bomb exactly 2 seconds after the bomb is released. With what initial speed should the projectile be fired in order to intercept the bomb at an altitude of 400 ft.?


Problem Solution:
This is a problem that involves Newton’s Second Law. Since no mention of air resistance is given in the problem description, let’s simply assume no air friction. In this case we have two objects in motion -- the bomb and the projectile. Let’s do separate analyses on each object.

Bomb Analysis:

A force balance on the bomb gives
\[ m_b \frac{dv_b}{dt} = -m_b g \]

This equation is separable and easily integrated to give
\[ v_b(t) = -gt + c \]

Applying the initial condition that \( v_b(0) = 0 \) gives \( c = 0 \). Therefore, \( v_b(t) \) is simply
\[ v_b(t) = -gt \]

Now considering the location of the bomb versus time gives
\[ \frac{dy_b}{dt} = v_b \quad \text{and} \quad y_b(t) = -\frac{1}{2}gt^2 + c \]

To determine the integration constant, we use the fact that \( y_b(0) = y_{b0} = 800 \text{ ft} \), so \( c = 800 \text{ ft} \) and
\[ y_b(t) = 800 - \frac{1}{2}gt^2 \]

From the problem statement, we want to know how long it takes for the bomb to reach 400 ft. Therefore,
\[ gt^2 = (800 - 400)2 = 800 \quad \text{and} \quad t = 10\sqrt{\frac{8}{g}} \text{ seconds} \]

With \( g = 32.2 \text{ ft/s}^2 \), the time of impact is \( t = t_f = 4.984 \text{ seconds} \).
**Projectile Analysis:**

Performing a similar analysis on the projectile gives a force balance,

\[ m_p \frac{dv_p}{dt} = -m_p g \]

with solution

\[ v_p(t) = -gt + c \]

In this case the initial condition is written as \( v_p(2) = v_{p0} \), which gives \( c = v_{p0} + 2g \), and

\[ v_p(t) = v_{p0} + (2 - t)g \text{ for } t \geq 2 \]

Now the position of the projectile is given by the solution of

\[ \frac{dy_p}{dt} = v_p \]

which gives

\[ y_p(t) = v_{p0}t + g\left(2t - \frac{1}{2}t^2\right) + c \text{ for } t \geq 2 \]

The projectile is on the ground at \( t = 2 \) seconds, which gives

\[ y_{p0}(2) = 0 = 2v_{p0} + 2g + c \]

or \( c = -2v_{p0} - 2g \). Thus, the projectile position versus time is given as

\[ y_p(t) = v_{p0}(t - 2) + g\left(-2 + 2t - \frac{1}{2}t^2\right) \]

and, for \( y_{p}(t_f) = 400 \text{ ft} \) and \( t_f = 4.984 \text{ s} \), we can solve for \( v_{p0} \), giving

\[ v_{p0} = \frac{1}{t_f - 2}\left[y_p(t_f) - g\left(-2 + 2t_f - \frac{1}{2}t_f^2\right)\right] \]

or

\[ v_{p0} = \frac{1}{2.984}\left[400 - 32.2\left(-4.452\right)\right] = 182 \frac{\text{ft}}{\text{sec}} \]

**A Slight Generalization:**

One can generalize the relationship between impact time and height and the required initial projectile velocity to achieve the desired impact parameters. For example, if we equate the expressions for \( y_b(t) \) and \( y_p(t) \) at \( t = t_f \), we get

\[ y_{b0} = \frac{1}{2}gt_f^2 = v_{p0}(t_f - 2) + g\left(-2 + 2t_f - \frac{1}{2}t_f^2\right) \text{ for } t_f \geq 2 \]

and solving this for the initial projectile velocity, \( v_{p0} \), gives
\[ v_{p0} = \frac{1}{t_f - 2} \left[ y_{bo} - g(t_f - 2) \right] \text{ for } t_f \geq 2 \]

This relationship is plotted below. The plot was generated by Matlab file BOMBS.M (see listing below). The upper curve shows how the impact time varies with initial projectile velocity. Using the expression for \( y_b(t) \), the collision height versus \( v_{p0} \) is also given in the lower curve. The student should review the plots and the Matlab m-file as a straightforward example of function evaluation and plotting in Matlab.
Listing of BOMBS.M

get started
   clear all; close all; nfig = 0;

define some initial parameters
   ybo = 800; % initial height of bomb (ft)
   g = 32.2; % gravitational acceleration constant (ft/s^2)

define range of reasonable collision times (seconds)
   T = linspace(3,7,20);

now evaluate desired quantities
   (Note: "Dot arithmetic" is used here to evaluate the desired quantities at 20
   different collision times in a single statement -- this vector processing
   capability is one of the many very nice features available in Matlab)
   vpo = (ybo - g*(2*T-2))/(T-2); % initial velocity for given collision times
   H = ybo -(1/2)*g*T.^2; % collision height vs collision time

plot desired quantities
   (Note: One usually does not plot two functions with different units on the same
   plot. One way to plot two different quantities against the same variable on a
   single page is to use Matlab's subplot capability. Here we plot collision time
   vs initial velocity in the upper plot and show the collision height vs initial
   velocity in the lower plot.)
   nfig = nfig+1; figure(nfig)
   subplot(2,1,1),plot(vpo,T,'r-','LineWidth',2), grid
   title('Collision Parameters for the "Bombs Away" Example')
   ylabel('Collision Time (sec)')

   subplot(2,1,2),plot(vpo,H,'g:','LineWidth',2), grid
   ylabel('Collision Height (ft)')
   xlabel('Initial Projectile Speed (ft/sec)')
   gtext('Delay Time = 2 sec')

end of program
Newton’s Law of Cooling

Basic Terminology

The relationship for the energy flow rate due to convective heat transfer at the interface between a solid object and a fluid environment is referred to as *Newton’s Law of Cooling*. If we denote the surface temperature of the solid as \( T_s \) and the bulk fluid temperature as \( T_\infty \), then the energy transfer rate between the solid and fluid is proportional to the surface area for heat transfer, \( A \), and the temperature difference between the solid surface and the bulk fluid temperature. The proportionality constant is often given the symbol \( h \), which is known as the heat transfer coefficient.

In equation form, the above statement for Newton's Law of Cooling can be written as

\[
q = hA(T_s - T_\infty)
\]

This assumes that energy flow is from the solid to the fluid (i.e. cooling of the solid). A units analysis for eqn. (1) gives

\[
W = J/s = (\text{units of } h)(m^2)(^\circ C)
\]

Thus, we see that the units of the heat transfer coefficient, \( h \), are \( W/m^2\cdot{}^\circ C \) and that the energy transfer rate, \( q \), has units of \( W \) or \( J/s \).

Often the expression for Newton’s Law of Cooling is divided by the heat transfer area and written as

\[
q'' = \frac{q}{A} = h(T_s - T_\infty)
\]

where the energy flow rate per unit area, \( q'' \), has units of \( W/m^2 \) in the mks system of units. Note here that the prime notation is sometimes used in the heat transfer literature to denote energy flow rate per unit area - it does not mean the second derivative.

Now assume that the solid object is relatively small and that it has a high thermal conductivity. These qualifications allow us to assume that the temperature variation within the object is not significant. Mathematically, we approximate the temperature distribution throughout the object as a single constant temperature, or

\[
T(\vec{r}, t) \Rightarrow T_s(t) \Rightarrow T(t)
\]

where \( T(t) \) denotes the spatially constant temperature throughout the object (only varies with time). This sequence of assumptions is often referred to as a Lumped Parameter Model for the system.

Lumped Parameter Energy Balance

An overall energy balance for the solid object can be written as

\[
\begin{bmatrix}
\text{rate of change} \\
\text{of energy in CV}
\end{bmatrix} = \begin{bmatrix}
\text{net flow rate of energy} \\
\text{across surface of CV}
\end{bmatrix} + \begin{bmatrix}
\text{energy generation} \\
\text{rate in CV}
\end{bmatrix}
\]

where the object’s volume is referred to as the control volume (CV).

To write this expression in equation form, we first need to make some definitions. Let’s denote the total energy (J) within control volume as $E_{CV}$, the mass (kg) of the object as $m$, and the energy per unit mass (J/kg) with the symbol $e$. Using this notation, we have the total energy given by

$$E_{CV} = me$$

(5)

In general the total energy can be the sum of several components (internal, kinetic, potential, magnetic, etc.). However, for a stationary solid object, the total energy is completely dominated by the internal motion of the molecules making up the solid (i.e. its internal energy). Also it makes sense that this molecular motion should be related to the temperature of the object. Therefore, we have

$$e \approx u = c_pT$$

(6)

where $u$ is the internal energy per unit mass (J/kg), $c_p$ is the specific heat of the solid material (J/kg·°C), and $T$ is the object’s temperature.

Now with all this background, we are ready to write a mathematical expression for the energy balance, as follows:

$$\frac{d}{dt}(me) = -hA(T_s - T_\infty) + 0$$

(7)

where we have assumed that there is no internal energy generation within the object. For constant mass, the LHS can be written as

$$\frac{d}{dt}(me) = m \frac{d}{dt}e = m \frac{d}{dt}u = m \frac{d}{dt}(c_pT)$$

(8)

Also, if the specific heat is constant over the temperature range (or a suitable average value is used), we have

$$\frac{d}{dt}(me) = m \frac{d}{dt}(c_pT) = mc_p \frac{d}{dt}T$$

(9)

With this simplification, the final energy balance for a lumped parameter system with constant properties becomes

$$mc_p \frac{d}{dt}T = -hA(T - T_\infty)$$

(10)

where the net energy transfer rate for convective cooling is governed by Newton’s Law of Cooling (the negative sign implies a net loss of energy from the solid).

Dividing by the constant parameters gives

$$\frac{d}{dt}T = \beta(T - T_\infty) \quad \text{where} \quad \beta = \frac{hA}{mc_p}$$

(11)

Note that this last expression is consistent with that given in the Differential Equations text by Edwards and Penny. A somewhat subtle difference is that most engineering references refer to
Newton’s Law of Cooling as the mathematical relationship given in eqn. (1), whereas most math texts use eqn. (11) as the mathematical representation of this law. In most applications of interest it is fairly obvious which expression should be used based on the problem context. Of importance, however, is the fact that eqn. (11) is formally derived using eqn. (1) and a series of assumptions that include a spatially constant temperature throughout the solid, constant material properties, and no internal heat generation. The insight from the more formal development given here should be invaluable in practical application of the physical and mathematical representations of Newton’s Law of Cooling.
When did he die?

Problem Description:
A homicide victim was found in a room that is kept at a constant temperature of 70 °F. A body temperature measurement was made at time $\tau$ and another was made one hour later. The results were:

$$T(\tau) = 80 \text{ °F} \quad \text{and} \quad T(\tau + 1) = 75 \text{ °F}$$

where time is measured in hours. Assuming that the victim’s temperature was 98.6 °F just before death, determine the time of death relative to time point $\tau$.


Problem Solution:
This is a problem that involves Newton’s Law of Cooling, where it is assumed that the body is cooled via convection to the room environment and that the body temperature can be characterized by a single value, $T$, that only varies with time. With these assumptions, the governing energy balance can be written as

$$\frac{dT}{dt} = -\beta(T - T_\infty) \quad (1)$$

where $T_\infty$ represents the environment temperature and $\beta$ is a proportionality constant that is related to the material properties of the body and the specific heat transfer environment.

The defining ODE in eqn. (1) is separable, or

$$\frac{dT}{T_\infty - T} = \beta dt \quad (2)$$

and integrating both sides gives

$$-\ln(T_\infty - T) = \beta t + c'$$

$$\ln(T_\infty - T) = -\beta t - c'$$

or

$$T_\infty - T(t) = e^{-\beta t}e^{-c'} = ce^{-\beta t} \quad (3)$$

From the problem statement, we know that the body temperature just before death at $t = 0$ was $T(0) = T_0 = 98.6 \text{ °F}$. Using this initial condition in eqn. (3) gives $c = T_\infty - T_0$, which leads to

$$T_\infty - T(t) = (T_\infty - T_0)e^{-\beta t}$$

or

$$\frac{T(t) - T_\infty}{T_0 - T_\infty} = e^{-\beta t} \quad (4)$$
This general expression for $T(t)$ represents the desired relationship for the body temperature as a function of time after death (at $t = 0$).

Now, upon arrival on the scene at $t = \tau$, the detective in charge of the homicide investigation measured the victim’s temperature to be $T(\tau) = 80 \degree F$. One hour later a second measurement gave $T(\tau + 1) = 75 \degree F$.

From these two measurements we can determine the proportionality constant in eqn. (4) and the time of death. To do this, we evaluate eqn. (4) at the two measurement times, giving

$$T(\tau) - T_\infty \over T_0 - T_\infty = e^{-\beta\tau}$$

(5)

and

$$T(\tau + \Delta t) - T_\infty \over T_0 - T_\infty = e^{-\beta(\tau + \Delta t)}$$

(6)

Now dividing eqn. (5) by eqn. (6) gives

$$\frac{T(\tau) - T_\infty}{T(\tau + \Delta t) - T_\infty} = \frac{e^{-\beta\tau}}{e^{-\beta(\tau + \Delta t)}} = e^{\beta\Delta t}$$

Therefore, solving this expression for $\beta$ gives

$$\beta = \frac{1}{\Delta t} \left[ \ln \left( \frac{T(\tau) - T_\infty}{T(\tau + \Delta t) - T_\infty} \right) \right]$$

(7)

Now with the given data

$\Delta t = 1 \text{ hr} \quad T(\tau) = 80 \degree F \quad T(\tau + \Delta t) = 75 \degree F$

we have,

$$\beta = \frac{1}{1 \text{ hr}} \left[ \ln \left( \frac{80 - 70}{75 - 70} \right) \right] = \ln 2 = 0.693 \text{ hr}^{-1}$$

With this proportionality constant, we can evaluate the time of death relative to time $t = 0$. In particular, solving eqn. (5) for $\tau$ gives

$$\tau = -\frac{1}{\beta} \ln \left( \frac{T(\tau) - T_\infty}{T_0 - T_\infty} \right) = -\frac{1}{\ln 2} \ln \left( \frac{80 - 70}{98.6 - 70} \right) = -\frac{1}{\ln 2} \ln \left( \frac{10}{28.6} \right) = 1.516 \text{ hr}$$

For example, if the first temperature measurement was made at 12 noon, then the death occurred at about 10:29 AM (a little over 1.5 hr before the detectives arrived on the scene).
Overview

Torricelli’s Law is a mathematical relationship between the flow rate of fluid from a draining tank and the height of fluid in the tank. The formal development of this relationship gives a lot of insight into the use of the resultant formula, including the inherent assumptions that place limitations on its application in certain situations.

We can develop the desired relationship by considering a tank containing a fluid that can be drained by removing a plug at the bottom of the tank -- a rough sketch of such a system is given below.

\[
\begin{align*}
A &= \text{surface area of tank} \\
V &= \text{volume of tank} = Ah \\
A_e &= \text{cross-sectional area at exit} \\
P_1 &= \text{pressure at bottom of tank}
\end{align*}
\]

Assume that the exhaust pressure is a given constant, \(P_a\), and that the inlet flow rate is zero for the current system. The fluid volume at any time is \(V(t)\) and the output of interest is the exit mass flow rate, \(\dot{m}_e(t)\). A set of mass and energy balances are performed on this system, as follows:

**Mass Balance**

As with any balance equation, the rate of change of the quantity of interest is equal to the difference in its production and loss rates within the desired volume. In this case, if we choose the mass of fluid in the tank as the quantity of interest, we have

\[
\begin{align*}
\frac{dM}{dt} &= \text{rate of change of mass in tank} \\
&= \text{mass flow rate into tank} - \text{mass flow rate out of tank}
\end{align*}
\]

The mass is simply the fluid density times the volume, \(M = \rho V\), with units of \((\text{kg/m}^3)(\text{m}^3) = \text{kg}\). The mass flow rate at the tank exit is given in terms of the density, exit area, and exit velocity of the fluid, or

\[
\dot{m}_e = \rho A_e v_e
\]

with consistent units, \((\text{kg/m}^3)(\text{m}^2)(\text{m/s}) = \text{kg/s}\).

For no inflow and constant density, eqn. (1) becomes

\[
\frac{dM}{dt} = \rho \frac{dV}{dt} = -\dot{m}_e = -\rho A_e v_e
\]

or

\[ \frac{dV}{dt} = -A_e v_e \]  \hspace{1cm} (4)

where \( V \) is the volume of fluid in the tank, \( A_e \) is the exit flow area, and \( v_e \) is the average exit fluid velocity.

**Energy Balance (on control volume at tank exit)**

Now we would like to relate the exit velocity, \( v_e \), to the height of the fluid in the tank. This can be done by performing an energy balance on a small control volume at the tank exit,

\[
\begin{bmatrix}
\text{rate of change of energy in CV} \\
\text{flow into CV}_1 \\
\text{flow out of CV}_2
\end{bmatrix} = \begin{bmatrix}
\text{rate of energy} \\
\text{flow into CV}_1 \\
\text{flow out of CV}_2
\end{bmatrix}
\]  \hspace{1cm} (5)

or

\[ \frac{d}{dt} E_{CV} = \dot{E}_1 - \dot{E}_2 \]  \hspace{1cm} (6)

where point 1 is at the inlet to the control volume (CV) and point 2 is at the exit as noted in the above sketch. If we assume no mass or energy accumulation in this small control volume at the bottom of the tank, then

\[ \frac{d}{dt} E_{CV} = 0 \quad \text{and} \quad \frac{d}{dt} M_{CV} = 0 \]  \hspace{1cm} (7)

The no mass accumulation assumption in the CV says that the mass flow rates at the inlet and exit of the CV are equal,

\[ \dot{m}_e = \dot{m}_1 = \dot{m}_2 \]  \hspace{1cm} (8)

and, similarly, with no energy accumulation, the change in total energy from the inlet to the exit of the CV is zero.

The total energy is usually written as the sum of four major components -- internal energy (U), kinetic energy (KE), potential energy (PE), and, for problems involving flow, a term that accounts for the work associated with moving the fluid into and out of the CV [i.e. the flow work (FW) term]. Here we are implicitly neglecting electrical or magnetic energy effects. Thus, the change in energy from the inlet to the exit of the CV can be written as

\[ \Delta E = E_1 - E_2 = 0 = \Delta U + \Delta KE + \Delta PE + \Delta FW \]  \hspace{1cm} (9)

Each of these terms is usually written as an energy flow rate with units of energy per unit time. For example, the internal energy per unit mass \((u_i)\) times the mass flow rate gives the internal energy flow rate at position \(i\) (where the \(i\) subscript denotes either the inlet or exit of the CV), or

\[ \begin{bmatrix}
\text{rate of change of internal energy in CV} \\
\end{bmatrix} = \dot{m}_i u_i - \dot{m}_2 u_2 = \dot{m}_e (u_1 - u_2) \]  \hspace{1cm} (10)

Using similar arguments, eqn. (9), written as a rate equation, becomes

\[ \Delta \dot{E} = \dot{E}_1 - \dot{E}_2 = 0 = \dot{m}_e (u_1 - u_2) + \frac{\dot{m}_e}{2} (v_1^2 - v_2^2) + \dot{m}_e g (z_1 - z_2) + \frac{\dot{m}_e}{\rho} (P_1 - P_2) \]  \hspace{1cm} (11)
where each of the four components have consistent units, as follows:

\[
\begin{align*}
\text{rate of change of internal energy in CV} &= \dot{m}_c (u_1 - u_2) \quad \Rightarrow \quad \left( \frac{\text{kg}}{\text{s}} \right) \left( \frac{\text{J}}{\text{kg}} \right) = \frac{\text{J}}{\text{s}} \quad = \frac{\text{N} \cdot \text{m}}{\text{s}} \\
\text{rate of change of kinetic energy in CV} &= \frac{\dot{m}_c}{2} (v_1^2 - v_2^2) \quad \Rightarrow \quad \left( \frac{\text{kg}}{\text{s}} \right) \left( \frac{\text{m}^2}{\text{s}^2} \right) = \left( \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \right) \frac{\text{m}}{\text{s}} = \frac{\text{N} \cdot \text{m}}{\text{s}} \\
\text{rate of change of potential energy in CV} &= \dot{m}_c g (z_1 - z_2) \quad \Rightarrow \quad \left( \frac{\text{kg}}{\text{s}} \right) \left( \frac{\text{m}^2}{\text{s}^2} \right) \text{(m)} = \frac{\text{N} \cdot \text{m}}{\text{s}} \\
\text{rate of change of flow work} &= \frac{\dot{m}_c}{\rho} (P_1 - P_2) \quad \Rightarrow \quad \left( \frac{\text{kg}}{\text{s}} \right) \left( \frac{\text{m}^3}{\text{kg} \cdot \text{m}^2} \right) \frac{\text{N}}{\text{m}} = \frac{\text{N} \cdot \text{m}}{\text{s}}
\end{align*}
\]

In the above expressions we have used the fact that work is the result of a force moving through some distance, or

\[ W = \int \vec{F} \cdot \text{d}\vec{s} = \int \vec{F}(x) \text{d}x \quad (\text{for 1-D problems}) \quad (12) \]

Thus a joule (J) is simply a force times distance (N-m), and the rate of change of energy has units of J/s or N-m/s in the metric system of units.

Now to simplify things considerably, we make the following arguments:

1. Since there is no heat addition into the CV, the change in internal energy in the CV is zero.
2. Since, for this part of the problem, there is no change in elevation, there is also no change in potential energy in the CV.
3. The fluid velocity at the bottom of the tank (at point 1) is small compared to the velocity at the small exit opening (at point 2). Note that, since there is no mass accumulation in the control volume, then

\[ \dot{m}_1 = \rho_1 A_1 v_1 = \dot{m}_2 = \rho_2 A_2 v_2 \quad (13) \]

If the density is essentially constant, \( \rho_1 = \rho_2 \), and

\[ v_2 = \frac{A_1}{A_2} v_1 \quad (14) \]

For a small discharge hole relative to the size of the tank, \( A_2 \) is usually much smaller than the cross-sectional area of the tank, \( A_1 \), and \( v_2 \gg v_1 \) (usually). Thus, one usually lets \( v_2^2 \gg v_1^2 \).

With these arguments (i.e. \( \Delta U = \Delta PE = 0 \) and \( v_2^2 \gg v_1^2 \)), the energy balance in eqn. (11) becomes

\[ v_e = v_2 = \sqrt{\frac{2}{\rho} (P_1 - P_2)} \quad (15) \]
This says that the change in pressure from the bottom of the tank to just outside the exit region is the driving force that increases the fluid velocity (i.e. the change in flow work is converted to an increase in kinetic energy of the fluid).

But the pressure at point 1 (at the bottom of the tank) is simply the sum of the pressure at the top of the tank plus the pressure due to the weight of a column of fluid of height $y = h$, or

$$P_1 = P_a + \rho gy \quad (16)$$

If point 2 is open to the atmosphere, we have $P_2 = P_a$. Therefore,

$$\Delta P = P_1 - P_2 = \rho gy \quad (17)$$

and

$$v_e = \sqrt{2gy} \quad (18)$$

where $y$ is the height of fluid in the tank. Equation (18) is the desired relationship we were after and, with this expression for the exit velocity, the mass balance in eqn. (4) becomes

$$\frac{dV}{dt} = -A_e \sqrt{2gy} \quad (19)$$

Finally, we note that actual flow from a nozzle varies somewhat from the above development. In particular, depending on the type of opening, the actual flow area tends to be smaller than the physical area, $A_e$. One often defines an orifice coefficient as the ratio of the observed area to the physical area of the exit nozzle, or

$$c = \frac{A_{\text{observed}}}{A_{\text{physical}}} \quad (20)$$

and the observed area, $cA_e$, is actually used in eqn. (19) instead of the physical area, $A_e$, giving

$$\frac{dV}{dt} = -cA_e \sqrt{2gy} \quad (21)$$

Note that, in practice, $A_{\text{observed}}$ is difficult to obtain in realistic problems so that $c$ is often determined experimentally.

Finally, letting $k = cA_e \sqrt{2g}$, eqn. (21) can be simplified to give

$$\frac{dV}{dt} = -k \sqrt{y} \quad (22)$$

This is **Torricelli’s Law**. It says that the rate of change of volume of fluid in a draining tank is proportional to the square root of the depth of fluid in the tank. Although this statement is all that is needed in many simple situations, the physical understanding and general insight into the use of this law are significantly enhanced by the development of the final mathematical model from basic principles. The student is encouraged to approach every new situation from the basics -- since starting from fundamentals always gives more insight into the particular application or problem of interest.
Design of a Water Clock

Problem Description:
A 12-hour water clock is to be designed with the dimensions shown in the sketch below. The shape of the clock is obtained by revolving the curve $y = f(x)$ around the y axis. Define the shape function, $f(x)$, and the radius of the circular hole at the bottom that gives a constant water level decrease of 4 in/hr.


Problem Solution:
This is a problem that involves Torricelli’s Law. Assuming that all the assumptions associated with Torricelli's law are valid, we have

$$\frac{dV}{dt} = -k \sqrt{y} \quad \text{with} \quad k = A_c c \sqrt{2g} = A_c \sqrt{2g}$$

where $A_c$ is the exit area, $c$ is the discharge coefficient (assumed to be unity), and $g$ is the gravitational acceleration.

To solve this ODE, we first must relate the container volume, $V$, to the height of fluid within the water clock. The volume of water in the container is given by

$$V(y) = \int_0^y A(\eta)d\eta$$

where $A(y)$ is the top surface area for any height, $y$, and the variable $\eta$ is just a dummy variable of integration. We can form the time derivative of the volume as

$$\frac{dV}{dt} = \frac{dV}{dy} \frac{dy}{dt} = A(y) \frac{dy}{dt}$$
Leibnitz Rule

The above implied relationship, \( dV/dy = A(y) \), can be developed formally using Leibnitz Rule, which is a theorem for treating derivatives of integrals with variable limits of integration. Leibnitz’s rule states that, given

\[
F(t) = \int_{a(t)}^{b(t)} \phi(x, t) dx
\]

then

\[
\frac{dF}{dt} = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} \phi(x, t) dx + \phi[b(t), t] \frac{db(t)}{dt} - \phi[a(t), t] \frac{da(t)}{dt}
\]

if \( a(t), b(t) \) and \( \phi(x, t) \) are continuous functions.

For the case of interest here, we have

\[
V(y) = \int_{\eta(0)=0}^{\eta(y)} A(\eta) d\eta
\]

and

\[
\frac{dV}{dy} = \int_{0}^{y} \frac{\partial}{\partial y} A(\eta) d\eta + A(y) \frac{dy}{dy} - A(0) \frac{dy}{dy} 0 + A(y) \times 1 - A_e \times 0
\]

or simply

\[
\frac{dV}{dy} = A(y)
\]

as implied in the above relationship.

Now, Torricelli’s law as expressed in eqn. (1) becomes

\[
A(y) \frac{dy}{dt} = -k \sqrt{y} \quad \text{or} \quad \frac{dy}{dt} = -\frac{k \sqrt{y}}{A(y)} \quad (4)
\]

From the original problem statement, however, the clock shape is obtained by revolving the curve \( y = f(x) \) around the y-axis. Thus, for any \( y \) value, \( A(y) = \pi x^2 \), where \( x \) is the radius at height \( y \). Substitution into eqn. (4) gives

\[
\frac{dy}{dt} = -\frac{k \sqrt{y}}{\pi x^2} \quad (5)
\]

But, based on the design statement, we require \( dy/dt \) to be constant. We can achieve this goal if

\[
y = f(x) = b^2 x^4 \quad (6)
\]

With this functional relationship, the balance equation gives
\[ \frac{dy}{dt} = -\frac{k\sqrt{y}}{\pi x^2} = -\frac{k\sqrt{b^2x^4}}{\pi x^2} = -\frac{kb}{\pi} = \text{const} \quad (7) \]

Thus, the specification that the rate of decrease in water height be constant leads to the shape function given by eqn. (6) – this defines the shape of the container for the water clock.

To explicitly determine the constant, \( b \), we note that at time zero the clock is full of water, and the above diagram gives

\[ y_0 = 4 \text{ ft} \quad \text{and} \quad x_0 = 1 \text{ ft} \]

Putting these values into eqn. (6) gives

\[ b^2 = \frac{y_0}{x_0} = \frac{4 \text{ ft}}{(1 \text{ ft})^4} = 4 \text{ ft}^{-3} \quad \text{or} \quad b = 2 \text{ ft}^{-\frac{3}{2}} \]

Now if we desire a water level decrease of 4 in/hr, we have

\[ \frac{dy}{dt} = -\frac{1 \text{ ft}}{3 \text{ hr}} = -\frac{kb}{\pi} = -\frac{A e b \sqrt{2g}}{\pi} = -\frac{\pi r_e^2 b \sqrt{2g}}{\pi} \]

or

\[ r_e^2 b \sqrt{2g} = \frac{1 \text{ ft}}{3 \text{ hr}} \]

Now, let's work with the units a little to compute a numerical value for the exit radius. The constant \( \sqrt{2g} \) can be written as

\[ \sqrt{2g} = \sqrt{2 \left( 32.2 \text{ ft/s}^2 \right) \left( \frac{3600 \text{ s}}{\text{hr}} \right)^2} = 2.889 \times 10^4 \text{ ft}^{\frac{3}{2}}/\text{hr} \]

and with \( b = 2 \text{ ft}^{-\frac{3}{2}} \), we have

\[ r_e^2 = \left( \frac{1 \text{ ft}}{3 \text{ hr}} \right) \left( \frac{1}{2 \text{ ft}^{-\frac{3}{2}}} \right) \left( \frac{1 \text{ hr}}{2.889 \times 10^4 \text{ ft}^{\frac{1}{2}}} \right) = \left( \frac{1}{3} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2.889 \times 10^4} \right) \text{ ft}^2 \]

or

\[ r_e = \left( 2.402 \times 10^{-3} \text{ ft} \right) \left( \frac{12 \text{ in}}{\text{ft}} \right) = 0.0288 \text{ in} \]

Therefore, the radius of the exit hole is very small indeed - only 0.0288 in!

Thus, we have finally completed the design of the water clock per the above specifications. In particular, our final design has a shape defined by \( y = bx^4 \) and an initial fluid height and radius of 4 ft and 1 ft, respectively. If this clock has a small circular hole at the bottom with a radius of about 0.0288 in for draining the fluid, the water level will decrease by 4 in per hour - which gives the desired 12-hr clock!
Old Moon Rocks

Problem Description:
A particular moon rock is analyzed and it is found to contain equal amounts of Potassium-19 and Argon-18, which are denoted as $^{40}_{19}$K and $^{40}_{18}$Ar, respectively. $^{40}_{19}$K is a radioactive isotope that can decay to $^{40}_{18}$Ar. Note that $T_{1/2} = 1.29 \times 10^9$ yr for potassium decay and that the yield of $^{40}_{18}$Ar is about 11% (i.e. about 1 out of every 9 $^{40}_{19}$K atoms decays to $^{40}_{18}$Ar). Assuming that the initial amount of argon was zero and that all of the argon is the result of potassium decay, estimate the age of the moon rock.


Problem Solution:

Background
This problem is a classical radioactive decay problem. The general balance equation is given by

$$\begin{bmatrix} \text{rate of change} \\ \text{of quantity} \end{bmatrix} = \begin{bmatrix} \text{production rate} \\ \text{of quantity} \end{bmatrix} - \begin{bmatrix} \text{loss rate} \\ \text{of quantity} \end{bmatrix}$$

(1)

where, for this application, the quantities of interest are the amounts of $^{40}_{19}$K and $^{40}_{18}$Ar present at any time. The key to problems of this type is to be able to write mathematical expressions for the production and loss rate terms in eqn. (1).

Radioactive decay is a very complicated physical process that is governed by a relatively simple mathematical relationship. Simply stated, the basic law for radioactive decay is that the decay rate is proportional to the amount present. This means that the decay rate (i.e. loss rate) associated with N atoms of a particular isotope is simply $\lambda N$, where $\lambda$ is a proportionality constant (referred to as the decay constant) and N(t) represents the amount present at time t.

Armed with this physical law, we can now attempt to solve the particular problem of interest. The basic transition scheme for the decay of $^{40}_{19}$K is sketched below:

$$^{40}_{19}K \rightarrow \begin{cases} \text{EC (11%)} \rightarrow ^{40}_{18}Ar \\ \beta^{-} (89\%) \rightarrow ^{40}_{20}Ca \end{cases}$$

This implies that there are two competing nuclear processes that can occur - one referred to as Electron Capture (EC) and the other is called Beta minus ($\beta^{-}$) decay. The electron capture route occurs with a yield of 0.11 and the alternative $\beta^{-}$ reaction occurs with a fractional probability of 0.89 (i.e. 89% of the time).

Note: The electron capture process effectively combines a proton and electron within the atom to form a neutron ($p^+ + e^- \rightarrow n^0$); thus, the proton number is reduced by one but the mass number (number of protons plus neutrons) is unchanged. The $\beta^{-}$ decay process, on the other
hand, converts a neutron into a proton and electron \( \left( n^0 \rightarrow p^+ + e^- \right) \). In this reaction, the mass number is again constant but, this time, the proton number increases by one. Thus, the two competing radioactive decay processes in \(^{40}\text{K}\) yield different daughter products depending upon the particular reaction that takes place (see above sketch of the \(^{40}\text{K}\) decay scheme).

\[ \text{^{40}K Balance} \]

With this background information, we can now set up the appropriate balance equations for this problem. Focusing first on the \(^{40}\text{K}\) isotope, we know that it is radioactive and its loss rate is governed by the basic radioactive decay law. Since no mention is made concerning the production of \(^{19}\text{K}\), we can assume that the production rate of this nuclide is zero. Thus, denoting the number of atoms of \(^{40}\text{K}\) present as \( K(t) \), the general isotope balance for \(^{40}\text{K}\) becomes

\[
\frac{d}{dt} K(t) = 0 - \lambda K(t) \tag{2}
\]

This is a separable equation which can be written as

\[
\frac{dK}{K} = -\lambda dt
\]

Integrating both sides gives

\[
\ln K = -\lambda t + c'
\]

and taking antilogarithms, we have

\[
K(t) = e^{-\lambda t + c'} = e^{c} e^{-\lambda t} = ce^{-\lambda t}
\]

Finally, specifying the initial condition as \( K(0) = K_0 \), we see that at \( t = 0 \), we get \( K(0) = K_0 = c \). Therefore, the number of atoms of \(^{40}\text{K}\) versus time is simply

\[
K(t) = K_0 e^{-\lambda t} \tag{3}
\]

\[ \text{^{40}Ar Balance} \]

Now, performing a similar analysis for \(^{40}\text{Ar}\), we first note that \(^{40}\text{Ar}\) is a stable isotope. Therefore it has no loss rate. Also, its only production path is from the decay of \(^{40}\text{K}\) with a yield of \( \gamma = 0.11 \). Thus, the balance equation for \(^{40}\text{Ar}\) is

\[
\frac{d}{dt} A(t) = \gamma \lambda K(t) - 0 \tag{4}
\]

where \( A(t) \) represents the number of atoms of \(^{40}\text{Ar}\) present at time \( t \), and \( \lambda K(t) \) is the total loss rate of \(^{40}\text{K}\) (notice that the production rate of \(^{40}\text{Ar}\) is simply the yield, \( \gamma \), times the total loss rate of \(^{40}\text{K}\)).
Equation (4) is easily solved by substitution of eqn. (3),
\[
\frac{dA}{dt} = \gamma \lambda K_0 e^{-\lambda t}
\]
and integration of both sides to give
\[
A(t) = \gamma \lambda K_0 \int e^{-\lambda t} dt = \gamma \lambda K_0 \left[ \frac{1}{-\lambda} \left( e^{-\lambda t} + c \right) \right] = c - \gamma K_0 e^{-\lambda t}
\]
Specifying $A(0) = A_0$ as the initial condition, we have
\[
at t = 0, \quad A(0) = A_0 = c - \gamma K_0 \quad \text{or} \quad c = A_0 + \gamma K_0
\]
Therefore, the desired expression for $A(t)$ becomes
\[
A(t) = A_0 + \gamma K_0 (1 - e^{-\lambda t}) \quad (5)
\]
For this specific case, we assume $A_0 = 0$ as given in the problem specification. Thus, eqn. (3) and eqn. (5) with $A_0 = 0$ are the desired result of our model development and analysis for this problem.

**So how old are the moon rocks?**

Now, to answer the specific question concerning the age of the moon rocks, we have the added information that the measured amounts of $^{40}K$ and $^{40}Ar$ are equal. This measurement was made at some time, $\tau$, relative to the moon’s origin at $t = 0$. Therefore, equating eqns. (3) and (5) at $t = \tau$ gives
\[
K(\tau) = K_0 e^{-\lambda \tau} = \gamma K_0 (1 - e^{-\lambda \tau}) = A(\tau)
\]
This relationship can be rearranged to give an explicit expression for $\tau$ as follows:
\[
e^{-\lambda \tau} = \gamma - \gamma e^{-\lambda \tau}
\]
\[
(1 + \gamma) e^{-\lambda \tau} = \gamma
\]
\[
-\lambda \tau = \ln \left( \frac{\gamma}{1 + \gamma} \right)
\]
or
\[
\tau = -\frac{1}{\lambda} \ln \left( \frac{\gamma}{1 + \gamma} \right) \quad (6)
\]
This expression represents an approximation for the age of the moon rocks in equation form. All we need to do now is to numerically evaluate this expression.

From the definition of half life (i.e. the time it takes for half of the isotope to decay), eqn. (3) gives
\[
\frac{K(T_{1/2})}{K_0} = \frac{1}{2} = e^{-\lambda T_{1/2}}
\]

or

\[-\ln 2 = -\lambda T_{1/2} \quad \text{and} \quad \lambda = \frac{\ln 2}{T_{1/2}}\]

For the decay of \(^{40}\text{K}\), \(T_{1/2} = 1.28\times10^9\) yr and \(\lambda = 5.415\times10^{-10}\) yr\(^{-1}\).

With this result and the specific value of yield from the problem statement \((\gamma = 0.11)\), the age of the moon rock can be estimated by numerical evaluation of eqn. (6) to be about \(4.25\times10^9\) years. Thus, the moon was probably formed about 4.3 billion years ago!
Mixture Problems

Basic Terminology

The physical mixing of two or more material species represents a common situation that occurs in many fields of interest (chemical and biological systems, environmental engineering, etc.). The specifics associated with each field may be quite different, but there is some general terminology that is common to many applications. For introduction here, let’s consider only two material species - a solid material which is dissolved in a fluid. We refer to the solid as the solute and to the liquid as the solvent. Consider a reservoir or container that has inflow and outflow as sketched in the diagram below.

Let $C_i$ and $C_e$ be the concentrations of the solute in the inlet and exit streams with units of mass of solute per unit volume of solvent (kg of solute/liter of solvent), and let $q_i$ and $q_e$ be the inlet and exit volumetric flow rates of the solvent (liter of solvent/sec). With this notation, the mass flow rate of the solute, $m_s$, becomes

$$m_s = qC$$

Mass Balance on the Solute

In general, a mass balance on any system of interest consisting of an arbitrary control volume (CV) can be written as

$$\text{rate of change of mass in CV} = \left( \frac{\ell \text{ of solvent}}{\text{sec}} \right) \left( \frac{\text{kg of solute}}{\ell \text{ of solvent}} \right) = \frac{\text{kg of solute}}{\text{sec}}$$

Let $C_i$ and $C_e$ be the concentrations of the solute in the inlet and exit streams with units of mass of solute per unit volume of solvent (kg of solute/liter of solvent), and let $q_i$ and $q_e$ be the inlet and exit volumetric flow rates of the solvent (liter of solvent/sec). With this notation, the mass flow rate of the solute, $m_s$, becomes

$$m_s = qC$$

Mass Balance on the Solute

In general, a mass balance on any system of interest consisting of an arbitrary control volume (CV) can be written as

$$\text{rate of change of mass in CV} = \left( \frac{\ell \text{ of solvent}}{\text{sec}} \right) \left( \frac{\text{kg of solute}}{\ell \text{ of solvent}} \right) = \frac{\text{kg of solute}}{\text{sec}}$$

This simple expression assumes no generation or destruction within the control volume (by chemical means, for example). Thus, the only way to get material into or out of the CV is by crossing the surface of the CV.

If the quantity of interest is the amount of solute dissolved within the solvent, then combining the notation from eqn. (1) with the generic mass balance in eqn. (2) gives

$$\frac{d}{dt} m_s = q_i C_i - q_e C_e$$

where $m_s$ is the mass of solute in the system and the solvent volume is the CV of interest. This simply says that the rate of change of solute in the system is equal to the net rate of solute crossing the system boundaries (inlet - outlet).
In eqn. (3) the inlet solute concentration and inlet solvent flow rate are independent forcing functions (i.e. they must be specified for a given problem). This leaves three variables, \( m_s \), \( C_e \), and \( q_e \), that need to be determined. Clearly, with only one constraint equation, this is not possible.

**Mass Balance on the Solvent**

This dilemma can be partially resolved by performing another mass balance on the second material species - in this case, the solvent material. If we denote the average solvent density as \( \rho \), the solvent mass within the system is simply \( \rho V \), and the inlet and exit mass flow rates are given by

\[
\dot{m}_i = (\rho q)_i \quad \text{and} \quad \dot{m}_e = (\rho q)_e
\]

Putting these relationships into eqn. (2) gives

\[
\frac{d}{dt} m = \dot{m}_i - \dot{m}_e \quad \text{or} \quad \frac{d}{dt} (\rho V) = (\rho q)_i - (\rho q)_e
\]

If there is no significant density variation within the system, then \( \rho = \rho_i = \rho_e \) and the mass balance equation becomes

\[
\frac{d}{dt} V = q_i - q_e
\]

In many situations, the inlet and exit volumetric flow rates may not be equal. When this happens, we either have partial filling or draining of the system, and the solution of eqn. (6) gives \( V(t) \) as the volume of the system versus time. For example, we have already seen a situation where the exit flow rate is related to the height of a column of fluid in a container (see the discussion on Torricelli’s Law). Knowledge of \( q_i \) and the relationship between \( q_e \) and \( V \) should allow solution of eqn. (6) -- thus, giving \( V(t) \). This information can then be incorporated into eqn. (3).

**Relating Exit and Average Conditions**

Even with known \( V(t) \) and a formal relationship for \( q_e \) as a function of \( V \) or \( m_s \), eqn. (3) still has two unknowns; the mass of solute and the exit concentration, \( m_s \) and \( C_e \), respectively. We somehow must introduce another constraint equation to resolve this conflict - that is, we must somehow relate the exit conditions of the fluid to the average conditions within the CV. Although there are a number of ways to do this, the easiest and the most common method is the so-called well-mixed or continuously stirred approximation. This modeling technique assumes that the exit concentration, \( C_e \), is equal to the average concentration, \( C \), within the volume, \( V \), where \( C \) is simply \( m_s/V \). This assumption occurs naturally in many problems with turbulent flow or with direct stirring of a mixture by mechanical means, and it is quite often a very good approximation. This simple relationship allows us to write the exit concentration in terms of the average concentration, or

\[
C_e \approx C = \frac{m_s}{V} = \frac{\text{mass of solute}}{\text{volume of solvent}}
\]
With eqn. (7), the general solute mass balance in eqn. (3) now becomes
\[
\frac{d}{dt} m_s = q_i C_i - \frac{q_e}{V} m_s
\]  
(8)

This system, along with the solvent mass balance in eqn. (6), represents a coupled set of first order equations that can be solved by traditional means.

**Note:** If the spatial dependence of the concentration distribution within the CV is negligible, then \( C(\bar{r}, t) \rightarrow C(t) \). If this is valid we have a **lumped parameter system** with only one independent variable - which gives an ordinary differential equation (ODE) as the defining equation. When it is necessary to treat the space and time dependent concentration in detail, the system is referred to as a **distributed parameter system**. This notation implies that there are two or more independent variables (space and time, for example), which leads to partial differential equations (PDEs) and these are considerably more difficult to solve. Thus, the well-stirred tank approximation, if it is appropriate for a given system, simplifies the mathematical model significantly.

**Constant Flow Rates**

A special case, of course, is when the inlet and exit flow rates are equal. In this situation, eqn. (6) for the solvent balance simply shows that \( \frac{dV}{dt} = 0 \) and \( V(t) = V \) is constant. This is indeed a common constraint in many problems and it certainly simplifies things considerably. Additionally, a constant flow rate, \( q = q_i = q_e \), also simplifies the solute mass balance in eqn. (8) giving
\[
\frac{d}{dt} m_s = q \left( C_i - \frac{1}{V} m_s \right)
\]  
(9)

With known \( q, V, \) and \( C_i \), this equation represents a single linear first order ODE that can be easily solved for the solute mass versus time, \( m_s(t) \).

Sometimes it may be convenient to treat the average concentration, \( C(t) \), as the dependent variable (instead of the solute mass). For example, if we divide each term in eqn. (9) by the constant volume, \( V \), we have
\[
\frac{d}{dt} C = \frac{q}{V} (C_i - C) = \frac{1}{\tau} (C_i - C)
\]  
(10)

where \( \tau \) is the fluid residence time, or
\[
\tau = \frac{V}{q} \Rightarrow \frac{\text{solute volume}}{\text{solvent volume per unit time}} \Rightarrow \text{residence time}
\]  
(11)

The solution here gives \( C(t) \), the average solute concentration versus time.
Two Salty Tanks

Problem Description:
Consider the cascade of two tanks as shown below.

Assume that the volumetric flow rate throughout the system is constant with \( q = 5 \text{ gal/s} \). With a constant flow rate the volumes of both Tank 1 and Tank 2 are also constant with \( V_1 = 100 \text{ gal} \) and \( V_2 = 200 \text{ gal} \). If the inlet to Tank 1 is pure water and the initial masses of the salt dissolved in the tanks are \( m_{10} = m_{20} = 50 \text{ lbm} \), determine the amount of salt in each tank versus time. Also determine the time and magnitude of the mass in Tank 2 when \( m_2 \) is at its highest value.


Problem Solution:

Background Discussion
This situation is a typical mixing problem (see previous discussion on Mixing Problems) that involves the dilution of a solute-solvent mixture by adding pure solvent to the system. In this case the salt is the solute and the water is the solvent. With only fresh water entering Tank 1 (i.e. the inlet concentration of salt is zero), we expect that the mass of salt in Tank 1, \( m_1 \), and the concentration of salt in Tank 1, \( C_1 = m_1/V_1 \), will decrease over time from their initial values of 50 lbm and 0.5 lbm/gal, respectively. Note, however, that even though Tank 2 has the same amount of salt as Tank 1 initially, because of its larger volume, the initial concentration of salt in Tank 2 is lower than that for Tank 1, with

\[
C_{20} = m_{20}/V_2 = 50 \text{ lbm}/200 \text{ gal} = 0.25 \text{ lbm/gal}
\]

Thus, we might expect \( C_2 \) to increase for some time before it eventually decreases to zero as the overall dilution of the system advances. The problem specification asks us to determine the time profile of the mass of salt in each tank and to address the mass or concentration peak that is expected to occur in Tank 2.
**Tank 1 Mass Balance**

To solve this problem we will follow the same procedure as outlined in our previous general discussion of Mixture Problems. In addition we will assume the well-stirred tank approximation in the model development for this problem. In particular, a mass balance on the salt in Tank 1 gives

\[
\frac{d}{dt} m_i = m_i - m_e = q(C_i - C_e) = -qC_e = -qC_i
\]

where the last equality is the result of the well-mixed tank approximation (i.e. the exit concentration is assumed to be approximately equal to the average concentration).

Since \( C_i = \frac{m_i}{V_i} \), for tank 1, we have

\[
\frac{d}{dt} m_i = -\frac{q}{V_i} m_i
\]

This is a separable equation, so

\[
\frac{dm_i}{m_i} = -\frac{q}{V_i} dt
\]

and integration gives

\[
\ln m_i = -\frac{q}{V_i} t + c'
\]

Taking antilogarithms gives

\[
m_i(t) = e^{-q/t/v_i} = e^{-q/t/v_i} e^{c'} = c e^{-q/t/v_i}
\]

Finally, applying the initial condition gives

at \( t = 0 \), \( m_i(0) = m_{i0} = 50 \text{ lbm} \) and \( c = m_{i0} \)

Therefore, the time profile for the salt mass in Tank 1 is

\[
m_i(t) = m_{i0} e^{-q/v_i}
\]

Putting in numerical values gives

\[
m_i(t) = 50e^{-t/20} \text{ lbm} \quad \text{(with } t \text{ in seconds)}
\]

**Tank 2 Mass Balance**

Setting up a similar mass balance for the salt in Tank 2 gives

\[
\frac{d}{dt} m_2 = q(C_1 - C_2)
\]

where again, we have assumed well-mixed tanks. Here the concentration leaving Tank 1, \( C_1 \), enters Tank 2, and \( C_2 \) is the average (and exit) concentration in Tank 2. Writing the concentrations as mass/volume gives
\[ \frac{d}{dt} m_2 = \frac{q}{V_1} m_1 - \frac{q}{V_2} m_2 \]

or in standard form
\[ \frac{d}{dt} m_2 + \frac{q}{V_2} m_2 = \frac{q}{V_1} m_1 \] (6)

This is a linear first order ODE with an integrating factor given by
\[ IF = e^{\int \frac{q}{V_2} dt} = e^{q t/V_2} \]

Multiplying eqn. (6) by the integrating factor gives
\[ e^{q t/V_2} \left[ \frac{d}{dt} m_2 + \frac{q}{V_2} m_2 \right] = \frac{d}{dt} \left[ e^{q t/V_2} m_2 \right] = \frac{q}{V_1} e^{q t/V_2} m_1 \]

Integrating both sides of this expression gives
\[ e^{q t/V_2} m_2 = \frac{q}{V_1} \int e^{q t/V_2} m_1(t) dt \]

But, \( m_1(t) \) is given explicitly by eqn. (3). Thus, we have
\[ e^{q t/V_2} m_2 = \frac{q m_{10}}{V_1} \int e^{q \left( \frac{1}{V_2} - \frac{1}{V_1} \right) t} dt \]

and this can be simplified, in several steps, to give
\[ e^{q t/V_2} m_2 = \frac{q m_{10}}{V_1} \int e^{q \left( \frac{V_1 - V_2}{V_1 V_2} \right) t} dt = \frac{q m_{10}}{V_1} \left( \frac{V_1 V_2}{q(V_1 - V_2)} \right) e^{q \left( \frac{V_1 - V_2}{V_1 V_2} \right) t} + c \]
\[ = m_{10} \left( \frac{V_2}{V_1 - V_2} \right) e^{q t/V_2} e^{-q t/V_1} + c \]

Now, premultiplying by \( e^{-q t/V_2} \) gives
\[ m_2 = m_{10} \left( \frac{V_2}{V_1 - V_2} \right) e^{-q t/V_1} + ce^{-q t/V_2} \]

Finally, we can use the initial condition \( [at t = 0, \; m_2(0) = m_{20} = 50 \text{ lbm}] \) to determine \( c \), or
\[ m_{20} = m_{10} \left( \frac{V_2}{V_1 - V_2} \right) + c \quad \text{and} \quad c = m_{20} - m_{10} \left( \frac{V_2}{V_1 - V_2} \right) \]

Therefore, the salt mass in Tank 2 versus time is given by
\[ m_2(t) = m_{20} e^{-q t/V_2} + m_{10} \left( \frac{V_2}{V_1 - V_2} \right) \left[ e^{-q t/V_1} - e^{-q t/V_2} \right] \] (7)
Putting in numerical values gives
\[ m_2(t) = 50e^{-t/40} - 100\left[e^{-t/20} - e^{-t/40}\right] = 150e^{-t/40} - 100e^{-t/20} \]  \hspace{1cm} (8)

*What about the peak conditions in Tank 2?*

To address the peak conditions in Tank 2, we note that the maximum of \( m_2(t) \) must occur when \( \frac{dm_2}{dt} = 0 \). Therefore, taking the derivative of eqn. (7) and setting the result to zero gives
\[ \frac{d}{dt} m_2 = -\frac{qm_{20}}{V_2} e^{-qt/V_2} - m_{10}\left(\frac{V_2}{V_1 - V_2}\right)\left[q e^{-qt/V_1} - \frac{q}{V_2} e^{-qt/V_2}\right] = 0 \]  \hspace{1cm} (9)

We could now solve this for \( t = t_{\text{max}} \), and then putting this value of \( t \) into eqn. (7) will yield \( m_{2\text{max}} \). However, as can be seen, this will be very messy algebraically.

Another alternative is to go back to the original balance equation given in eqn. (5). With \( \frac{dm_2}{dt} = 0 \), eqn. (5) tells us that \( C_1 = C_2 \) at peak conditions. This makes sense physically, since as long as \( C_1 > C_2 \), the concentration of Tank 2 should continue to increase. If \( C_1 < C_2 \) the amount of salt in Tank 2 will decrease since the inflow concentration is less that the outflow concentration. Thus, clearly the peak conditions in Tank 2 occur when \( C_1 = C_2 \). Thus, we have the valid relationship,
\[ \frac{1}{V_1} m_{10}e^{-qt/V_1} = \frac{1}{V_2}\left[m_{20}e^{-qt/V_2} + m_{10}\left(\frac{V_2}{V_1 - V_2}\right)\left[e^{-qt/V_1} - e^{-qt/V_2}\right]\right] \]  \hspace{1cm} (10)

at peak conditions. Well, although this alternative approach is insightful, this last expression is no easier to evaluate for \( t = t_{\text{max}} \) than eqn. (9).

*MATLAB to the Rescue!*

Not being a big fan of a lot of tedious algebra, one approach is to simply plot the solutions given by eqn. (3) and eqn. (7) and observe visually where the peak conditions occur. This procedure has been implemented in the first part of Matlab file STANKS2.M. This Matlab program actually does quite a bit of work, and it is broken into three functional parts as follows:

Part I - Function evaluation and plotting of the analytical solution.

Part II - Solution and analysis of the ODEs using Matlab’s symbolic capability.

Part III - **Numerical** solution and analysis of the ODEs using Matlab’s built-in ODE solver.

Thus, STANKS2.M represents a good illustration of a variety of methods (also see ODE23 function file, STANKS2_EQNS.M, which is needed in Part III of STANKS2.M). The reader is strongly encouraged to study these programs carefully (see listings of STANKS2.M and STANKS2_EQNS.M below) and to become familiar with the various Matlab capabilities illustrated here. Actually running the program and observing the output edit and plots would also be very helpful!

Summary results from the program are given below in a single Matlab plot with two subplots; one that shows the mass versus time profiles and another that gives the concentrations versus time. The subplots are very similar, since the masses and concentrations are simply related...
through a normalization factor (i.e. \( C = \frac{m}{V} \)). However, the concentration curves clearly show the fact that \( C_1 = C_2 \) at the peak of the \( C_2(t) \) profile (see above discussion). Thus, from these data, we see that peak conditions in Tank 2 occur at about 11.5 seconds after dilution begins and that, at this maximum point, Tank 2 has about 56.2 lbm of salt dissolved in the water. At this point the salt concentration in both tanks is about 0.281 lbm/gal. In general, all of the results shown here agree with our \textit{a priori} expectations for this system.

\textbf{Note:} Although not shown explicitly here, the solution curves for the three solution methods - analytical, symbolic, and numerical - are all exactly the same (you can run the program to see this). This clearly is as expected and it gives confidence in our ability to use the three methods to accurately solve problems of this type. Clearly, however, once the basic syntax and code algorithm has been worked out for the Matlab solutions, it becomes quite easy to apply either the symbolic or numerical computational methodology to quickly solve similar, but different problems. We may also be able to solve a new problem analytically, but the Matlab solutions - either the symbolic or numeric - offer a much easier and more efficient approach. In addition, if we plan to evaluate and plot the solutions anyway, we might as well simply solve the complete problem in the same code. In practical applications, computer solution and analysis of ODEs is clearly the method of choice and, in many cases where analytical solutions by hand are very tedious or not even possible, it is the only way to go!
Listing of STANKS2.M

STANKS2.M  Solution of the "Two Salty Tanks" problem (Differential Equations)

This Matlab file implements the solution to the "Two Salty Tanks" problem
that was discussed as part of the Class Notes for Differential Equations (92.236).
The problem involves the dilution of a salt-water mixture in a two tank
arrangement. This problem was derived from Prob. 38 pg. 50 in Edwards & Penny

The solution to this problem is given using three different methods and the Matlab
file is broken into three separate parts - one for each method - as follows:
- Part I - Function evaluation and plotting of ANALYTICAL solution
- Part II - Solution of the ODEs using Matlab's SYMBOLIC capability
- Part III - NUMERICAL solution of the defining ODEs using Matlab's ODE23 routine

The defining ODEs that represent a mass balance on the salt in the two tanks are:

Tank 1 ->   \( \frac{dm_1}{dt} = -\frac{q}{V_1}m_1 \)  \( m_1(0) = m_{10} \)
Tank 2 ->   \( \frac{dm_2}{dt} + \frac{q}{V_2}m_2 = \frac{q}{V_1}m_1 \)  \( m_2(0) = m_{20} \)

where
- \( m_1, m_2 \) - salt mass in tanks 1 and 2
- \( q \) - volumetric flow rate of pure water (constant)
- \( V_1, V_2 \) - volumes of water in tanks 1 and 2 (constants)

Also of interest is the concentration of salt in the tanks, where \( C = \frac{m}{V} \).

Both the salt masses and concentrations are plotted versus time as the primary
results of this problem.

There are two important goals for this demonstration:
1. Solution and analysis of the particular mixing problem given here.
2. Demonstration of the various methods available for solving ODEs of this type.


going to get started

\[
\text{clear all; close all; nfig = 0; }
\]
define some base parameters for the problem

\[
m_{10} = 50; \quad m_{20} = 50; \quad % \text{ initial masses of salt in tanks (lbm)}
V_1 = 100; \quad V_2 = 200; \quad % \text{ constant volumes of the tanks (gal)}
q = 5; \quad \% \text{ constant volumetric flow rate (gal/s)}
\]

Part I  Function evaluation and plotting of ANALYTICAL solution

define range of times for plotting solution

to = 0; \quad tf = 90; \quad Nta = 91; \quad ta = linspace(to,tf,Nta);

evaluate masses and concentrations [see "Two Salty Tanks" for derivations]

\[
m_{1a} = m_{10} \exp\left(-\frac{q}{V_1}ta\right);
m_{2a} = m_{20} \exp\left(-\frac{q}{V_2}ta\right) + m_{10} \exp\left(-\frac{q}{V_1}ta\right) \exp\left(-\frac{q}{V_2}ta\right) / (V_1-V_2);
\]

c_{1a} = \frac{m_{1a}}{V_1}; \quad c_{2a} = \frac{m_{2a}}{V_2};

find magnitude and time for peak conditions in Tank 2

\[
[m_{2amax},index] = \text{max}(m_{2a}); \quad t_{amax} = \text{ta}(index);
c_{1amax} = c_{1a}(index); \quad c_{2amax} = c_{2a}(index);
\]

fprintf(1,\"\nSolution to the "Two Salty Tanks" Problem (see Class Notes) \n\"");

fprintf(1,\'\nPeak Conditions in Tank 2 from ANALYTICAL Solution \n\');

fprintf(1,\'\nMethod #1 -- by searching vector of values \n\');

fprintf(1,\'Time of peak : \ tmax = %6.2f sec \n\',t_{amax});

fprintf(1,\'Salt mass at peak: \ m2max = %6.2f lbm \n\',m_{2amax});

fprintf(1,\'Salt conc in Tank 1: \ c1max = %6.3f lbm/gal \n\',c_{1amax});

fprintf(1,\'Salt conc in Tank 2: \ c2max = %6.3f lbm/gal \n\n\n\n\',c_{2amax});

plot ANALYTICAL solutions for the masses and concentrations

nfig = nfig+1; \ figure(nfig)

subplot(2,1,1),plot(ta,m_{1a},r-',ta,m_{2a},g--','LineWidth',2), grid
title(""Two Salty Tanks" ANALYTICAL Results")
ylabel('Salt Mass (lbm)'),'legend(''Tank 1'', 'Tank 2'')
Part II  Solution of the ODEs using Matlab's SYMBOLIC capability

solve the given ODEs for the masses using symbolic manipulations

\[
S = \text{dsolve}('Dm1s = -qs\,m1s/V1s','Dm2s = qs\,m1s/V1s - qs\,m2s/V2s', ...
\]
\[
'm1s(ts0) = m10s','m2s(ts0) = m20s','ts');
\]

set symbolic variables to numerical values (except for the time variable - ts)

to = 0;   tf = 90;
ts0 = to;   m10s = m10;   m20s = m20;
qs = q;     V1s = V1;     V2s = V2;

print solutions

\[
fprintf(1,'\n  The SYMBOLIC solution for m1(t) is: \n');
\]
m1s = S.m1s;   pretty(m1s)

\[
fprintf(1,'\n  and with appropriate variable substitutions, this becomes: \n');
\]
m1s = subs(m1s);   pretty(m1s)

\[
fprintf(1,'\n  The SYMBOLIC solution for m2(t) is: \n');
\]
m2s = S.m2s;   pretty(m2s)

\[
fprintf(1,'\n  and with appropriate variable substitutions, this becomes: \n');
\]
m2s = subs(m2s);   pretty(m2s)

find magnitude and time for peak conditions in Tank 2

Method 1 -- searching vector of values

\[
[m2pmax, index] = \text{max}(m2p);   tmax = \text{tp(index)};
\]

\[
c1pmax = \text{c1p(index)};   c2pmax = \text{c2p(index)};
\]

\[
fprintf(1,'\n\n  Peak Conditions in Tank 2 from SYMBOLIC Solution \n');
\]

\[
fprintf(1,'Method #1 -- by searching vector of values \n');
\]

\[
fprintf(1,'\n  Time of peak :        tmax  =  %6.2f sec 
',tmax);
\]

\[
fprintf(1,'\n  Salt mass at peak:    m2max =  %6.2f lbm 
',m2max);
\]

\[
fprintf(1,'\n  Salt conc in Tank 1:  c1max =  %6.3f lbm/gal 
',c1max);
\]

\[
fprintf(1,'\n  Salt conc in Tank 2:  c2max =  %6.3f lbm/gal 
',c2max);
\]

Method 2 -- by solving SYMBOLIC expression: C1 - C2 = 0

\[
\text{tsmax} = \text{solve}(m1s/V1s-m2s/V2s,ts);
\]

\[
m1max = \text{double}(\text{subs}(m1s,ts,tsmax));   m2max = \text{double}(\text{subs}(m2s,ts,tsmax));
\]

\[
c1max = \text{m1max}/V1;   c2max = \text{m2max}/V2;
\]

\[
fprintf(1,'\nMethod #2 -- by solving SYMBOLIC expression: C1 - C2 = 0 \n');
\]

\[
fprintf(1,'\n  Time of peak :        tmax  =  %6.2f sec 
',\text{double}(tsmax));
\]

\[
fprintf(1,'\n  Salt mass at peak:    m2max =  %6.2f lbm 
',\text{m2max});
\]

\[
fprintf(1,'\n  Salt conc in Tank 1:  c1max =  %6.3f lbm/gal 
',\text{c1max});
\]

\[
fprintf(1,'\n  Salt conc in Tank 2:  c2max =  %6.3f lbm/gal 
',\text{c2max});
\]

plot SYMBOLIC solutions for the masses and concentrations

\[
f1g - f1g+1;   \text{figure}(f1g)
\]

\[
\text{subplot}(2,1,1),\text{plot}(\text{tp},\text{m1p},'r-',\text{tp},\text{m2p},'g--','\text{LineWidth}',2), \text{grid}
\]

\[
\text{ylabel}('\text{Salt Mass (lbm)}'),\text{legend}('\text{Tank 1}','\text{Tank 2}')
\]

\[
\text{xlabel}('\text{Time (sec)}')
\]

Part III  NUMERICAL solution of the defining ODEs using Matlab's ODE23 routine

define domain of interest, initial condition vector, and options for ODE23

to = 0;   tf = 90;   zo = [m10 m20]';
Ntn = 91;   tn = \text{linspace}(to,tf,Ntn);
tol = 0.0001;   options = \text{odeset}('\text{RelTol}',tol);

evaluate masses and concentrations [use built-in ODE solver]

\[
[\text{tn},\text{zn}] = \text{ode23}('\text{stanks2_eqns}',\text{tn},\text{zo},\text{options},\text{q},V1,V2);
\]
\begin{verbatim}
m1n = zn(:,1); m2n = zn(:,2);
c1n = m1n/V1; c2n = m2n/V2;

% find magnitude and time for peak conditions in Tank 2
[m2nmax,index] = max(m2n); tnmax = tn(index);
c1nmax = c1n(index); c2nmax = c2n(index);
fprintf(1,'  Peak Conditions in Tank 2 from NUMERICAL Solution 
');
fprintf(1,'     Method #1 -- by searching vector of values \n');
fprintf(1,'     Time of peak :        tmax  =  %6.2f sec 
',tnmax);
fprintf(1,'     Salt mass at peak:    m2max =  %6.2f lbm 
',m2nmax);
fprintf(1,'     Salt conc in Tank 1:  c1max =  %6.3f lbm/gal 
',c1nmax);
fprintf(1,'     Salt conc in Tank 2:  c2max =  %6.3f lbm/gal 
',c2nmax);

% plot NUMERICAL solutions for the masses and concentrations
nfig = nfig+1; figure(nfig)
subplot(2,1,1),plot(tn,m1n,'r-',tn,m2n,'g--','LineWidth',2), grid
title('Two Salty Tanks NUMERICAL Results')
ylabel('Salt Mass (lbm)'),legend('Tank 1','Tank 2')

subplot(2,1,2),plot(tn,c1n,'r-',tn,c2n,'g--','LineWidth',2), grid
ylabel('Salt Conc. (lbm/gal)'),legend('Tank 1','Tank 2')
xlabel('Time (sec)')

% end of program
\end{verbatim}

Listing of \texttt{STANKS2_EQNS.M}

\begin{verbatim}

% STANKS2_EQNS.M  Function evaluation for ODE step in STANKS2.M
% Defining differential equations for the "Two Salty Tanks" problem
% Tank 1 ->   dm1/dt = -(q/V1)m1               m1(0) = m10
% Tank 2 ->   dm2/dt + (q/V2)m2 = (q/V1)m1     m2(0) = m20
% Here we have z(1) -> m1  (salt mass in tank 1)
% z(2) -> m2  (salt mass in tank 2)
% t  -> time in seconds (this is the independent variable)
% flag -> dummy variable needed as a place holder for ODE23 (not used)
% Also, for this example, we have passed in three arguments within the
% function call list, as follows:
% q  = constant volume flow rate for both tanks
% V1 = fixed volume of tank 1
% V2 = fixed volume of tank 2

function zp = eqnfile(t,z,flag,q,V1,V2)
zp = zeros(length(z),1);
zp(1) = -(q/V1)*z(1);
zp(2) = -(q/V2)*z(1) + (q/V1)*z(2);
\end{verbatim}

Population Models

Basic Terminology

General population models can always be written in the form of a standard balance equation,

\[
\frac{\text{rate of change}}{\text{of quantity}} = \left[\frac{\text{production rate}}{\text{of quantity}}\right] - \left[\frac{\text{loss rate}}{\text{of quantity}}\right]
\]

(1)

where, for this application, the quantity of interest relates to the number of members of a given population. Also, for many studies, the production and loss rates are simply the birth and death rates associated with a given population. With this terminology and \(P(t)\) representing the amount of the species of interest present at time \(t\), eqn. (1) can be written as

\[
\frac{d}{dt} P(t) = \text{birth rate} - \text{death rate} = BP - DP = (B - D)P
\]

(2)

where \(B\) and \(D\) are the normalized birth and death rates, respectively. These normalized rates are defined precisely as follows:

\[
B = \frac{\text{birth rate}}{P} = \text{number of births per unit time per unit population}
\]

\[
D = \frac{\text{death rate}}{P} = \text{number of deaths per unit time per unit population}
\]

Note that \(B\) and \(D\) can be functions of time or directly related to the population, \(P\), depending on the specific situation of interest. A few special cases are illustrated below:

Constant Birth and Death Rates

If both \(B\) and \(D\) are constant, then \(k = (B - D)\) is a constant, and the population balance in eqn. (2) becomes

\[
\frac{dP}{dt} = kP
\]

(3)

Note that \(k\) can be positive or negative depending upon which normalized rate term is dominant. This first-order ODE is easily solved as a separable equation, leading to a general solution of the form

\[
P(t) = P_0 e^{kt}
\]

(4)

where \(P_0\) represents the initial population size.

Birth Rate Decreases with Increasing Population (\(D = \text{constant}\))

Most populations are limited in some fashion, often by restricted food supplies or physical space. A simple mathematical model that takes this observation into account models the normalized birth rate as a linearly decreasing function of the population size, or

\[
B = B_0 - B_1P
\]

(5)
where $B_0$ and $B_1$ are positive constants. This is simply one approximation that allows for a decreasing birth rate as the population increases. If we incorporate this birth rate representation with a constant normalized death rate, $D_0$, the population balance in eqn. (2) becomes

$$\frac{d}{dt} P(t) = B_0 P - B_1 P^2 - D_0 P = (B_0 - D_0)P - B_1 P^2 \quad (6)$$

To simplify this expression a little, we define two new constants,

$$k = B_1 \quad \text{and} \quad M = \frac{(B_0 - D_0)}{B_1} \quad (7)$$

With these definitions, eqn. (6) can be written as

$$\frac{d}{dt} P = kMP - kP^2 \quad (8)$$

This is often referred to as the **Logistic Equation**. This expression is separable and its detailed solution as a separable equation is developed in your text.

**Solution of the Logistic Equation as a Bernoulli Equation**

The logistic equation is also a Bernoulli equation and we will solve it here using this technique - as another example of this solution scheme. First we rewrite the nonlinear population balance in standard form

$$\frac{d}{dt} P - kMP = -kP^2 \quad (9)$$

and then recognize this as a Bernoulli equation of the form,

$$y' + p(x)y = g(x)y^a \quad (10)$$

Since letting $u(x) = y^{1-a}$ converts the nonlinear Bernoulli equation into a $1^{st}$ order linear system, we let $u = P^{1-a} = P^{-1}$ for the current problem (since $a = 2$ here). With this substitution we have

$$u(t) = \frac{1}{P(t)} = P^{-1} \quad \text{and} \quad \frac{d}{dt} u = -P^{-2} \frac{d}{dt} P \quad (11)$$

Before substitution, let's first multiply eqn. (9) by $-P^{-2}$, giving

$$-P^{-2} \frac{d}{dt} P + kMP^{-1} = k$$

Now, making the indicated substitutions from eqn. (11), this last expression simplifies to

$$\frac{d}{dt} u + kMu = k \quad (12)$$

This first order linear ODE has an integrating factor given by

$$g(t) = e^{\int kMdt} = e^{kMt}$$
Now, multiplying by the integrating factor gives
\[ e^{kMt} \left[ \frac{d}{dt} u + kMu \right] = \frac{d}{dt} \left[ e^{kMt} u \right] = ke^{kMt} \]
and one final integration gives
\[ e^{kMt} u = k \int e^{kMt} dt = k \left[ \frac{1}{kM} e^{kMt} \right] + c' \]
or
\[ u = \frac{1}{M} + c' e^{-kMt} = \frac{1 + ce^{-kMt}}{M} \]
Finally, since \( u = 1/P \), we can write the general solution to the logistic equation as
\[ P(t) = \frac{M}{1 + ce^{-kMt}} \]  
(13)
This is the general solution to the original population model stated in eqn. (8). To obtain a unique solution we simply apply the initial condition, \( P(0) = P_0 \). Doing this systematically gives
\[ P_0 = \frac{M}{1 + c} \]
or
\[ c = \frac{M}{P_0} - 1 = \frac{M - P_0}{P_0} \]
Putting this result for \( c \) into the general solution gives
\[ P(t) = \frac{M}{1 + \left( \frac{M - P_0}{P_0} \right) e^{-kMt}} \]
or
\[ P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}} \]  
(14)
This is the final unique solution for this mathematical model.
As a final note, we see that as time becomes large (i.e. \( t \to \infty \)), the population approaches a limiting value, or
\[ \text{as } t \to \infty \quad P(t) \to \frac{MP_0}{P_0} = M \]  
(15)
The quantity, \( M \), which was defined in terms of the coefficients in the birth and death rate expressions [see eqn. (7)], is referred to as the limiting population. This result was expected and it is consistent with the physical model for this system.
A Fish Story

Problem Description:
Consider the fish population, $P(t)$, in a fresh-water man-made pond. Assume that the initial population is set at 100 fish by stocking the pond with a combination of mature male and female fresh-water trout. Based on various environmental factors (food supply, water conditions, etc.), a reasonable approximation to the population dynamics for low population sizes suggests that the normalized fish birth and death rates are both inversely proportional to the square root of the population present at time $t$.

Based on the above description, derive a mathematical model for the fish population and find a general solution for $P(t)$. Given that the initial population is 100 fish and that the measured population after six months is 169 fish, estimate how many fish will be in the pond after the first year. Using the same dynamics model, what will be the population size after five years? Based on your results, what can you say about the accuracy of the mathematical model for large population sizes?

Reference: This problem was derived from Prob. 2 on page 76 of the Differential Equations text by Edwards and Penny (Prentice Hall Inc., 1996).

Problem Solution:

Model Development

This is a classical problem in population dynamics. Since the birth and death rates are normalized, we can write the balance equation as

$$\frac{d}{dt} P(t) = \text{birth rate} - \text{death rate} = BP - DP$$

where $B$ and $D$ are the normalized birth and death rates, respectively.

For the mathematical model described above, the birth and death rates are inversely proportional to the square root of the population size, or

$$B = \frac{k_1}{\sqrt{P}} \quad \text{and} \quad D = \frac{k_2}{\sqrt{P}}$$

Therefore, the population balance equation becomes

$$\frac{dP}{dt} = (B - D)P = (k_1 - k_2) \frac{P}{\sqrt{P}} = k\sqrt{P}$$

where $k$ is constant. Thus, the mathematical model for this system is simply $P' = k\sqrt{P}$.
Analytical Solution

This is a separable equation whose solution can be developed as follows. We first rewrite the balance equation as

\[
\frac{1}{\sqrt{P}} \, dP = k \, dt \quad \text{or} \quad P^{-\frac{1}{2}} \, dP = k \, dt
\]

Integrating both sides gives

\[
2P^{\frac{1}{2}} = kt + c'
\]

and, after some simple algebra, the general solution for the fish population becomes

\[
P(t) = \left(\frac{1}{2} kt + c\right)^2
\]  

(4)

Now, applying the initial condition gives \( P(0) = P_0 = c^2 \); therefore \( c = \sqrt{P_0} \) and the unique solution is

\[
P(t) = \left[\frac{1}{2} kt + \sqrt{P_0}\right]^2
\]  

(5)

The growth rate constant, \( k \), can be determined from the data given in the problem specification. We know that \( P_0 = 100 \) fish and after 6 months, \( P_6 = 169 \) fish. Evaluating eqn. (5) at \( t = 6 \) months gives

\[
P_6 = 169 = \left[\frac{1}{2} k(6) + \sqrt{100}\right]^2
\]

and solving this for \( k \) gives \( k = 1 \text{ fish}^{1/2} / \text{month} \) (note that the odd units here are needed to give consistency in the original balance equation -- since \( k \sqrt{P} \) must have units of fish per month).

With the growth rate constant known, we can use the unique solution in eqn. (5) as a predictive model. Evaluating this expression at \( t = 1 \) year (12 months) and again at 5 years (60 months) gives

\[
P_{12} = \left[\frac{1}{2} (1)(12) + 10\right]^2 = 16^2 = 256 \quad \text{and} \quad P_{60} = \left[\frac{1}{2} (1)(60) + 10\right]^2 = 40^2 = 1600
\]

Thus, we see that the fish population grows rather rapidly, especially at larger values of \( P \) (since the rate of change of \( P \) is proportional to \( \sqrt{P} \)). This model, however, predicts unbounded growth as time becomes large, and this is physically not possible for a pond of finite size and finite food supplies. The model may be adequate over a period of several years, but eventually its unbounded nature would lead to large errors in a real limited environment. Periodic comparison of the model’s predictive capability and the actual fish population would alert the user of the need to modify the mathematical model for this ecosystem.
This problem was solved rather easily using standard analytical techniques. It can also be solved numerically in Matlab, which we illustrate here. For use with a standard numerical ODE solver (like ODE23 in Matlab), one must have numerical values for all the constants within the mathematical model. In this problem, the only constant is the growth rate factor, k, but it is initially an unknown parameter. The usual way to handle this situation in numerical solutions is to simply guess a range of k values, solve the given IVP for each value of k, and then determine which value fits all the data given for the problem of interest.

This basic solution scheme was implemented within Matlab files FISHSTORY.M and FISHSTORY_EQNS.M (see listings below). Four values of k were chosen, ODE23 was used to solve eqn. (3) for each k value with an initial fish population of 100 trout, and the resultant P(t) profiles were plotted. A single point, representing the known fish population at 6 months (i.e. \( P_6 = 169 \) fish), was also indicated on the plot of possible solution curves. The P(t) profile that passes through this known data point represents the correct solution, identifying the value of k that is consistent with the data for this problem. As seen in the plot shown below, a numerical value of \( k = 1.0 \) gives the desired solution, and this agrees with the analytical solution determined above. With a known value for the growth rate, the mathematical model now can be use as a predictive tool in future environmental studies.

This example is typical of many real problems where all the model coefficients are not known prior to solution of the defining balance equations. Instead, they are determined from actual measured data that are obtained from the physical system under study. In these situations, one must search for the unknown parameters by systematically guessing values for the unknown coefficients until the solution of the balance equations matches the known measured data. When this occurs, a proper set of coefficients has been identified, and a predictive model is now available. For problems with one unknown parameter, this search procedure is very straightforward (as illustrated in this example). For problems with higher dimensionality, the basic concept is similar, but actual implementation is more complicated because of the number of combinations that are available (formal optimization methods and search algorithms are available for these cases).
Listing of **FISHSTORY.M**

FISHSTORY.M  Find fish population versus time for "Fish Story" Example

This problem is described in some detail in the Class Notes for Differential Equations. The basic idea was derived from Prob. 2 pg. 76 of Edwards and Penny (Prentice Hall Inc., 1996). It addresses the NUMERICAL solution of a particular Population Model that approximates the fish population in a man-made pond.

The population balance equation for this problem is

\[ \frac{dP}{dt} = k \sqrt{P} \]

where \( k \) is a constant that needs to be determined based on data for the particular problem. We will solve the problem for a range of \( k \)'s.

Calls FISHSTORY_EQNS.M to evaluate desired function \( P' = f(t,P) \).


getting started

\[ \text{clear all; close all; nfig = 0;} \]
\[ \text{global kk} \]

define integration variables and initial condition

\[ \text{to = 0; tf = 30; Po = 100; \% time in months and P in numbers of fish} \]

let's define 4 values of \( k \) to investigate (since we don't know this value)

\[ \text{k = [0.50 1.00 1.50 2.00]; \% growth rate constant (units of \( \sqrt{\text{fish}}/\text{month} \)} \]

\[ \text{cols = ['r- ';'g: ';'b-.';'m--'];} \]

evaluate \( P(t) \) for each value of \( k \)

(\( \text{Note: A single experimental value of the fish population at a specific time point should be sufficient to determine the unknown growth coefficient, } k, \text{ in the mathematical model. However, since we are solving this problem numerically, our approach is to search for the value of } k \text{ that gives a solution curve that passes directly through the known experimental value. If this occurs, then we have } \text{"guessed" the correct value of } k, \text{ and we have the correct population profile vs time. In this problem, we pick four (4) values of } k \text{ and plot the resultant solution to the given IVP, as well as the known experimental point. From here, we can easily pick an appropriate value for } k. \text{ As seen in the resultant plots, } k = 1.0 \text{ gives essentially the exact solution.}) \]

\[ \text{Nk = length(k);} \]
\[ \text{for } n = 1:Nk \]
\[ \text{kk = k(n);} \]
\[ \text{[t,P] = ode23('fishstory_eqns',[to tf],Po);} \]
\[ \text{plot(t,P,cols(n,:),'LineWidth',2),hold on} \]
\[ \text{gtext([''k = ',num2str(kk,'%2.1f')])} \]
\[ \text{end} \]
\[ \text{plot(6,169,'kx','LineWidth',2),grid \% mark the known experimental point} \]
\[ \text{gtext('expt')} \]
\[ \text{hold off} \]
\[ \text{title('Fish Population Versus Time for the "Fish Story" Example')} \]
\[ \text{xlabel('Time (months)'),ylabel('Number of Fish')} \]

end of problem

Listing of **FISHSTORY_EQNS.M**

FISHSTORY_EQNS.M  Function file for solution of the "Fish Story" example

This file simply evaluates \( y_p = f(x,y) \) for the input \( x \) and \( y \) values.

For this problem: \( x \rightarrow \text{time} \quad \text{and} \quad y \rightarrow \text{fish population} \)


\[ \text{function yp = eqnfile(x,y)} \]
\[ \text{global kk} \]
\[ \text{yp = kk*sqrt(y);} \]

end of function